EXPLICIT FORMULAS AND PERIODIC BEHAVIOR OF SOLUTIONS FOR A CLASS OF RATIONAL RECURSIVE SEQUENCES

ABSTRACT. This paper investigates the qualitative behavior and explicit solutions of a system of rational difference equations of the form

$$u_{p+1} = \frac{u_p v_{p-2}}{a_1 v_{p-1} + a_2 v_{p-2}}, \qquad v_{p+1} = \frac{u_{p-2} v_p}{a_3 u_{p-1} + a_4 u_{p-2}}, \qquad p = 0, 1, 2, \ldots,$$

with arbitrary initial conditions $u_{-2}, u_{-1}, u_0, v_{-2}, v_{-1}, v_0 \in \mathbb{R}$. The constants $a_i \in \{-1, 1\}$ for $i = 1, 2, 3, 4$. Four special cases of the system are examined based on the sign combinations in the denominators. For each case, we prove that all solutions are periodic with period twelve and derive explicit closed-form formulas for the solutions in terms of the initial values. The proofs are established using mathematical induction. Numerical examples are provided to verify and illustrate the theoretical results, demonstrating the periodic nature of the solutions. This work contributes to the understanding of higher-order rational difference equations and their periodic behavior.

Keywords: Difference equations, Recursive sequences, Periodicity, Rational difference equations, System of difference equations.

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1. Introduction

Difference equations serve as fundamental mathematical tools for modeling discrete phenomena across numerous scientific disciplines. Unlike differential equations which describe continuous change, difference equations naturally capture the evolution of systems where observations or measurements occur at discrete time intervals. This inherent discreteness makes them particularly relevant in fields such as economics, biology, physics, and engineering, where data collection typically happens at specific time points rather than continuously. Moreover, difference equations play a crucial role in numerical analysis as they emerge naturally from discretization methods applied to differential equations, providing approximate solutions through finite difference schemes.

The study of rational difference equations, especially those of order greater than one, presents both significant challenges and rewarding opportunities in the field of discrete dynamical systems. These equations are characterized by ratios of polynomial expressions in the dependent variables, and they frequently arise in mathematical models describing real-world phenomena. Their importance stems from

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several factors: they serve as prototypes for developing general theories of nonlinear difference equations, they exhibit rich dynamical behaviors including periodicity, stability, and bifurcation, and they often admit closed-form solutions in special cases that provide insights into more complex systems (see [1–10]).

Despite decades of intensive research, the global behavior of higher-order rational difference equations remains largely unexplored territory. While numerous results exist for first-order equations and certain special classes, no unified theory adequately addresses the comprehensive dynamics of systems with order greater than one. This knowledge gap is particularly significant because many practical applications naturally lead to higher-order formulations. The complexity arises from the intricate interplay between multiple delayed arguments, nonlinear coupling, and the rational structure of the equations, which can produce unexpectedly sophisticated behaviors even in seemingly simple systems.

The literature contains numerous investigations into specific classes of rational difference equations and their systems. For instance, Cinar [11] examined periodic solutions in systems of the form $u_{p+1} = m/v_p$, $v_{p+1} = pv_p/(u_{p-1}v_{p-1})$, establishing fundamental results about their oscillatory behavior. Din and colleagues [6] conducted a comprehensive analysis of fourth-order systems, investigating equilibrium points, local and global stability characteristics, and periodic properties of positive solutions for equations incorporating multiple delayed terms. Ahmed [12] contributed to the understanding of global dynamics in systems where each equation contains linear denominators with constant terms. Bao [13] explored second-order nonlinear systems exhibiting rich dynamical structures, while Din [14] further extended these investigations to fourth-order formulations with more complex coupling mechanisms.

These studies collectively demonstrate that rational difference equations can support a wide spectrum of dynamical behaviors, from simple equilibrium solutions to complex periodic and possibly chaotic dynamics. However, each investigation typically addresses specific functional forms, and the systematic classification of behaviors based on equation structure remains an open research direction. Particularly noteworthy is the observation that minor modifications in equation parameters or signs can fundamentally alter solution characteristics, a phenomenon that merits careful investigation.

The present work aims to contribute to this growing body of knowledge by examining a system of rational difference equations characterized by the presence of both positive and negative signs in the denominators. Specifically, we consider the system

$$(1.1)\quad u_{p+1} = \frac{u_p v_{p-2}}{a_1 v_{p-1} + a_2 v_{p-2}}, \qquad v_{p+1} = \frac{u_{p-2} v_p}{a_3 u_{p-1} + a_4 u_{p-2}}, \qquad p = 0, 1, 2, \ldots,$$

with arbitrary initial conditions $u_{-2}, u_{-1}, u_0, v_{-2}, v_{-1}, v_0 \in \mathbb{R}$. The constants $a_i \in \{-1, 1\}$ for $i = 1, 2, 3, 4$. This system exhibits several interesting features that warrant detailed investigation. The third-order nature (involving indices $p-2$, $p-1$, and p) introduces memory effects that can support periodic solutions. The symmetric structure between the u and v equations suggests potential conservation properties or reciprocal relationships. Most significantly, the choice of signs in the denominators creates four distinct special cases, each potentially exhibiting unique dynamical characteristics.

The primary objectives of this paper are threefold: first, to derive explicit closed-form solutions for each of the four sign combinations in System (1.1); second, to establish the periodic nature of these solutions and determine their fundamental periods; and third, to validate the theoretical findings through numerical simulations. Our analysis reveals that regardless of the sign configuration, all solutions exhibit periodicity with period twelve, though the exact functional forms of the solutions differ substantially between cases. This periodicity result is particularly noteworthy given the relatively high order of the system and the complexity of the intermediate expressions.

The methodology employed combines inductive proofs with algebraic manipulation of the recursive definitions. By establishing explicit formulas for all twelve terms in a fundamental period, we demonstrate that the system returns to its initial state after twelve iterations. The proofs carefully track the algebraic relationships between successive terms, revealing an underlying structure that persists despite the apparent complexity of the intermediate expressions. Numerical examples accompanying each theoretical result confirm the analytical findings and provide visual illustrations of the periodic behavior.

Beyond the specific results presented, this work contributes to the broader understanding of rational difference equations by demonstrating that systematic sign variations in denominators can lead to predictable periodic behavior with explicit closed-form representations. Such results may prove valuable in applications where periodic dynamics are expected, such as population cycles in ecology, economic fluctuations, or oscillatory phenomena in physical systems. Furthermore, the techniques developed here may inform investigations of similar higher-order systems with rational structure.

The paper is organized as follows. Section 2 examines the case where both denominators contain negative signs, deriving complete solution formulas and establishing periodicity. Section 3 investigates the configuration with a positive sign in the first denominator and negative in the second. Section 4 addresses the case with a negative sign in the first denominator and positive in the second. Section 5 analyzes the fully positive sign configuration. Each of these sections includes numerical verification of the theoretical results. Finally, Section 6 provides concluding remarks and discusses potential directions for future research.

2. First Case: On the Difference Equation $u_{p+1} = \frac{u_p v_{p-2}}{-v_{p-1} - v_{p-2}}, \quad v_{p+1} = \frac{v_p u_{p-2}}{-u_{p-1} - u_{p-2}}$

In this section, we investigate the first special case of System (1.1), where both denominators contain negative signs with addition. This configuration, given by System (2.1) below, represents the fully negative sign scenario and serves as the foundation for understanding the effects of sign variations in subsequent cases. We establish that all solutions are periodic with period twelve and derive explicit closed-form formulas expressing each term of the solution sequences in terms of the six initial values. Let us consider the following special case of System (1.1):

$$(2.1)\qquad u_{p+1} = \frac{u_p v_{p-2}}{-v_{p-1} - v_{p-2}}, \quad v_{p+1} = \frac{u_{p-2} v_p}{-u_{p-1} - u_{p-2}}, \quad p = 0, 1, \ldots,$$

where the initial conditions $u_{-2} = c, u_{-1} = b, u_0 = a, v_{-2} = g, v_{-1} = e, v_0 = d$ are arbitrary real numbers such that $(e+g)(d+e)\,b\,(b+c)(a+b) \neq 0$.

Theorem 2.1. *Let $\{u_p, v_p\}_{p=-2}^{\infty}$ be a solution of System (2.1). Then all solutions of System (2.1) are periodic with period twelve, and explicit formulas are given as follows:*

$$\begin{aligned}
u_{12p-2} &= c, & u_{12p-1} &= b,\\
u_{12p} &= a, & u_{12p+1} &= -\frac{ag}{(e+g)},\\
u_{12p+2} &= \frac{age}{(e+g)(d+e)}, & u_{12p+3} &= -\frac{age(b+c)}{(e+g)(d+e)b},\\
u_{12p+4} &= \frac{ge(b+c)(a+b)}{(e+g)(d+e)b}, & u_{12p+5} &= \frac{e^2(b+c)(a+b)}{(e+g)(d+e)b},\\
u_{12p+6} &= \frac{ed(b+c)(a+b)}{(e+g)(d+e)b}, & u_{12p+7} &= -\frac{ecd(a+b)}{(e+g)(d+e)b},\\
u_{12p+8} &= \frac{cde}{(e+g)(d+e)}, & u_{12p+9} &= -\frac{cd}{(d+e)},
\end{aligned}$$

and

$$\begin{aligned}
v_{12p-2} &= g, & v_{12p-1} &= e,\\
v_{12p} &= d, & v_{12p+1} &= -\frac{cd}{(b+c)},\\
v_{12p+2} &= \frac{bcd}{(b+c)(a+b)}, & v_{12p+3} &= -\frac{bcd(e+g)}{(b+c)(a+b)e},\\
v_{12p+4} &= \frac{bc(d+e)(e+g)}{(b+c)(a+b)e}, & v_{12p+5} &= \frac{b^2(d+e)(e+g)}{(b+c)(a+b)e},\\
v_{12p+6} &= \frac{ab(d+e)(e+g)}{(b+c)(a+b)e}, & v_{12p+7} &= -\frac{bga(d+e)}{(b+c)(a+b)e},\\
v_{12p+8} &= \frac{gab}{(b+c)(a+b)}, & v_{12p+9} &= -\frac{ag}{(a+b)}.
\end{aligned}$$

provided the denominators are nonzero, where $u_{-2} = c, u_{-1} = b, u_0 = a, v_{-2} = g, v_{-1} = e, v_0 = d$.

Proof. We use an inductive proof for this rational recursive sequences. One can see that for $p = 0$, the result holds. Suppose that $p > 0$ and that the assumption is satisfied for $p-1$. That is;

$$u_{12p-14}=c, \qquad u_{12p-13}=b,$$
$$u_{12p-12}=a, \qquad u_{12p-11}=-\frac{ag}{(e+g)},$$
$$u_{12p-10}=\frac{age}{(e+g)(d+e)}, \qquad u_{12p-9}=-\frac{age(b+c)}{(e+g)(d+e)b},$$
$$u_{12p-8}=\frac{ge(b+c)(a+b)}{(e+g)(d+e)b}, \qquad u_{12p-7}=\frac{e^2(b+c)(a+b)}{(e+g)(d+e)b},$$
$$u_{12p-6}=\frac{ed(b+c)(a+b)}{(e+g)(d+e)b}, \qquad u_{12p-5}=-\frac{ecd(a+b)}{(e+g)(d+e)b},$$
$$u_{12p-4}=\frac{cde}{(e+g)(d+e)}, \qquad u_{12p-3}=-\frac{cd}{(d+e)},$$

and

$$v_{12p-14}=g, \qquad v_{12p-13}=e,$$
$$v_{12p-12}=d, \qquad v_{12p-11}=-\frac{cd}{(b+c)},$$
$$v_{12p-10}=\frac{bcd}{(b+c)(a+b)}, \qquad v_{12p-9}=-\frac{bcd(e+g)}{(b+c)(a+b)e},$$
$$v_{12p-8}=\frac{bc(d+e)(e+g)}{(b+c)(a+b)e}, \qquad v_{12p-7}=\frac{b^2(d+e)(e+g)}{(b+c)(a+b)e},$$
$$v_{12p-6}=\frac{ab(d+e)(e+g)}{(b+c)(a+b)e}, \qquad v_{12p-5}=-\frac{bga(d+e)}{(b+c)(a+b)e},$$
$$v_{12p-4}=\frac{gab}{(b+c)(a+b)}, \qquad v_{12p-3}=-\frac{ag}{(a+b)}.$$

Now we find from Eq. (2.1) that

$$u_{12p-2}=\frac{u_{12p-3}v_{12p-5}}{-v_{12p-4}-v_{12p-5}}=\frac{-\frac{cd}{(d+e)}\times-\frac{bga(d+e)}{(b+c)(a+b)e}}{-\frac{gab}{(b+c)(a+b)}+\frac{bga(d+e)}{(b+c)(a+b)e}}=\frac{\frac{cdbga}{(b+c)(a+b)e}}{\frac{bgad}{(b+c)(a+b)e}}=c.$$

Similarly,

$$v_{12p-2}=\frac{u_{12p-5}v_{12p-3}}{-u_{12p-4}-u_{12p-5}}=\frac{-\frac{ecd(a+b)}{(e+g)(d+e)b}\times-\frac{ag}{(a+b)}}{-\frac{cde}{(e+g)(d+e)}+\frac{ecd(a+b)}{(e+g)(d+e)b}}=\frac{\frac{agecd}{(e+g)(d+e)b}}{\frac{ecda}{(e+g)(d+e)b}}=g.$$

Similarly, one can obtain the other relations. Thus, the proof is completed. □

For confirming the results of this section, we consider numerical example for $u_{-2}=-1.3, u_{-1}=1.2, u_0=2.6, v_{-2}=2.8, v_{-1}=1.1, v_0=-1.5$ (See Figure 1).

3. Second Case: On the Difference Equation $u_{p+1}=\frac{u_pv_{p-2}}{v_{p-1}+v_{p-2}}, \quad v_{p+1}=\frac{u_{p-2}v_p}{u_{p-1}-u_{p-2}}$

In this section, we examine the second special case of System (1.1), characterized by a positive sign in the denominator of the u-equation and a negative sign in the

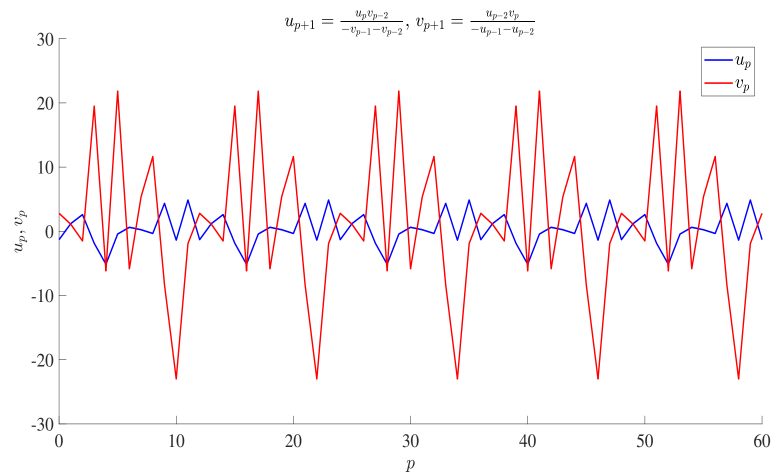

FIGURE 1. Numerical simulations for System (2.1) where $u_{-2} = -1.3, u_{-1} = 1.2, u_0 = 2.6, v_{-2} = 2.8, v_{-1} = 1.1, v_0 = -1.5$

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denominator of the v-equation. This mixed sign configuration, presented in System (3.1) below, demonstrates how asymmetry in the sign pattern affects the solution structure while preserving the fundamental twelve-periodicity. We provide complete explicit formulas for all solution terms and verify the results through numerical simulation. Let us consider the following special case of System (1.1):

$$u_{p+1} = \frac{u_p v_{p-2}}{v_{p-1} + v_{p-2}}, \quad v_{p+1} = \frac{u_{p-2} v_p}{u_{p-1} - u_{p-2}}, \quad p = 0, 1, \dots, \tag{3.1}$$

with real initial conditions $u_{-2} = c, u_{-1} = b, u_0 = a, v_{-2} = g, v_{-1} = e, v_0 = d$ are arbitrary real numbers such that $(e+g)(d+e)\, b\, (b-c)(a-b) \neq 0$.

Theorem 3.1. *Let $\{u_p, v_p\}_{p=-2}^{\infty}$ be a solution of System (3.1). Then all solutions of System (3.1) are periodic with period twelve, and explicit formulas are given as follows:*

$$u_{12p-2} = c, \qquad u_{12p-1} = b,$$
$$u_{12p} = a, \qquad u_{12p+1} = \frac{ag}{(e+g)},$$
$$u_{12p+2} = \frac{age}{(e+g)(d+e)}, \qquad u_{12p+3} = \frac{age(b-c)}{(e+g)(d+e)b},$$
$$u_{12p+4} = \frac{ge(b-c)(a-b)}{(e+g)(d+e)b}, \qquad u_{12p+5} = -\frac{e^2(b-c)(a-b)}{(e+g)(d+e)b},$$
$$u_{12p+6} = \frac{ed(b-c)(a-b)}{(e+g)(d+e)b}, \qquad u_{12p+7} = -\frac{ecd(a-b)}{(e+g)(d+e)b},$$
$$u_{12p+8} = \frac{cde}{(e+g)(d+e)}, \qquad u_{12p+9} = \frac{cd}{(d+e)},$$

and

$$v_{12p-2} = g, \qquad v_{12p-1} = e,$$
$$v_{12p} = d, \qquad v_{12p+1} = \frac{cd}{(b-c)},$$
$$v_{12p+2} = \frac{bcd}{(b-c)(a-b)}, \qquad v_{12p+3} = -\frac{bcd(e+g)}{(b-c)(a-b)e},$$
$$v_{12p+4} = \frac{bc(d+e)(e+g)}{(b-c)(a-b)e}, \qquad v_{12p+5} = -\frac{b^2(d+e)(e+g)}{(b-c)(a-b)e},$$
$$v_{12p+6} = \frac{ab(d+e)(e+g)}{(b-c)(a-b)e}, \qquad v_{12p+7} = -\frac{bga(d+e)}{(b-c)(a-b)e},$$
$$v_{12p+8} = \frac{gab}{(b-c)(a-b)}, \qquad v_{12p+9} = -\frac{ag}{(a-b)}.$$

provided the denominators are nonzero, where $u_{-2} = c, u_{-1} = b, u_0 = a, v_{-2} = g, v_{-1} = e, v_0 = d$.

Proof. We use an inductive proof for this rational recursive sequences. One can see that for $p = 0$, the result holds. Suppose that $p > 0$ and that the assumption is satisfied for $p - 1$. That is;

$$u_{12p-14} = c, \qquad u_{12p-13} = b,$$
$$u_{12p-12} = a, \qquad u_{12p-11} = \frac{ag}{(e+g)},$$
$$u_{12p-10} = \frac{age}{(e+g)(d+e)}, \qquad u_{12p-9} = \frac{age(b-c)}{(e+g)(d+e)b},$$
$$u_{12p-8} = \frac{ge(b-c)(a-b)}{(e+g)(d+e)b}, \qquad u_{12p-7} = -\frac{e^2(b-c)(a-b)}{(e+g)(d+e)b},$$
$$u_{12p-6} = \frac{ed(b-c)(a-b)}{(e+g)(d+e)b}, \qquad u_{12p-5} = -\frac{ecd(a-b)}{(e+g)(d+e)b},$$
$$u_{12p-4} = \frac{cde}{(e+g)(d+e)}, \qquad u_{12p-3} = \frac{cd}{(d+e)},$$

and

$$v_{12p-14} = g, \qquad v_{12p-13} = e,$$
$$v_{12p-12} = d, \qquad v_{12p-11} = \frac{cd}{(b-c)},$$
$$v_{12p-10} = \frac{bcd}{(b-c)(a-b)}, \qquad v_{12p-9} = -\frac{bcd(e+g)}{(b-c)(a-b)e},$$
$$v_{12p-8} = \frac{bc(d+e)(e+g)}{(b-c)(a-b)e}, \qquad v_{12p-7} = -\frac{b^2(d+e)(e+g)}{(b-c)(a-b)e},$$
$$v_{12p-6} = \frac{ab(d+e)(e+g)}{(b-c)(a-b)e}, \qquad v_{12p-5} = -\frac{bga(d+e)}{(b-c)(a-b)e},$$
$$v_{12p-4} = \frac{gab}{(b-c)(a-b)}, \qquad v_{12p-3} = -\frac{ag}{(a-b)}.$$

Now we find from Eq. (3.1) that

$$u_{12p-2} = \frac{u_{12p-3}v_{12p-5}}{v_{12p-4}+v_{12p-5}} = \frac{\frac{cd}{(d+e)} \times -\frac{bga(d+e)}{(b-c)(a-b)e}}{\frac{gab}{(b-c)(a-b)} - \frac{bga(d+e)}{(b-c)(a-b)e}} = \frac{-\frac{cdbga}{(b-c)(a-b)e}}{-\frac{bgad}{(b-c)(a-b)e}} = c.$$

Similarly,

$$v_{12p-2} = \frac{u_{12p-5}v_{12p-3}}{u_{12p-4}-u_{12p-5}} = \frac{-\frac{ecd(a-b)}{(e+g)(d+e)b} \times -\frac{ag}{(a-b)}}{\frac{cde}{(e+g)(d+e)} + \frac{ecd(a-b)}{(e+g)(d+e)b}} = \frac{\frac{agecd}{(e+g)(d+e)b}}{\frac{ecda}{(e+g)(d+e)b}} = g.$$

Similarly, one can obtain the other relations. Thus, the proof is completed. □

For confirming the results of this section, we consider numerical example for $u_{-2} = -1.3, u_{-1} = 1.2, u_0 = 2.6, v_{-2} = 2.8, v_{-1} = 1.1, v_0 = -1.5$ (See Figure 2).

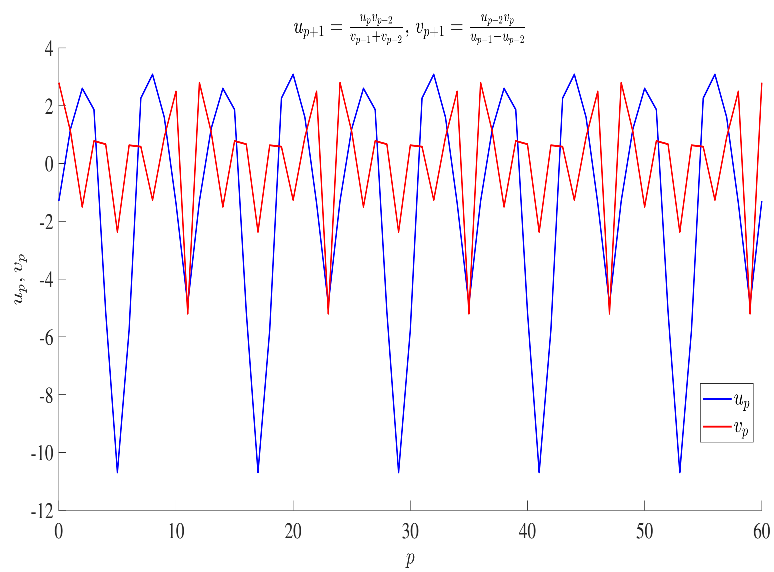

Figure 2. Numerical simulations for System (3.1) where $u_{-2} = -1.3, u_{-1} = 1.2, u_0 = 2.6, v_{-2} = 2.8, v_{-1} = 1.1, v_0 = -1.5$
.

4. Third Case: On the Difference Equation $u_{p+1} = \frac{u_p v_{p-2}}{v_{p-1} - v_{p-2}}, \quad v_{p+1} = \frac{v_p u_{p-2}}{u_{p-1} + u_{p-2}}$

In this section, we analyze the third special case of System (1.1), where the sign pattern is reversed from the previous section: a negative sign appears in the denominator of the u-equation while a positive sign appears in the denominator of the v-equation. This configuration, specified in System (4.1) below, further illustrates the symmetrical nature of the sign variations and their impact on the solution formulas. We establish that all solutions maintain period twelve and derive the corresponding explicit expressions, complemented by numerical verification. Let us consider the following special case of System (1.1):

$$u_{p+1} = \frac{u_p v_{p-2}}{v_{p-1} - v_{p-2}}, \quad v_{p+1} = \frac{v_p u_{p-2}}{u_{p-1} + u_{p-2}}, \quad p = 0, 1, \ldots, \tag{4.1}$$

with real initial conditions $u_{-2} = c, u_{-1} = b, u_0 = a, v_{-2} = g, v_{-1} = e, v_0 = d$ are arbitrary real numbers such that $(e-g)(d-e)\,b\,(b+c)(a+b) \neq 0$.

Theorem 4.1. *Let $\{u_p, v_p\}_{p=-2}^{\infty}$ be a solution of System (4.1). Then all solutions of System (4.1) are periodic with period twelve, and explicit formulas are given as follows:*

$$\begin{aligned}
u_{12p-2} &= c, & u_{12p-1} &= b,\\
u_{12p} &= a, & u_{12p+1} &= \frac{ag}{(e-g)},\\
u_{12p+2} &= \frac{age}{(e-g)(d-e)}, & u_{12p+3} &= -\frac{age(b+c)}{(e-g)(d-e)b},\\
u_{12p+4} &= \frac{ge(b+c)(a+b)}{(e-g)(d-e)b}, & u_{12p+5} &= -\frac{e^2(b+c)(a+b)}{(e-g)(d-e)b},\\
u_{12p+6} &= \frac{ed(b+c)(a+b)}{(e-g)(d-e)b}, & u_{12p+7} &= -\frac{ecd(a+b)}{(e-g)(d-e)b},\\
u_{12p+8} &= \frac{cde}{(e-g)(d-e)}, & u_{12p+9} &= -\frac{cd}{(d-e)},
\end{aligned}$$

and

$$\begin{aligned}
v_{12p-2} &= g, & v_{12p-1} &= e,\\
v_{12p} &= d, & v_{12p+1} &= \frac{cd}{(b+c)},\\
v_{12p+2} &= \frac{bcd}{(b+c)(a+b)}, & v_{12p+3} &= \frac{bcd(e-g)}{(b+c)(a+b)e},\\
v_{12p+4} &= \frac{bc(d-e)(e-g)}{(b+c)(a+b)e}, & v_{12p+5} &= -\frac{b^2(d-e)(e-g)}{(b+c)(a+b)e},\\
v_{12p+6} &= \frac{ab(d-e)(e-g)}{(b+c)(a+b)e}, & v_{12p+7} &= -\frac{bga(d-e)}{(b+c)(a+b)e},\\
v_{12p+8} &= \frac{gab}{(b+c)(a+b)}, & v_{12p+9} &= \frac{ag}{(a+b)},
\end{aligned}$$

provided the denominators are nonzero, where $u_{-2} = c, u_{-1} = b, u_0 = a, v_{-2} = g, v_{-1} = e, v_0 = d$.

Proof. We use an inductive proof for this rational recursive sequences. One can see that for $p = 0$, the result holds. Suppose that $p > 0$ and that the assumption is satisfied for $p-1$. That is;

$$u_{12p-14}=c, \qquad u_{12p-13}=b,$$
$$u_{12p-12}=a, \qquad u_{12p-11}=\frac{ag}{(e-g)},$$
$$u_{12p-10}=\frac{age}{(e-g)(d-e)}, \qquad u_{12p-9}=-\frac{age(b+c)}{(e-g)(d-e)b},$$
$$u_{12p-8}=\frac{ge(b+c)(a+b)}{(e-g)(d-e)b}, \qquad u_{12p-7}=-\frac{e^2(b+c)(a+b)}{(e-g)(d-e)b},$$
$$u_{12p-6}=\frac{ed(b+c)(a+b)}{(e-g)(d-e)b}, \qquad u_{12p-5}=-\frac{ecd(a+b)}{(e-g)(d-e)b},$$
$$u_{12p-4}=\frac{cde}{(e-g)(d-e)}, \qquad u_{12p-3}=-\frac{cd}{(d-e)},$$

and

$$v_{12p-14}=g, \qquad v_{12p-13}=e,$$
$$v_{12p-12}=d, \qquad v_{12p-11}=\frac{cd}{(b+c)},$$
$$v_{12p-10}=\frac{bcd}{(b+c)(a+b)}, \qquad v_{12p-9}=\frac{bcd(e-g)}{(b+c)(a+b)e},$$
$$v_{12p-8}=\frac{bc(d-e)(e-g)}{(b+c)(a+b)e}, \qquad v_{12p-7}=-\frac{b^2(d-e)(e-g)}{(b+c)(a+b)e},$$
$$v_{12p-6}=\frac{ab(d-e)(e-g)}{(b+c)(a+b)e}, \qquad v_{12p-5}=-\frac{bga(d-e)}{(b+c)(a+b)e},$$
$$v_{12p-4}=\frac{gab}{(b+c)(a+b)}, \qquad v_{12p-3}=\frac{ag}{(a+b)}.$$

Now we find from Eq. (4.1) that

$$u_{12p-2}=\frac{u_{12p-3}v_{12p-5}}{v_{12p-4}-v_{12p-5}}=\frac{\frac{cd}{(d-e)}\times\frac{bga(d-e)}{(b+c)(a+b)e}}{\frac{gab}{(b+c)(a+b)}+\frac{bga(d-e)}{(b+c)(a+b)e}}=\frac{\frac{cdbga}{(b+c)(a+b)e}}{\frac{bgad}{(b+c)(a+b)e}}=c.$$

Similarly,

$$v_{12p-2}=\frac{u_{12p-5}v_{12p-3}}{u_{12p-4}+u_{12p-5}}=\frac{-\frac{ecd(a+b)}{(e-g)(d-e)b}\times\frac{ag}{(a+b)}}{\frac{cde}{(e-g)(d-e)}-\frac{ecd(a+b)}{(e-g)(d-e)b}}=\frac{\frac{-agecd}{(e-g)(d-e)b}}{-\frac{ecda}{(e-g)(d-e)b}}=g.$$

Similarly, one can obtain the other relations. Thus, the proof is completed. □

For confirming the results of this section, we consider numerical example for $u_{-2}=-1.3, u_{-1}=1.2, u_0=2.6, v_{-2}=2.8, v_{-1}=1.1, v_0=-1.5$ (See Figure 3).

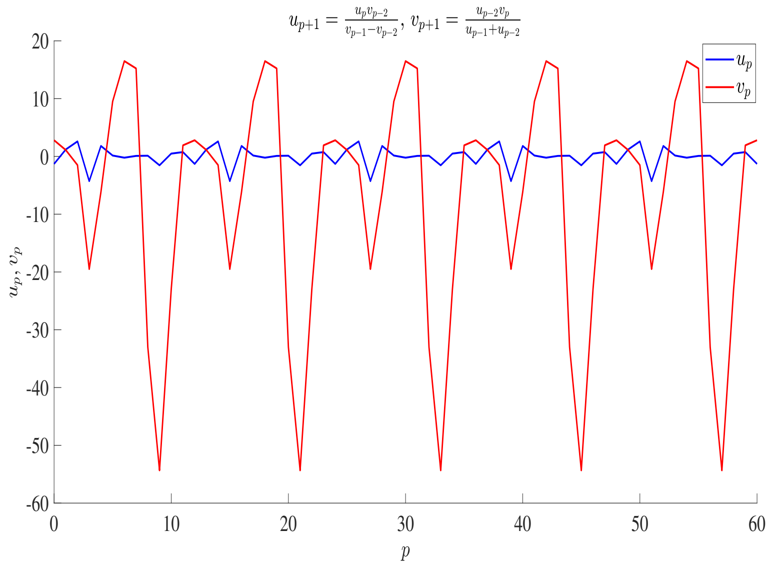

FIGURE 3. Numerical simulations for System (4.1) where $u_{-2} = -1.3, u_{-1} = 1.2, u_0 = 2.6, v_{-2} = 2.8, v_{-1} = 1.1, v_0 = -1.5$
.

5. Fourth Case: On the Difference Equation $u_{p+1} = \frac{u_p v_{p-2}}{-v_{p-1} + v_{p-2}}, \quad v_{p+1} = \frac{v_p u_{p-2}}{-u_{p-1} + u_{p-2}}$

n this section, we investigate the fourth and final special case of System (1.1), characterized by both denominators containing negative signs combined with addition. While this configuration bears superficial resemblance to the first case, the subtle difference in denominator structure leads to distinct solution formulas while preserving the twelve-periodicity property. System (5.1) below is analyzed in detail, with complete explicit solutions provided and validated through numerical simulations, thereby completing our systematic examination of all possible sign combinations. Let us consider the following special case of System (1.1):

$$u_{p+1} = \frac{u_p v_{p-2}}{-v_{p-1} + v_{p-2}}, \quad v_{p+1} = \frac{v_p u_{p-2}}{-u_{p-1} + u_{p-2}}, \quad p = 0, 1, \ldots, \tag{5.1}$$

with real initial conditions $u_{-2} = c, u_{-1} = b, u_0 = a, v_{-2} = g, v_{-1} = e, v_0 = d$ are arbitrary real numbers such that $(e - g)(d - e)\, b\, (b - c)(a - b) \neq 0$.

Theorem 5.1. *Let $\{u_p, v_p\}_{p=-2}^{\infty}$ be a solution of System (5.1). Then all solutions of System (5.1) are periodic with period twelve, and explicit formulas are given as* *follows:*

$$\begin{aligned}
u_{12p-2} &= c, & u_{12p-1} &= b,\\
u_{12p} &= a, & u_{12p+1} &= -\frac{ag}{(e-g)},\\
u_{12p+2} &= \frac{age}{(e-g)(d-e)}, & u_{12p+3} &= \frac{age(b-c)}{(e-g)(d-e)b},\\
u_{12p+4} &= \frac{ge(b-c)(a-b)}{(e-g)(d-e)b}, & u_{12p+5} &= \frac{e^2(b-c)(a-b)}{(e-g)(d-e)b},\\
u_{12p+6} &= \frac{ed(b-c)(a-b)}{(e-g)(d-e)b}, & u_{12p+7} &= -\frac{ecd(a-b)}{(e-g)(d-e)b},\\
u_{12p+8} &= \frac{cde}{(e-g)(d-e)}, & u_{12p+9} &= \frac{cd}{(d-e)},
\end{aligned}$$

and

$$\begin{aligned}
v_{12p-2} &= g, & v_{12p-1} &= e,\\
v_{12p} &= d, & v_{12p+1} &= -\frac{cd}{(b-c)},\\
v_{12p+2} &= \frac{bcd}{(b-c)(a-b)}, & v_{12p+3} &= \frac{bcd(e-g)}{(b-c)(a-b)e},\\
v_{12p+4} &= \frac{bc(d-e)(e-g)}{(b-c)(a-b)e}, & v_{12p+5} &= \frac{b^2(d-e)(e-g)}{(b-c)(a-b)e},\\
v_{12p+6} &= \frac{ab(d-e)(e-g)}{(b-c)(a-b)e}, & v_{12p+7} &= -\frac{bga(d-e)}{(b-c)(a-b)e},\\
v_{12p+8} &= \frac{gab}{(b-c)(a-b)}, & v_{12p+9} &= \frac{ag}{(a-b)},
\end{aligned}$$

provided the denominators are nonzero, where $u_{-2} = c, u_{-1} = b, u_0 = a, v_{-2} = g, v_{-1} = e, v_0 = d$.

Proof. We use an inductive proof for this rational recursive sequences. One can see that for $p = 0$, the result holds. Suppose that $p > 0$ and that the assumption is satisfied for $p - 1$. That is;

$$\begin{aligned}
u_{12p-14} &= c, & u_{12p-13} &= b,\\
u_{12p-12} &= a, & u_{12p-11} &= -\frac{ag}{(e-g)},\\
u_{12p-10} &= \frac{age}{(e-g)(d-e)}, & u_{12p-9} &= \frac{age(b-c)}{(e-g)(d-e)b},\\
u_{12p-8} &= \frac{ge(b-c)(a-b)}{(e-g)(d-e)b}, & u_{12p-7} &= \frac{e^2(b-c)(a-b)}{(e-g)(d-e)b},\\
u_{12p-6} &= \frac{ed(b-c)(a-b)}{(e-g)(d-e)b}, & u_{12p-5} &= -\frac{ecd(a-b)}{(e-g)(d-e)b},\\
u_{12p-4} &= \frac{cde}{(e-g)(d-e)}, & u_{12p-3} &= \frac{cd}{(d-e)},
\end{aligned}$$

and

$$v_{12p-14} = g, \qquad v_{12p-13} = e,$$
$$v_{12p-12} = d, \qquad v_{12p-11} = -\frac{cd}{(b-c)},$$
$$v_{12p-10} = \frac{bcd}{(b-c)(a-b)}, \qquad v_{12p-9} = \frac{bcd(e-g)}{(b-c)(a-b)e},$$
$$v_{12p-8} = \frac{bc(d-e)(e-g)}{(b-c)(a-b)e}, \qquad v_{12p-7} = \frac{b^2(d-e)(e-g)}{(b-c)(a-b)e},$$
$$v_{12p-6} = \frac{ab(d-e)(e-g)}{(b-c)(a-b)e}, \qquad v_{12p-5} = -\frac{bga(d-e)}{(b-c)(a-b)e},$$
$$v_{12p-4} = \frac{gab}{(b-c)(a-b)}, \qquad v_{12p-3} = \frac{ag}{(a-b)}.$$

Now we find from Eq. (5.1) that

$$u_{12p-2} = \frac{u_{12p-3}v_{12p-5}}{-v_{12p-4}+v_{12p-5}} = \frac{\frac{cd}{(d-e)} \times -\frac{bga(d-e)}{(b-c)(a-b)e}}{-\frac{gab}{(b-c)(a-b)} - \frac{bga(d-e)}{(b-c)(a-b)e}} = \frac{\frac{cdbga}{(b-c)(a-b)e}}{\frac{bgad}{(b-c)(a-b)e}} = c.$$

Similarly,

$$v_{12p-2} = \frac{u_{12p-5}v_{12p-3}}{-u_{12p-4}+u_{12p-5}} = \frac{-\frac{ecd(a-b)}{(e-g)(d-e)b} \times \frac{ag}{(a-b)}}{-\frac{cde}{(e-g)(d-e)} - \frac{ecd(a-b)}{(e-g)(d-e)b}} = \frac{-\frac{agecd}{(e-g)(d-e)b}}{-\frac{ecda}{(e-g)(d-e)b}} = g.$$

Similarly, one can obtain the other relations. Thus, the proof is completed. □

For confirming the results of this section, we consider numerical example for $u_{-2} = -1.3, u_{-1} = 1.2, u_0 = 2.6, v_{-2} = 2.8, v_{-1} = 1.1, v_0 = -1.5$ (See Figure 4).

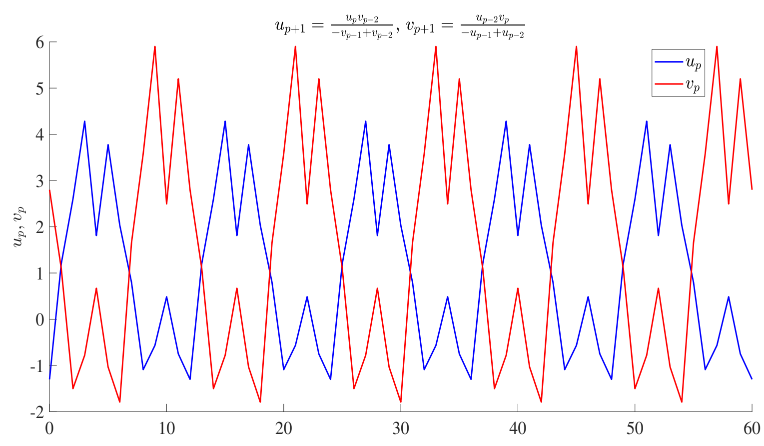

FIGURE 4. Numerical simulations for System (5.1) where $u_{-2} = -1.3, u_{-1} = 1.2, u_0 = 2.6, v_{-2} = 2.8, v_{-1} = 1.1, v_0 = -1.5$

.

6. Conclusion

This paper has presented a comprehensive investigation into the qualitative behavior and explicit solutions of a system of third-order rational difference equations characterized by sign variations in the denominators. The system under consideration,

$$u_{p+1} = \frac{u_p v_{p-2}}{a_1 v_{p-1} + a_2 v_{p-2}}, \qquad v_{p+1} = \frac{u_{p-2} v_p}{a_3 u_{p-1} + a_4 u_{p-2}}, \qquad p = 0, 1, 2, \ldots,$$

was analyzed for four distinct sign configurations, each yielding unique dynamical properties while maintaining an underlying structural similarity.

The principal findings of this work can be summarized as follows. For each of the four cases examined, we have demonstrated through rigorous inductive proofs that all solutions are periodic with period twelve, provided the denominators remain nonzero throughout the evolution. Explicit closed-form formulas were derived for every term within the fundamental period, expressed entirely in terms of the six initial values $u_{-2} = c$, $u_{-1} = b$, $u_0 = a$, $v_{-2} = g$, $v_{-1} = e$, and $v_0 = d$. These formulas reveal the intricate algebraic relationships that govern the system's dynamics and provide complete characterization of solution behavior for all real initial conditions.

The four cases investigated correspond to the following sign combinations:

- **Case I:** $u_{p+1} = \dfrac{u_p v_{p-2}}{-v_{p-1} - v_{p-2}}$, $v_{p+1} = \dfrac{u_{p-2} v_p}{-u_{p-1} - u_{p-2}}$
- **Case II:** $u_{p+1} = \dfrac{u_p v_{p-2}}{v_{p-1} + v_{p-2}}$, $v_{p+1} = \dfrac{u_{p-2} v_p}{u_{p-1} - u_{p-2}}$
- **Case III:** $u_{p+1} = \dfrac{u_p v_{p-2}}{v_{p-1} - v_{p-2}}$, $v_{p+1} = \dfrac{v_p u_{p-2}}{u_{p-1} + u_{p-2}}$
- **Case IV:** $u_{p+1} = \dfrac{u_p v_{p-2}}{-v_{p-1} + v_{p-2}}$, $v_{p+1} = \dfrac{v_p u_{p-2}}{-u_{p-1} + u_{p-2}}$

Despite the different sign patterns, all four cases share the remarkable property of twelve-periodicity, suggesting an underlying mathematical structure that transcends the specific sign choices. This uniformity in period length, combined with the diversity of intermediate expressions, highlights the rich algebraic structure inherent in such rational recursive systems.

Numerical simulations were provided for each case using the representative initial values $u_{-2} = -1.3$, $u_{-1} = 1.2$, $u_0 = 2.6$, $v_{-2} = 2.8$, $v_{-1} = 1.1$, $v_0 = -1.5$. These computational experiments visually confirm the theoretical predictions, clearly illustrating the twelve-periodic nature of the solutions and validating the derived explicit formulas. The agreement between analytical and numerical results provides strong evidence for the correctness of our mathematical derivations.

Several observations merit further discussion. First, the periodic behavior persists across all sign configurations, indicating that the system possesses inherent oscillatory characteristics that are robust to sign variations in the denominators. Second, the explicit formulas reveal that certain intermediate terms may become singular for particular parameter choices, necessitating the denominator nonvanishing conditions stated in the theorems. Third, the symmetric structure between the u and v sequences is evident in the formulas, with corresponding terms exhibiting analogous algebraic forms after appropriate variable substitutions.

The significance of these results extends beyond the specific system investigated. They contribute to the broader understanding of higher-order rational difference equations by demonstrating that systematic sign variations can lead to predictable periodic behavior with explicit closed-form representations. Such findings may inform investigations of similar systems and potentially guide the development of more general theories for rational recursive sequences. Moreover, the techniques employed—inductive proof combined with careful algebraic manipulation of recursive definitions—provide a methodological framework applicable to other classes of difference equations.

Several directions for future research emerge from this work. Natural extensions include investigating systems with different orders, such as fourth-order or fifth-order formulations, to determine whether similar periodic phenomena occur. The role of the coefficients could be explored by introducing parameters multiplying the numerator or denominator terms. Questions of stability and bifurcation behavior under parameter variation remain open for investigation. Additionally, exploring the possibility of longer periods or even chaotic behavior for different sign combinations or initial condition domains would be valuable. The connection between these discrete systems and continuous dynamical systems through appropriate limiting procedures presents another intriguing avenue for research.

In conclusion, this paper has provided a complete characterization of the periodic solutions for four special cases of a third-order rational difference equation system. The explicit formulas derived, together with the numerical verification, offer both theoretical insight and practical tools for understanding these dynamical systems. The results contribute to the growing literature on rational difference equations and may find applications in fields where such equations arise as mathematical models of real-world phenomena.

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