

# A Note on Separation Number of Graphs

## Abstract

In this paper, the concept of total domination is used to introduce a new graph parameter, called the *separation number*  $S_n(G)$  of a graph  $G$ . We study the fundamental properties of this parameter and establish general lower and upper bounds for  $S_n(G)$ . Furthermore, the separation number is determined for various classes of graphs, including complete graphs and star graphs. We also examine the behavior of  $S_n(G)$  under specific graph operations such as the corona of graphs and the conormal product of complete graphs.

*Keywords:* total domination set; separation number; complete graphs.

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## 1 Introduction

A graph  $G = (V(G), E(G))$ , where  $V(G)$  is a finite non-empty set and  $E(G)$  is a collection of vertices called edges. A *total dominating set* of a graph  $G$  is a set  $S$  of vertices of  $G$  such that every vertex of  $G$  is adjacent to a vertex in  $S$ . Latheesh Kumar [7] introduced a new separation axiom in graphs using the concept of total domination in analogues to the separation axioms in topology, [6],[8]. A simple connected graph  $G$  is said to satisfy the  $T_0$  axiom [7] if for any two distinct vertices  $u$  and  $v$  of  $G$ , there exists a total dominating set  $S$  such that either  $u \in S$  but not  $v$  or  $v \in S$  but not  $u$ . Using this a new parameter called separation number of graphs is introduced.

## 2 Main Results

**Definition 2.1.** A separating set  $S(G)$  of a graph  $G$  is a set of total dominating sets that separates all the vertices of  $G$ .

**Definition 2.2.** A separating set is called a minimal separating set if no proper subset of it is a separating set.

**Definition 2.3.** A minimum separating set is a separating set containing minimum number of sets.

**Definition 2.4.** The separation number  $S_n(G)$  of a graph  $G$  is the number of sets in a minimum separating set of  $G$ .

There is need to add definition of more terms used in the text like  $T_0$  graph, Complete graph, Star graph, Wheel graph, Bipartite graph, Corona of graph, Conormal product..

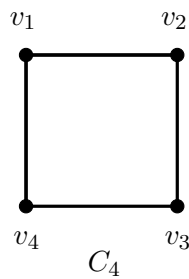


Figure 1:  $C_4$

The separating sets of  $C_4$  shown in Figure 1 are given below:

$$\begin{aligned}
 S_1 &= \{\{v_1, v_2\}, \{v_2, v_3\}\}, \\
 S_2 &= \{\{v_2, v_3\}, \{v_3, v_4\}\}, \\
 S_3 &= \{\{v_3, v_4\}, \{v_4, v_1\}\}, \\
 S_4 &= \{\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \{v_3, v_4, v_1\}\}, \\
 S_5 &= \{\{v_2, v_3, v_4\}, \{v_3, v_4, v_1\}, \{v_4, v_1, v_2\}\}, \\
 S_6 &= \{\{v_3, v_4, v_1\}, \{v_4, v_1, v_2\}, \{v_1, v_2, v_3\}\}, \\
 S_7 &= \{\{v_4, v_1, v_2\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\}.
 \end{aligned}$$

Here  $S_4, S_5, S_6$  and  $S_7$  are minimal separating sets while  $S_1, S_2$  and  $S_3$  are minimum separating set of  $C_4$ . Therefore  $S_n(C_4) = 2$ .

**Theorem 2.1.** For a  $T_0$  graph  $G$  with  $n$  vertices,  $\lceil \log_2 n \rceil \leq S_n(G) \leq n - 1$ .

*Proof.* Let  $V(G) = \{v_1, v_2, \dots, v_n\}$ . Let  $k = \lceil \log_2 n \rceil$ , so that  $2^k \geq n$ .

List the first  $n$  binary strings of length  $k$  (i.e., strings of the form  $(b_1, b_2, \dots, b_k)$  where  $b_i \in \{0, 1\}$ ). Assign these strings to the vertices as follows: Let  $v_1$  be the first binary string,  $v_2$  be the second binary string,  $\dots, v_n$  be the  $n^{th}$  binary string.

Define  $S = \{S_1, S_2, \dots, S_k\}$ , where  $S_i = \{v_j \mid \text{the } i\text{-th bit of } v_j \text{ is } 0\}$ . In this construction, for any distinct vertices  $v_p$  and  $v_q$ , their assigned binary strings differ in at least one coordinate, say the  $i$ -th bit. Then,  $S_i$  contains exactly one of  $v_p$  and  $v_q$ , thus separating them. Hence,  $k = \lceil \log_2 n \rceil \leq S_n(G)$ .

Next we show that  $S_n(G) \leq n - 1$ .

**Case (i)**  $G$  has no vertices of degree one

$S(G) = \{V - \{v_i\} \mid i = 1, 2, \dots, n - 1\}$  is a separating set of  $G$ .

**Case (ii)** Let  $v_1, v_2, \dots, v_r$  be the pendant vertices of  $G$ . Since  $G$  is a  $T_0$  graph all the pendant vertices are adjacent to a single vertex. Let it be  $v_n$ . Then  $S(G) = \{V - \{v_i\} \mid i = 1, 2, \dots, n - 1\}$  is a separating set of  $G$  with  $n - 1$  sets. Therefore  $S_n(G) \leq n - 1$ . This completes the proof. □

How does one determine sharpness of bounds?

*Remark 2.1.* The bounds of the above theorem are sharp. For the complete graphs  $K_3$  and  $K_4$ ,  $S(K_3) = \{\{v_1, v_2\}, \{v_2, v_3\}\} = S(K_4)$ . Therefore  $S_n(K_3) = 2 = n - 1$  and  $S_n(K_4) = 2 = \lceil \log_2 4 \rceil$ .

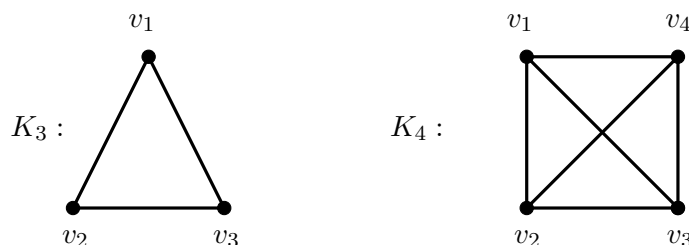


Figure 2:  $K_3$  and  $K_4$

**Theorem 2.2.** For the path  $P_3$ ,  $S_n(P_3) = 2$ .

*Proof.* Let  $P_3$  be as shown in the figure.

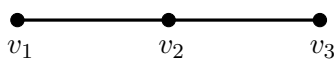


Figure 3:  $P_3$

Then the set  $S(P_3) = \{\{v_1, v_2\}, \{v_2, v_3\}\}$  separates all the vertices of  $P_3$ . Since  $\lceil \log_2 3 \rceil = 2$ , it follows that  $S_n(P_3) = 2$ . □

**Theorem 2.3.** The separation number of the complete graph  $K_n$ , with  $n \geq 3$  vertices is precisely  $S_n(K_n) = \lceil \log_2 n \rceil$ .

*Proof.* Every two element subset of the complete graph  $K_n$  constitutes a total dominating set. Consequently, the statement follows from Theorem 2.1. □

**Theorem 2.4.** The separation number of the star graph  $K_{1,n}$ , is  $S_n(K_{1,n}) = \lceil \log_2 (n + 1) \rceil$ .

*Proof.* In the proof of Theorem 2.1, the binary string  $(0, 0, \dots, 0)$  is assigned to the vertex  $v_1$ . The separating sets are defined as

$$S_i = \{v_j \mid \text{the } i\text{-th bit of } v_j \text{ is } 0\}.$$

Thus,  $v_1 \in S_i$  for all  $i$ .

Let  $v_1$  be the central vertex of degree  $n$  in the star graph  $K_{1,n}$ . By construction, each separating set  $S_i$  contains at least two vertices, ensuring that  $|S_i| \geq 2$ .

Consequently, each  $S_i$  forms a total dominating set of  $K_{1,n}$ . This completes the proof.  $\square$

**Theorem 2.5.** *The separation number of the wheel graph  $W_{1,n}$ , is  $S_n(W_{1,n}) = \lceil \log_2(n+1) \rceil$ .*

*Proof.* Let  $v_1$  be the vertex of degree  $n$  in  $W_{1,n}$ . Then proceeding as in Theorem 2.4 we can prove the result.  $\square$

**Theorem 2.6.** *The separation number of the complete bipartite graph  $K_{m,n}$ , is  $\lceil \log_2(m+n) \rceil$ .*

*Proof.* The separation number of the star graph is already obtained. Also,  $S_n(K_{2,2}) = 2$ . So we exclude star graphs and  $K_{2,2}$  in this discussion. Let  $m+n \geq 5$ . Let  $X = \{v_1, v_2, \dots, v_m\}$  and  $Y = \{u_1, u_2, \dots, u_n\}$  be the partite sets of  $K_{m,n}$ . We arrange the vertices as a sequence such that  $P = (v_1, v_2, v_3, u_1, u_2, u_3, v_4, v_5, v_6, u_4, u_5, u_6, \dots)$ . Consider the first  $m+n$  binary strings of length  $k = \lceil \log_2(m+n) \rceil$  of 0's and 1's. These binary strings serve as labels for the vertices in the sequence  $P$ . Assign the first binary string to  $v_1$ , the second to  $v_2$  and continue this process so that each vertex in the sequence  $P$  receives a unique binary string as its label corresponding to its position in the sequence. We form the separation set  $S(G) = \{S_1, S_2, \dots, S_k\}$  such that  $S_i = \{w_r | i^{th} \text{ bit of } w_r \text{ is } 0\}$ . This construction guarantees that each  $S_i$  includes at least one vertex from both  $X$  and  $Y$ , thereby ensuring that  $S_i$  forms a total dominating set of the complete bipartite graph  $K_{m,n}$ . In this construction, each vertex is assigned a unique binary string of length  $k$ . For any pair of distinct vertices, their corresponding binary strings differ in at least one position say, the  $i$ -th bit. This difference ensures that there exists a set which contains exactly one of the two vertices, thereby satisfying the separation property. This completes the proof.  $\square$

**Theorem 2.7.** *Let  $G$  be a graph with  $n \geq 2$  vertices. Then the separation number of the corona of  $G$  with complete graph  $K_m$  is given by*

$$S_n(G \circ K_m) = \begin{cases} n \lceil \log_2 m \rceil + n - 1 & \text{if } m \equiv 0 \pmod{4} \\ n \lceil \log_2(m+1) \rceil & \text{otherwise} \end{cases}$$

*Proof. Case i: When  $m \not\equiv 0 \pmod{4}$ .* For the complete graph  $K_m$ ,  $S_n(K_m) = \lceil \log_2 m \rceil$ . While building the separating set, the vertex  $v_1$  is assigned as the first binary string,  $v_2$  is assigned as the second binary string and so on. Since  $m \not\equiv 0 \pmod{4}$ , the string  $(1, 1, 1, \dots, 1)$  is excluded from labelling the vertices. So all vertices of  $K_m$  are included in atleast one of the separating sets. In  $G \circ K_m$ , all vertices of the  $i^{th}$  copy of  $K_m$  are adjacent to the  $i^{th}$  vertex of  $G$ . So there are  $n$  copies of  $K_{m+1}$  adjoined to the  $n$  vertices of  $G$ . So the union of the separating families of each copy of  $K_{m+1}$  will be a minimal separating family of  $K_m \circ G$ . Therefore  $S_n(G \circ K_m) = n \lceil \log_2(m+1) \rceil$

*Case ii: When  $m \equiv 0 \pmod{4}$ .* Proceeding as in Case (i), the vertex  $v_1$  is assigned the first binary string,  $v_2$  the second, and so on, with  $v_n$  corresponding to the binary string  $(1, 1, \dots, 1)$ . For each  $i$ , define

$$S_j = \{u \mid \text{the } j^{th} \text{ bit of the binary string assigned to } u \text{ is } 0\}.$$

It is evident that the vertex  $v_n$  does not belong to any of the sets  $S_j$ . As in Case (i) if we select the union of the separating sets then the excluded vertices of the  $n$  copies of  $K_m$  can not be separated. So for  $i = 1, 2, \dots, n-1$ . we set the separating set of the  $i^{th}$  copy of  $K_m$  as  $S^i = \{S_1, S_2, \dots, S_{\lceil \log_2 n \rceil}, V^i(K_m)\}$ , where  $V^i(K_m)$  is the vertex set of the  $i^{th}$  copy of  $K_m$ . For the  $n^{th}$  copy of  $K_m$ , the separating set is  $S^i = \{S_1, S_2, \dots, S_{\lceil \log_2 n \rceil}\}$ . Then the union of all these separating sets will form a separating set of  $G \circ K_m$ . Therefore  $S_n(G \circ K_m) = (n-1)[\lceil \log_2 m \rceil + 1] + \lceil \log_2 m \rceil = n \lceil \log_2 m \rceil + n - 1$ . This completes the proof.  $\square$

**Theorem 2.8.** *The separation number of the join of  $K_m$  with a graph  $G$  is given by  $S_n(K_m \vee G) = \lceil \log_2(m+k) \rceil$ , where  $k$  is the number of vertices in  $G$ .*

*Proof.* Let  $V(K_m) = \{v_1, v_2, \dots, v_m\}$  and  $V(G) = \{u_1, u_2, \dots, u_k\}$ . The first  $m$  binary strings are assigned to the vertices  $v_1, v_2, \dots, v_m$  and the next  $k$  binary strings are assigned to the vertices  $u_1, u_2, \dots, u_k$ . Let  $S = \{S_1, S_2, \dots, S_{\lceil \log_2(m+k) \rceil}\}$ , where  $S_j = \{w \mid \text{the } j^{\text{th}} \text{ bit of the binary string assigned to } w \text{ is } 0\}$ . Then  $S$  is a minimum separating set of  $K_m \vee G$  consisting of total dominating sets. This completes the proof.  $\square$

**Theorem 2.9.** *The separation number of the conormal product or disjunctive product of  $K_m$  with a graph  $G$  with  $k$  vertices is given by  $S_n(K_m * G) = \lceil \log_2(mk) \rceil$ .*

*Proof.* Let  $x_1, x_2$  be two vertices of  $K_m$ . Since they are adjacent in  $K_m$ , any two pair of vertices  $(x_1, y_1)$  and  $(x_2, y_2)$  in  $K_m * G$ , are adjacent. Therefore proceeding as in Theorem 2.8 the result can be proved.  $\square$

**Theorem 2.10.** *The separation number of the conormal product of two star graphs  $K_{1,n}$  and  $K_{1,m}$  is given by  $S_n(K_{1,n} * K_{1,m}) = \lceil \log_2(mn) \rceil$ .*

*Proof.* Let  $V(K_{1,n}) = \{u, x_1, x_2, \dots, x_n\}$  and  $V(K_{1,m}) = \{u, y_1, y_2, \dots, y_m\}$ , where  $d(u) = m$  and  $d(v) = m$ . Consider the first  $mn$  binary strings of length  $k$ , where  $k = \lceil \log_2(mn) \rceil$ . Assign the first string to the vertex  $(u, v)$  and the remaining strings to other vertices. Then if  $S$  is a subset of  $V(K_{1,n} * K_{1,m})$  with  $(u, v) \in S$  and  $|S| \geq 2$ , then  $S$  is a total domination set of  $K_{1,n} * K_{1,m}$ . Then the proof follows by proceeding as in Theorem 2.8.  $\square$

## Conclusion

In this paper, we introduced the separation number  $S_n(G)$  of a graph  $G$  and its exact values for certain classes of graphs such as complete graphs, and the conormal product of complete graphs. The results provide parameters and contribute to the structural understanding of graphs. Future work may focus on extending the concept to other graph products and exploring algorithmic methods for computing  $S_n(G)$  in complex networks.

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