# Analyzing Uncertainty through Nonlinear Rational Contractions in Perturbed Metric Spaces


**Abstract**

In the framework of perturbed metric spaces introduced by Jleli and Samet, where the effective distance is determined by the exact metric $d = \mathscr{D} - \mathscr{P}$, we investigate rational-type contractive conditions that incorporate the perturbation component. First, we introduce a perturbed rational contraction formulated directly in terms of the exact metric $d$ and establish a Banach-type fixed point theorem guaranteeing the existence and uniqueness of a fixed point together with convergence of


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the Picard iteration. Next, we propose a nonlinear perturbed rational contraction in which the contractive control is expressed through a rational inequality involving the perturbation functional $\mathscr{P}$. Under the parameter restriction $\lambda(1+2a)<1$, we prove a corresponding Banach-type fixed point theorem and derive an explicit linear convergence estimate for the iterative sequence. Several examples are presented to illustrate the applicability of the proposed contractive conditions. As an application, we study a nonlinear Volterra fractional integral equation and show that the associated solution operator admits a unique fixed point under a natural Lipschitz-type condition on the kernel, leading to existence and uniqueness of the solution in an appropriate function space. These results extend classical rational contraction principles to perturbed metric structures and contribute to the development of fixed point theory in generalized metric frameworks involving perturbations.

**Keywords:** perturbed metric space; nonlinear rational contraction; fixed point; Picard iteration; uncertainty modeling.

**MSC 2020:** 47H10, 54H25.

# 1 Introduction

Fixed point theory plays a fundamental role in nonlinear analysis and provides a powerful framework for studying iterative processes arising in differential equations, optimization, and mathematical modeling. One of the most influential results in this direction is the Banach contraction principle [1], which guarantees the existence and uniqueness of a fixed point together with the convergence of the Picard iteration. Since the appearance of this theorem, numerous generalizations of contractive mappings have been developed in order to treat nonlinear phenomena that do not satisfy strict Lipschitz conditions. Important contributions include the Kannan contraction [2], the Chatterjea contraction [3], and rational-type contractions introduced by Dass and Gupta [4]. Further refinements and comparisons of various contractive mappings were investigated by Geraghty [5] and Rhoades [6]. Other significant extensions involve altering distance functions [7], simulation functions [15], and cyclical contractive mappings [8]. These developments illustrate the broad scope of fixed point theory and its ability to model increasingly complex nonlinear phenomena.

Alongside these developments, many researchers have extended fixed point results to generalized distance structures. For instance, fuzzy metric spaces introduced by Kramosil and Michalek [9] and later developed by George and Veeramani [10] have been widely used to analyze uncertainty in mathematical models. Similarly, partial metric spaces introduced by Matthews [11] and $b$-metric spaces studied by Czerwik [12] provide flexible frameworks in which classical metric assumptions are relaxed. Other generalized structures, such as intuitionistic fuzzy metric spaces, have also attracted considerable attention in the literature [16, 17]. Further developments include graphical metric frameworks and modular-type structures that broaden the scope of nonlinear analysis [13, 14]. Additional generalizations involving simulation functions and related contractive techniques have also been investigated in recent years [25, 26].

Recently, Jleli and Samet [18] introduced the concept of a *perturbed metric space*, which has attracted considerable interest because of its ability to model uncertainty and perturbations in distance measurements. In this framework, the effective distance is represented as

$$d(\xi, \eta) = \mathscr{D}(\xi, \eta) - \mathscr{P}(\xi, \eta),$$

where $\mathscr{D}$ and $\mathscr{P}$ are nonnegative functionals defined on $X \times X$. The function $d$ serves as the exact metric governing convergence and the induced topology, while the perturbation component $\mathscr{P}$ captures deviations, noise, or measurement errors. This decomposition provides a flexible mathematical structure for analyzing systems in which the observed distance is affected by uncertainty. Following the introduction of this concept, several authors have proposed new contractive conditions and extensions within perturbed metric spaces; see, for example, [19–22].

Motivated by the perturbed metric framework of Jleli and Samet [18] and the rational contraction technique introduced by Dass and Gupta [4], the present paper develops rational-type contractions adapted to perturbed metric spaces. We establish Banach-type fixed point theorems for these contractions and derive explicit convergence estimates for the associated Picard iteration. The obtained results extend classical rational contraction principles to perturbed metric structures and provide a useful analytical tool for studying nonlinear models involving uncertainty.

## 2 Preliminaries

**Definition 2.1** (Perturbed metric space [18]). *Let $X$ be a nonempty set and let $\mathscr{D}, \mathscr{P} : X \times X \to [0, \infty)$ be two functions. Define*

$$d(\xi, \eta) := \mathscr{D}(\xi, \eta) - \mathscr{P}(\xi, \eta), \qquad \xi, \eta \in X. \tag{1}$$

*The triple $(X, \mathscr{D}, \mathscr{P})$ is called a* perturbed metric space *if the function $d : X \times X \to [0, \infty)$ defined by (1) satisfies the following axioms for all $\xi, \eta, \zeta \in X$:*

(P1) $d(\xi, \eta) \geq 0$*;*

(P2) $d(\xi, \eta) = 0$ *if and only if* $\xi = \eta$*;*

(P3) $d(\xi, \eta) = d(\eta, \xi)$;

(P4) $d(\xi, \zeta) \leq d(\xi, \eta) + d(\eta, \zeta)$.

*In this case, $d$ is called the* exact metric *associated with the pair* $(\mathscr{D}, \mathscr{P})$.

**Definition 2.2** (Convergence, Cauchy property, completeness)**.** *Let $(X, \mathscr{D}, \mathscr{P})$ be a perturbed metric space with exact metric $d$ given by* (1).

(a) *A sequence $\{\xi_n\} \subset X$ converges to $\xi \in X$ if $d(\xi_n, \xi) \to 0$ as $n \to \infty$.*

(b) *A sequence $\{\xi_n\} \subset X$ is called* Cauchy *if $d(\xi_m, \xi_n) \to 0$ as $m, n \to \infty$.*

(c) *The perturbed metric space $(X, \mathscr{D}, \mathscr{P})$ is said to be* complete *if the metric space $(X, d)$ is complete.*

*Remark* 2.3. All analytical notions such as convergence, Cauchy sequences, completeness, continuity, and the induced topology are determined entirely by the exact metric $d = \mathscr{D} - \mathscr{P}$.

**Example 2.4.** Let $X = [0, 1]$ and define $\mathscr{D}, \mathscr{P} : X \times X \to [0, \infty)$ by

$$\mathscr{D}(\xi, \eta) = |\xi - \eta| + \xi\eta, \qquad \mathscr{P}(\xi, \eta) = \xi\eta.$$

Then $d(\xi, \eta) = \mathscr{D}(\xi, \eta) - \mathscr{P}(\xi, \eta) = |\xi - \eta|$, so $(X, \mathscr{D}, \mathscr{P})$ is a perturbed metric space.

**Example 2.5.** Let $X = \mathbb{R}$ and define $\mathscr{D}, \mathscr{P} : X \times X \to [0, \infty)$ by

$$\mathscr{D}(\xi, \eta) = |\xi - \eta| + |\xi||\eta|, \qquad \mathscr{P}(\xi, \eta) = |\xi||\eta|.$$

Then $d(\xi, \eta) = |\xi - \eta|$, hence $(X, \mathscr{D}, \mathscr{P})$ is a perturbed metric space.

*Remark* 2.6. Since $(X, \mathscr{D}, \mathscr{P})$ is a perturbed metric space, $(X, d)$ is a metric space. Hence: (i) every $d$-convergent sequence is $d$-Cauchy; (ii) the triangle inequality holds for $d$; (iii) $d(\xi, \eta) \leq \mathscr{D}(\xi, \eta)$ because $\mathscr{P} \geq 0$.

## 3 Main Results

**Definition 3.1** (Perturbed rational contraction)**.** *Let $(X, \mathscr{D}, \mathscr{P})$ be a perturbed metric space with exact metric $d = \mathscr{D} - \mathscr{P}$. A mapping $\mathscr{T} : X \to X$ is called a* perturbed rational contraction *if there exist constants*

$$\lambda \in (0, 1), \qquad \alpha \geq 0,$$

*such that for all $\xi, \eta \in X$,*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq \lambda \frac{d(\xi, \eta) + \alpha\, d(\xi, \mathscr{T}\xi) d(\eta, \mathscr{T}\eta)}{1 + d(\xi, \eta)}. \tag{2}$$

**Theorem 3.2** (Banach-type theorem for perturbed rational contractions)**.** *Let $(X, \mathscr{D}, \mathscr{P})$ be a complete perturbed metric space with exact metric $d = \mathscr{D} - \mathscr{P}$. Assume that $\mathscr{T} : X \to X$ satisfies (2) for some $\lambda \in (0, 1)$ and $\alpha \geq 0$, and suppose that*

$$\lambda\alpha < 1.$$

*Then $\mathscr{T}$ possesses a unique fixed point $\xi^* \in X$. Moreover, for any $\xi_0 \in X$, the Picard iteration $\xi_{n+1} = \mathscr{T}\xi_n$ converges to $\xi^*$ in the metric $d$.*

*Proof.* Fix $\xi_0 \in X$ and define the Picard sequence $\xi_{n+1} = \mathscr{T}\xi_n$ for $n \geq 0$. Set $t_n := d(\xi_n, \xi_{n+1})$. If $t_0 = 0$, then $\xi_0$ is a fixed point and there is nothing to prove. Assume

$t_0 > 0$. Applying (2) with $(\xi,\eta) = (\xi_n,\xi_{n-1})$ (for $n \geq 1$) and using $\mathscr{T}\xi_n = \xi_{n+1}$, $\mathscr{T}\xi_{n-1} = \xi_n$ gives

$$t_n \leq \lambda \frac{t_{n-1} + \alpha\, t_n t_{n-1}}{1 + t_{n-1}}.$$

Hence

$$t_n\big(1 + t_{n-1}(1 - \lambda\alpha)\big) \leq \lambda t_{n-1}.$$

Since $\lambda\alpha < 1$, we have $1 + t_{n-1}(1-\lambda\alpha) \geq 1$, and therefore $t_n \leq \lambda t_{n-1}$. Iterating yields $t_n \leq \lambda^n t_0 \to 0$.

For $m > n$, the triangle inequality implies

$$d(\xi_m,\xi_n) \leq \sum_{k=n}^{m-1} d(\xi_{k+1},\xi_k) = \sum_{k=n}^{m-1} t_k \leq \sum_{k=n}^{\infty} \lambda^k t_0 = \frac{\lambda^n}{1-\lambda} t_0,$$

so $\{\xi_n\}$ is Cauchy in $(X,d)$. Completeness yields $\xi^* \in X$ with $d(\xi_n,\xi^*) \to 0$.

Applying (2) with $(\xi,\eta) = (\xi^*,\xi_n)$ gives

$$d(\mathscr{T}\xi^*,\xi_{n+1}) \leq \lambda \frac{d(\xi^*,\xi_n) + \alpha\, d(\xi^*,\mathscr{T}\xi^*)\, d(\xi_n,\xi_{n+1})}{1 + d(\xi^*,\xi_n)}.$$

Letting $n \to \infty$ and using $d(\xi^*,\xi_n) \to 0$ and $d(\xi_n,\xi_{n+1}) = t_n \to 0$ yields $d(\mathscr{T}\xi^*,\xi^*) = 0$, hence $\mathscr{T}\xi^* = \xi^*$.

For uniqueness, if $\zeta_1,\zeta_2$ are fixed points, then (2) gives

$$d(\zeta_1,\zeta_2) = d(\mathscr{T}\zeta_1,\mathscr{T}\zeta_2) \leq \lambda \frac{d(\zeta_1,\zeta_2)}{1 + d(\zeta_1,\zeta_2)},$$

which forces $d(\zeta_1,\zeta_2) = 0$ and hence $\zeta_1 = \zeta_2$. □

**Definition 3.3** (Nonlinear perturbed rational contraction)**.** *Let* $(X,\mathscr{D},\mathscr{P})$ *be a perturbed* *metric space with exact metric* $d = \mathscr{D} - \mathscr{P}$. *A mapping* $\mathscr{T} : X \to X$ *is called a* nonlinear perturbed rational contraction *if there exist constants*

$$\lambda \in (0, 1), \qquad a \geq 0, \qquad b \geq 0,$$

*such that for all* $\xi, \eta \in X$,

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq \lambda \, \frac{d(\xi, \eta) + a\big(d(\xi, \mathscr{T}\xi) + d(\eta, \mathscr{T}\eta)\big)}{1 + b\, \mathscr{P}(\xi, \eta)}. \tag{3}$$

**Theorem 3.4** (Banach-type fixed point theorem)**.** *Let* $(X, \mathscr{D}, \mathscr{P})$ *be a complete perturbed metric space with exact metric* $d = \mathscr{D} - \mathscr{P}$. *Assume that* $\mathscr{T} : X \to X$ *satisfies* (3) *for some* $\lambda \in (0, 1)$, $a \geq 0$, *and* $b \geq 0$, *and that*

$$\lambda(1 + 2a) < 1.$$

*Then* $\mathscr{T}$ *has a unique fixed point* $\xi^* \in X$. *Moreover, for any* $\xi_0 \in X$, *the Picard iteration* $\xi_{n+1} = \mathscr{T}\xi_n$ *converges to* $\xi^*$ *in* $(X, d)$, *and the convergence is at least linear.*

*Proof.* Fix $\xi_0 \in X$ and define $\xi_{n+1} = \mathscr{T}\xi_n$ for $n \geq 0$. Set $t_n := d(\xi_n, \xi_{n+1})$. If $t_0 = 0$, then $\xi_0$ is a fixed point. Assume $t_0 > 0$.

For $n \geq 1$, applying (3) with $(\xi, \eta) = (\xi_n, \xi_{n-1})$ gives

$$t_n = d(\mathscr{T}\xi_n, \mathscr{T}\xi_{n-1}) \leq \lambda \, \frac{d(\xi_n, \xi_{n-1}) + a\big(d(\xi_n, \mathscr{T}\xi_n) + d(\xi_{n-1}, \mathscr{T}\xi_{n-1})\big)}{1 + b\, \mathscr{P}(\xi_n, \xi_{n-1})}.$$

Using $\mathscr{T}\xi_n = \xi_{n+1}$ and $\mathscr{T}\xi_{n-1} = \xi_n$, we obtain

$$t_n \le \lambda \frac{t_{n-1} + a(t_n + t_{n-1})}{1 + b\,\mathscr{P}(\xi_n, \xi_{n-1})} \le \lambda\big((1+a)t_{n-1} + at_n\big),$$

since $1 + b\,\mathscr{P}(\xi_n, \xi_{n-1}) \ge 1$. Hence

$$(1 - \lambda a)t_n \le \lambda(1+a)t_{n-1}.$$

Because $\lambda(1+2a) < 1$, we have $1 - \lambda a > 0$ and the constant

$$q := \frac{\lambda(1+a)}{1 - \lambda a}$$

satisfies $q \in (0,1)$. Therefore $t_n \le q\,t_{n-1}$ for all $n \ge 1$, and hence $t_n \le q^n t_0 \to 0$.

For $m > n$, the triangle inequality yields

$$d(\xi_m, \xi_n) \le \sum_{k=n}^{m-1} d(\xi_{k+1}, \xi_k) = \sum_{k=n}^{m-1} t_k \le \sum_{k=n}^{\infty} t_k \le t_0 \sum_{k=n}^{\infty} q^k = t_0 \frac{q^n}{1-q},$$

so $\{\xi_n\}$ is Cauchy in $(X,d)$. Completeness implies the existence of $\xi^* \in X$ such that $d(\xi_n, \xi^*) \to 0$.

To prove $\mathscr{T}\xi^* = \xi^*$, apply (3) with $(\xi, \eta) = (\xi^*, \xi_n)$:

$$d(\mathscr{T}\xi^*, \xi_{n+1}) = d(\mathscr{T}\xi^*, \mathscr{T}\xi_n) \le \lambda \frac{d(\xi^*, \xi_n) + a\big(d(\xi^*, \mathscr{T}\xi^*) + d(\xi_n, \xi_{n+1})\big)}{1 + b\,\mathscr{P}(\xi^*, \xi_n)} \le \lambda\Big(d(\xi^*, \xi_n) + a\,d(\xi^*, \mathscr{T}\xi^*) + c$$

Letting $n \to \infty$ and using $d(\xi^*, \xi_n) \to 0$, $t_n \to 0$, and $d(\xi_{n+1}, \xi^*) \to 0$ gives

$$d(\mathscr{T}\xi^*, \xi^*) \le \lambda a\, d(\xi^*, \mathscr{T}\xi^*) = \lambda a\, d(\mathscr{T}\xi^*, \xi^*).$$

Thus $(1-\lambda a)d(\mathscr{T}\xi^*,\xi^*) \le 0$, and since $1-\lambda a > 0$, we obtain $d(\mathscr{T}\xi^*,\xi^*) = 0$, hence $\mathscr{T}\xi^* = \xi^*$.

For uniqueness, if $\zeta_1, \zeta_2$ are fixed points, then (3) yields

$$d(\zeta_1,\zeta_2) = d(\mathscr{T}\zeta_1,\mathscr{T}\zeta_2) \le \lambda \frac{d(\zeta_1,\zeta_2)}{1+b\,\mathscr{P}(\zeta_1,\zeta_2)} \le \lambda\, d(\zeta_1,\zeta_2),$$

which forces $d(\zeta_1,\zeta_2)=0$ and hence $\zeta_1=\zeta_2$.

Finally, the estimates above give

$$d(\xi_n,\xi^*) \le \sum_{k=n}^{\infty} t_k \le t_0 \frac{q^n}{1-q},$$

so the convergence is at least linear with ratio $q \in (0,1)$. □

# 4 Illustrative examples

**Example 4.1.** Let $X=\mathbb{R}$ and define $\mathscr{T}(\xi)=\xi/2$. Define $\mathscr{D},\mathscr{P} : X\times X \to [0,\infty)$ by

$$\mathscr{D}(\xi,\eta) = |\xi-\eta| + \xi^2\eta^2, \qquad \mathscr{P}(\xi,\eta) = \xi^2\eta^2.$$

Then $(X,\mathscr{D},\mathscr{P})$ is a perturbed metric space with exact metric $d(\xi,\eta)=|\xi-\eta|$, and $\mathscr{T}$ satisfies (3) with parameters $(\lambda,a,b)=\left(\frac{1}{2},0,0\right)$.

*Proof.* For all $\xi,\eta\in\mathbb{R}$,

$$d(\xi,\eta) = \mathscr{D}(\xi,\eta) - \mathscr{P}(\xi,\eta) = \left(|\xi-\eta| + \xi^2\eta^2\right) - \xi^2\eta^2 = |\xi-\eta|.$$

Hence $(X,d)=(\mathbb{R},|\cdot|)$ is complete, so $(X,\mathscr{D},\mathscr{P})$ is a complete perturbed metric space.

Moreover,

$$d(\mathscr{T}\xi, \mathscr{T}\eta) = \left|\frac{\xi}{2} - \frac{\eta}{2}\right| = \frac{1}{2}|\xi - \eta| = \frac{1}{2}\, d(\xi, \eta).$$

Taking $a = 0$ and $b = 0$, inequality (3) reduces to

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \le \lambda\, d(\xi, \eta),$$

which holds with $\lambda = \frac{1}{2}$. Also $\lambda(1+2a) = \frac{1}{2} < 1$, so Theorem 3.4 applies. Solving $\mathscr{T}\xi = \xi$ gives $\xi^* = 0$. □

**Example 4.2.** Let $X = [0, 1]$ and define $\mathscr{T}(\xi) = \xi/3$. Define $\mathscr{D}, \mathscr{P} : X \times X \to [0, \infty)$ by

$$\mathscr{D}(\xi, \eta) = |\xi - \eta| + \xi\eta, \qquad \mathscr{P}(\xi, \eta) = \xi\eta.$$

Then $(X, \mathscr{D}, \mathscr{P})$ is a perturbed metric space with exact metric $d(\xi, \eta) = |\xi - \eta|$, and $\mathscr{T}$ satisfies (3).

*Proof.* For all $\xi, \eta \in [0, 1]$,

$$d(\xi, \eta) = \mathscr{D}(\xi, \eta) - \mathscr{P}(\xi, \eta) = \big(|\xi - \eta| + \xi\eta\big) - \xi\eta = |\xi - \eta|.$$

Hence $(X, d) = ([0, 1], |\cdot|)$ is complete, so $(X, \mathscr{D}, \mathscr{P})$ is a complete perturbed metric space.

Moreover,

$$d(\mathscr{T}\xi, \mathscr{T}\eta) = \left|\frac{\xi}{3} - \frac{\eta}{3}\right| = \frac{1}{3}|\xi - \eta| = \frac{1}{3}\, d(\xi, \eta).$$

Taking $a = 0$ and $b = 0$, inequality (3) becomes

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \le \lambda\, d(\xi, \eta),$$

which holds with $\lambda = \frac{1}{3}$. Also $\lambda(1+2a) = \frac{1}{3} < 1$, so Theorem 3.4 applies. Solving $\mathscr{T}\xi = \xi$ gives $\xi^* = 0$. $\square$

## 5 Application to a Volterra Fractional Integral Equation

Consider the nonlinear Volterra fractional integral equation

$$\xi(t) = f(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K\big(t, s, \xi(s)\big)\, ds, \qquad t \in [0,1],\ 0 < \alpha < 1, \tag{4}$$

where $f : [0,1] \to \mathbb{R}$ is continuous, $K : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, and $\Gamma(\cdot)$ is the Gamma function.

Let $X = C([0,1], \mathbb{R})$ endowed with the supremum metric

$$d(\xi, \eta) = \sup_{t\in[0,1]} |\xi(t) - \eta(t)|.$$

Define $\mathscr{P} : X \times X \to [0, \infty)$ by

$$\mathscr{P}(\xi, \eta) = \sup_{t\in[0,1]} |\xi(t)\eta(t)|, \qquad \mathscr{D}(\xi, \eta) = d(\xi, \eta) + \mathscr{P}(\xi, \eta).$$

Then for all $\xi, \eta \in X$,

$$\mathscr{D}(\xi, \eta) - \mathscr{P}(\xi, \eta) = d(\xi, \eta),$$

and since $d$ is a metric on $X$, it follows that $(X, \mathscr{D}, \mathscr{P})$ is a perturbed metric space with exact metric $d$. Moreover, $(X, d)$ is complete, hence $(X, \mathscr{D}, \mathscr{P})$ is complete.

Define the operator $\mathscr{T} : X \to X$ by

$$(\mathscr{T}\xi)(t) = f(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K\big(t, s, \xi(s)\big)\, ds, \qquad t \in [0,1]. \tag{5}$$

Since $f$ and $K$ are continuous and the kernel $(t-s)^{\alpha-1}$ is integrable on $[0,t]$ for $0 < \alpha < 1$, the function $\mathscr{T}\xi$ is continuous on $[0,1]$ whenever $\xi \in X$, so $\mathscr{T}$ is well-defined on $X$.

**Theorem 5.1.** *Assume that $K$ satisfies the Lipschitz condition*

$$|K(t,s,u) - K(t,s,v)| \le L|u-v|$$

*for all $t, s \in [0,1]$ and $u, v \in \mathbb{R}$, where $L > 0$. If*

$$\frac{L}{\Gamma(\alpha+1)} < 1,$$

*then (4) admits a unique solution in $X$.*

*Proof.* Let $\xi, \eta \in X$ and $t \in [0,1]$. Using (5) and the Lipschitz condition on $K$, we obtain

$$|(\mathscr{T}\xi)(t) - (\mathscr{T}\eta)(t)| = \frac{1}{\Gamma(\alpha)} \left| \int_0^t (t-s)^{\alpha-1} \Big( K(t,s,\xi(s)) - K(t,s,\eta(s)) \Big)\, ds \right| \le \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\xi(s) - \eta$$

Since $|\xi(s) - \eta(s)| \le d(\xi, \eta)$ for all $s \in [0,1]$, it follows that

$$|(\mathscr{T}\xi)(t) - (\mathscr{T}\eta)(t)| \le \frac{L}{\Gamma(\alpha)}\, d(\xi,\eta) \int_0^t (t-s)^{\alpha-1}\, ds.$$

A direct computation gives

$$\int_0^t (t-s)^{\alpha-1}\,ds = \frac{t^\alpha}{\alpha}.$$

Hence, for all $t \in [0,1]$,

$$|(\mathscr{T}\xi)(t) - (\mathscr{T}\eta)(t)| \le \frac{L}{\Gamma(\alpha)} \cdot \frac{t^\alpha}{\alpha}\, d(\xi,\eta) \le \frac{L}{\Gamma(\alpha)} \cdot \frac{1}{\alpha}\, d(\xi,\eta) = \frac{L}{\Gamma(\alpha+1)}\, d(\xi,\eta),$$

where we used $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. Taking the supremum over $t \in [0,1]$ yields

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \le \frac{L}{\Gamma(\alpha+1)}\, d(\xi,\eta).$$

Thus $\mathscr{T}$ is a contraction on the complete metric space $(X,d)$ with contraction constant $c := \frac{L}{\Gamma(\alpha+1)} \in (0,1)$. By the Banach contraction principle, $\mathscr{T}$ admits a unique fixed point $\xi^* \in X$, and the Picard iteration $\xi_{n+1} = \mathscr{T}\xi_n$ converges to $\xi^*$ in $(X,d)$ for any initial $\xi_0 \in X$.

Finally, $\xi^*$ is a fixed point of $\mathscr{T}$ if and only if it satisfies (4), so $\xi^*$ is precisely the unique solution of the Volterra fractional integral equation (4) in $X$. □

# 6 Conclusion

In this paper, we investigated fixed point results in the framework of perturbed metric spaces $(X, \mathscr{D}, \mathscr{P})$ introduced by Jleli and Samet [18], where the analytical structure is governed by the exact metric $d = \mathscr{D} - \mathscr{P}$. Two rational-type contractive frameworks were developed and analyzed.

First, we introduced a perturbed rational contraction condition expressed in terms of the exact metric $d$ and established a Banach-type fixed point theorem guaranteeing the existence and uniqueness of a fixed point together with convergence of the Picard iteration in the complete metric space $(X, d)$. Second, we proposed a nonlinear perturbed rational contraction in which the nominal functional $\mathscr{D}$ is controlled by a rational expression whose denominator involves the perturbation component $\mathscr{P}$. Under the explicit parameter restriction $\lambda(1 + 2a) < 1$, we proved the existence and uniqueness of the fixed point and obtained a linear convergence estimate for the corresponding iterative sequence.

The presented examples illustrate that the proposed contractive conditions are compatible with the structure of perturbed metric spaces and provide practical criteria for verifying the hypotheses of the main results. Furthermore, an application to a nonlinear Volterra fractional integral equation demonstrates that the developed theory can be effectively applied to obtain existence and uniqueness results for integral models involving fractional kernels.

The results obtained in this work extend classical rational contraction principles to the perturbed metric setting and contribute to the growing literature on fixed point theory in generalized metric structures. Future research directions may include the study of multivalued or coupled mappings, extensions to fractional integro-differential equations, and the development of similar rational-type contractions in related generalized spaces such as perturbed $b$-metric or parametric metric spaces.

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