

Wang-Type Fixed Point Theorems for Expansive Maps in Hemi-Metric Spaces

Abstract

In this paper, we establish Wang-type fixed point results for expansive mappings in the framework of hemi-metric spaces. In particular, we prove that a surjective expansive mapping on an h -complete hemi-metric space admits a unique fixed point, and we also obtain a corresponding local version on invariant h -complete subsets. Unlike existing results that are confined to pairwise distance structures, our approach extends expansion-type fixed point theory to $(m + 1)$ -point distance structures (hemi-metrics) via an inverse contraction technique. Several illustrative examples are presented to demonstrate the applicability of the results, and an application to integral equations is included to highlight the relevance of the developed theory.

Keywords: Fixed point; expansive maps; hemi-metric space.

1 Introduction

Fixed point theory is one of the most fundamental and widely studied areas of nonlinear analysis, with significant applications in differential equations, optimization theory, economics, and dynamical systems. A fixed point of a mapping is an element that remains invariant under the mapping, and the existence of such points often ensures the stability and solvability of various mathematical models [10].

A cornerstone result in this area is the celebrated Banach contraction principle, which provides a simple yet powerful criterion for the existence and uniqueness of fixed points.

Theorem 1.1 (Banach Contraction Principle [10]). *Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be a mapping satisfying*

$$d(Tx, Ty) \leq k d(x, y), \quad \text{for all } x, y \in X,$$

where $0 < k < 1$. Then T has a unique fixed point $x^ \in X$, and for any $x_0 \in X$, the iterative sequence defined by $x_{n+1} = T(x_n)$ converges to x^* .*

This principle has been extended in numerous directions to accommodate more general settings, including partially ordered metric spaces, nonlinear contractions, and hybrid contractive conditions [27, 28, 1]. In particular, modern approaches based on simulation functions and generalized contraction schemes have significantly broadened the scope of fixed point theory [14, 19].

In contrast to contraction mappings, expansive mappings increase the distance between points. Although this behavior appears to hinder convergence, it has been shown that fixed point results can still be obtained under suitable structural assumptions. The pioneering work of Wang et al. [13] initiated the study of expansive mappings, which was subsequently developed by Daffer and Kaneko [2], Rhoades [24], and Ćirić [26].

A typical formulation of such results is given by the following theorem.

Theorem 1.2 (Wang-type Expansive Mapping Principle [13]). *Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be a surjective mapping satisfying*

$$d(Tx, Ty) \geq a d(x, y), \quad \text{for all } x, y \in X,$$

where $a > 1$. Then T admits a unique fixed point in X .

These results demonstrate that expansiveness, when combined with structural conditions such as surjectivity and completeness, can still guarantee the existence and uniqueness of fixed points.

Parallel to these developments, there has been substantial progress in extending the concept of metric spaces. Early contributions by Dhage [4, 5] and Gähler [6, 7] introduced generalized and multi-dimensional distance structures. Later, Mustafa and Sims [11] proposed G -metric spaces, while Daheriya et al. [3] and Jain et al. [9] investigated expansive-type mappings in dislocated and parametric metric spaces. These generalizations provide a flexible framework for studying nonlinear problems beyond classical metric assumptions.

More recently, attention has been directed toward generalized metric-type spaces such as partial metric spaces and b -metric spaces, where fixed point results for expansive mappings have also been established [8, 17, 22]. In addition, unified approaches based on generalized contractions have been developed in recent literature [16, 18, 21, 20, 23], further enriching the theory.

Despite these advances, most existing results are confined to pairwise distance structures. The extension of expansive mapping principles to hemi-metric spaces, where distance is defined on $(m + 1)$ -tuples of points, remains largely unexplored.

Recently, hemi-metric spaces have emerged as an important generalization in which distance is defined on tuples of points rather than pairs. This multi-point framework allows a more comprehensive analysis of interactions among multiple elements. The work of Ozturk and Radenovic [12] highlights the significance of hemi-metric spaces in extending classical fixed point results.

To the best of our knowledge, such results have not been established in hemi-metric spaces.

Motivated by the above developments, the aim of this paper is to extend Wang-type fixed point theorems to the setting of hemi-metric spaces. Using an inverse mapping technique, we establish both global and local fixed point results for expansive mappings in this generalized framework. The obtained results not only generalize several known theorems but also provide new insights into the interplay between expansive mappings and multi-point distance structures.

For clarity, the main contributions of this paper are summarized as follows:

- We extend Wang-type expansive mapping principles from metric spaces to hemi-metric spaces.
- We employ an inverse contraction technique to establish existence and uniqueness of fixed points.
- We obtain both global and local fixed point results under expansive conditions.
- We provide illustrative examples and an application to integral equations to demonstrate the applicability of the theory.

2 Preliminaries

In this section, we recall some basic definitions and concepts related to hemi-metric spaces, following the framework introduced in [12].

Definition 2.1 ([12]). *Let H be a nonempty set and let $m \in \mathbb{N}$. A function*

$$d_h : H^{m+1} \rightarrow [0, \infty)$$

is called an m -hemi-metric on H if, for all $x_1, x_2, \dots, x_{m+1} \in H$, the following conditions hold:

(i) $d_h(x_1, x_2, \dots, x_{m+1}) \geq 0$;

(ii) $d_h(x, x, \dots, x) = 0$ for all $x \in H$, and conversely, if

$$d_h(x_1, x_2, \dots, x_{m+1}) = 0,$$

then $x_1 = x_2 = \dots = x_{m+1}$;

(iii) $d_h(x_1, x_2, \dots, x_{m+1})$ is invariant under any permutation of its arguments;

(iv) for all $x_1, \dots, x_{m+1}, y \in H$,

$$d_h(x_1, \dots, x_{m+1}) \leq \sum_{i=1}^{m+1} d_h(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_{m+1}).$$

The pair (H, d_h) is called a hemi-metric space.

Remark 2.2. *Unlike classical metric spaces, where the distance is defined between two points, a hemi-metric measures the collective dispersion among $(m + 1)$ points. This multi-point structure provides a richer framework for the study of nonlinear problems and fixed point theory.*

Definition 2.3 ([12]). *A sequence $(x_n) \subset H$ is said to converge to a point $x \in H$ if*

$$d_h(x_n, x, x, \dots, x) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Definition 2.4 ([12]). *A sequence $(x_n) \subset H$ is called Cauchy (or hemi-Cauchy) if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all indices $n_1, n_2, \dots, n_{m+1} \geq N$,*

$$d_h(x_{n_1}, x_{n_2}, \dots, x_{n_{m+1}}) < \varepsilon.$$

Definition 2.5 ([12]). *A hemi-metric space (H, d_h) is said to be h-complete if every hemi-Cauchy sequence in H converges to a point of H .*

Definition 2.6 ([12]). *Let (H, d_h) be a hemi-metric space. A point $x \in H$ is called a fixed point of a mapping $F : H \rightarrow H$ if $F(x) = x$.*

Definition 2.7 ([12]). *A mapping $F : H \rightarrow H$ is said to be continuous if, whenever $(x_n) \subset H$ converges to $x \in H$, the sequence $(F(x_n))$ converges to $F(x)$.*

Definition 2.8. *A mapping $F : H \rightarrow H$ is said to be surjective if, for every $y \in H$, there exists $x \in H$ such that $F(x) = y$.*

3 Examples of Hemi-Metric Spaces

In this section, we present several examples of hemi-metric spaces to illustrate the generality of the concept.

Example 3.1. Let $H = \mathbb{R}$ and define

$$d_h(x_1, \dots, x_{m+1}) = \max_{1 \leq i, j \leq m+1} |x_i - x_j|.$$

Then (H, d_h) is a hemi-metric space.

Proof. Conditions (i)–(iii) follow directly from the properties of the absolute value and the symmetry of the maximum. For (iv), let $y \in \mathbb{R}$. Using the triangle inequality,

$$|x_i - x_j| \leq |x_i - y| + |y - x_j|,$$

and taking the maximum over all i, j , we obtain

$$d_h(x_1, \dots, x_{m+1}) \leq 2 \max_{1 \leq k \leq m+1} |x_k - y|.$$

Moreover, for each i ,

$$d_h(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_{m+1}) \geq \max_{1 \leq k \leq m+1} |x_k - y|.$$

Summing over $i = 1, \dots, m+1$, we obtain the desired inequality. Hence (H, d_h) is a hemi-metric space. \square

Example 3.2. Let $H = \mathbb{R}$ and define

$$d_h(x_1, \dots, x_{m+1}) = \sum_{1 \leq i < j \leq m+1} |x_i - x_j|.$$

Then (H, d_h) is a hemi-metric space.

Proof. Conditions (i)–(iii) are immediate. For (iv), using

$$|x_i - x_j| \leq |x_i - y| + |y - x_j|,$$

and summing over all pairs $i < j$, we obtain

$$d_h(x_1, \dots, x_{m+1}) \leq m \sum_{k=1}^{m+1} |x_k - y|.$$

Furthermore, for each i ,

$$d_h(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_{m+1}) \geq \sum_{k \neq i} |x_k - y|.$$

Summing over $i = 1, \dots, m+1$ yields the required inequality. Hence (H, d_h) is a hemi-metric space. \square

Example 3.3. Let $H = \mathbb{R}$ and define

$$d_h(x_1, \dots, x_{m+1}) = \sum_{i=1}^{m+1} |x_i - \bar{x}|, \quad \text{where } \bar{x} = \frac{1}{m+1} \sum_{k=1}^{m+1} x_k.$$

Then (H, d_h) is a hemi-metric space.

Proof. Conditions (i)–(iii) are straightforward. For (iv), let $y \in \mathbb{R}$. Using

$$|x_i - \bar{x}| \leq |x_i - y| + |y - \bar{x}|,$$

and summing over i , we obtain

$$d_h(x_1, \dots, x_{m+1}) \leq \sum_{i=1}^{m+1} |x_i - y| + (m+1)|y - \bar{x}|.$$

Since

$$|y - \bar{x}| = \left| \frac{1}{m+1} \sum_{k=1}^{m+1} (y - x_k) \right| \leq \frac{1}{m+1} \sum_{k=1}^{m+1} |x_k - y|,$$

it follows that

$$(m+1)|y - \bar{x}| \leq \sum_{k=1}^{m+1} |x_k - y|.$$

Thus,

$$d_h(x_1, \dots, x_{m+1}) \leq 2 \sum_{k=1}^{m+1} |x_k - y|,$$

which yields the required inequality. Hence (H, d_h) is a hemi-metric space. \square

Remark 3.4. *The above examples illustrate different types of hemi-metrics, including maximum-type, sum-type, and mean-based structures. These examples show that hemi-metric spaces naturally generalize classical metric spaces and provide flexibility in modeling multi-point interactions.*

4 Main Results

In this section, we introduce the notion of expansive mappings in hemi-metric spaces and establish global and local fixed point results.

Definition 4.1. *Let (H, d_h) be a hemi-metric space. A mapping $F : H \rightarrow H$ is called expansive if there exists a constant $a > 1$ such that*

$$d_h(F(x_1), F(x_2), \dots, F(x_{m+1})) \geq a d_h(x_1, x_2, \dots, x_{m+1})$$

for all $(x_1, x_2, \dots, x_{m+1}) \in H^{m+1}$.

Theorem 4.2. *Let (H, d_h) be an h -complete hemi-metric space of order m , and let $F : H \rightarrow H$ be a mapping. Suppose that there exists a constant $a > 1$ such that*

$$d_h(F(u_1), \dots, F(u_{m+1})) \geq a d_h(u_1, \dots, u_{m+1})$$

for all $(u_1, \dots, u_{m+1}) \in H^{m+1}$. If F is surjective, then F has a unique fixed point in H .

Proof. First, we show that F is injective. Let $x, y \in H$ such that $F(x) = F(y)$. Consider the $(m+1)$ -tuple $(x, \dots, x, y) \in H^{m+1}$. Then

$$d_h(F(x), \dots, F(x), F(y)) = 0.$$

By the expansivity condition,

$$0 \geq a d_h(x, \dots, x, y).$$

Since $a > 1$ and $d_h \geq 0$, it follows that

$$d_h(x, \dots, x, y) = 0,$$

and hence $x = y$. Therefore, F is injective.

Since F is also surjective, it is bijective. Let $G := F^{-1} : H \rightarrow H$ denote its inverse.

Next, we show that G is a contraction. Let $(v_1, \dots, v_{m+1}) \in H^{m+1}$ and define $u_i = G(v_i)$ for $i = 1, \dots, m+1$. Then $F(u_i) = v_i$, and hence

$$d_h(v_1, \dots, v_{m+1}) = d_h(F(u_1), \dots, F(u_{m+1})) \geq a d_h(u_1, \dots, u_{m+1}),$$

which implies

$$d_h(G(v_1), \dots, G(v_{m+1})) \leq \frac{1}{a} d_h(v_1, \dots, v_{m+1}).$$

Thus, G is a contraction on (H, d_h) with contraction constant $k = \frac{1}{a} \in (0, 1)$.

Fix $x_0 \in H$ and define a sequence (x_n) by $x_{n+1} = G(x_n)$. Then, for any $p \geq 1$, repeated application of the contraction property yields

$$d_h(x_n, x_{n+1}, \dots, x_{n+p}) \leq k^n d_h(x_0, x_1, \dots, x_p).$$

Since $0 < k < 1$, it follows that

$$d_h(x_n, x_{n+1}, \dots, x_{n+p}) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and hence (x_n) is a hemi-Cauchy sequence.

Since (H, d_h) is h -complete, there exists $x^* \in H$ such that $x_n \rightarrow x^*$.

We now show that x^* is a fixed point of F . From $x_{n+1} = G(x_n)$ and the contractive inequality, it follows that G is sequentially continuous. Hence,

$$x_{n+1} = G(x_n) \rightarrow G(x^*).$$

But also $x_{n+1} \rightarrow x^*$, and therefore

$$G(x^*) = x^*.$$

Applying F to both sides, we obtain

$$F(x^*) = x^*.$$

Finally, we prove uniqueness. Suppose that $y^* \in H$ is another fixed point of F . Then

$$d_h(x^*, y^*, \dots) = d_h(F(x^*), F(y^*), \dots) \geq a d_h(x^*, y^*, \dots).$$

Since $a > 1$, this implies

$$d_h(x^*, y^*, \dots) = 0,$$

and hence $x^* = y^*$. Therefore, the fixed point is unique. □

Remark 4.3. *The above result generalizes the classical Wang-type expansive mapping principle from metric spaces to hemi-metric spaces, where the distance is defined on $(m+1)$ -tuples of points.*

Theorem 4.4. *Let (H, d_h) be a hemi-metric space and let $K \subset H$ be a nonempty subset such that $F(K) \subset K$. Suppose that:*

(i) there exists a constant $a > 1$ such that

$$d_h(F(u_1), \dots, F(u_{m+1})) \geq a d_h(u_1, \dots, u_{m+1})$$

for all $(u_1, \dots, u_{m+1}) \in K^{m+1}$;

(ii) $F : K \rightarrow K$ is surjective;

(iii) (K, d_h) is h -complete.

Then F has a unique fixed point in K .

Proof. First, we show that F is injective on K . Let $x, y \in K$ such that $F(x) = F(y)$. Consider the $(m + 1)$ -tuple $(x, \dots, x, y) \in K^{m+1}$. Then

$$d_h(F(x), \dots, F(x), F(y)) = 0.$$

By the expansivity condition (i), we obtain

$$0 \geq a d_h(x, \dots, x, y).$$

Since $a > 1$ and $d_h \geq 0$, it follows that

$$d_h(x, \dots, x, y) = 0,$$

and hence $x = y$. Therefore, F is injective on K .

Since $F : K \rightarrow K$ is both injective and surjective, it is bijective. Let $G := F^{-1} : K \rightarrow K$ denote its inverse.

Next, we show that G is a contraction. Let $(v_1, \dots, v_{m+1}) \in K^{m+1}$ and define $u_i = G(v_i)$ for $i = 1, \dots, m + 1$. Then $F(u_i) = v_i$, and hence

$$d_h(v_1, \dots, v_{m+1}) = d_h(F(u_1), \dots, F(u_{m+1})) \geq a d_h(u_1, \dots, u_{m+1}),$$

which implies

$$d_h(G(v_1), \dots, G(v_{m+1})) \leq \frac{1}{a} d_h(v_1, \dots, v_{m+1}).$$

Thus, G is a contraction on (K, d_h) with contraction constant $k = \frac{1}{a} \in (0, 1)$.

Fix $x_0 \in K$ and define a sequence (x_n) in K by

$$x_{n+1} = G(x_n), \quad n \geq 0.$$

Then, for any $p \geq 1$, repeated application of the contraction property yields

$$d_h(x_n, x_{n+1}, \dots, x_{n+p}) \leq k^n d_h(x_0, x_1, \dots, x_p).$$

Since $0 < k < 1$, it follows that

$$d_h(x_n, x_{n+1}, \dots, x_{n+p}) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence (x_n) is a hemi-Cauchy sequence in K .

Since (K, d_h) is h -complete, there exists $x^* \in K$ such that

$$x_n \rightarrow x^*.$$

We now show that x^* is a fixed point of F . From the definition of the sequence,

$$x_{n+1} = G(x_n).$$

It follows from the contractive-type inequality that G is sequentially continuous. Hence,

$$x_{n+1} = G(x_n) \rightarrow G(x^*).$$

But also $x_{n+1} \rightarrow x^*$, and therefore

$$G(x^*) = x^*.$$

Applying F to both sides, we obtain

$$F(x^*) = x^*,$$

so x^* is a fixed point of F in K .

Finally, we prove uniqueness. Suppose that $x^*, y^* \in K$ are two fixed points of F . Then

$$d_h(x^*, y^*, \dots) = d_h(F(x^*), F(y^*), \dots) \geq a d_h(x^*, y^*, \dots).$$

Since $a > 1$, this implies

$$d_h(x^*, y^*, \dots) = 0,$$

and hence $x^* = y^*$. Therefore, the fixed point is unique in K . □

Remark 4.5. *The above theorem provides a localized version of the main result, ensuring the existence and uniqueness of a fixed point within invariant subsets.*

Theorem 4.6. *Let (H, d_h) be an h -complete hemi-metric space and let $F : H \rightarrow H$ be a surjective mapping. Suppose that there exists a function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that:*

(i) $\varphi(t) > t$ for all $t > 0$,

(ii) φ is continuous and non-decreasing,

(iii) $\varphi(0) = 0$,

and

$$d_h(F(u_1), \dots, F(u_{m+1})) \geq \varphi(d_h(u_1, \dots, u_{m+1}))$$

for all $(u_1, \dots, u_{m+1}) \in H^{m+1}$. Then F has a unique fixed point in H .

Proof. First, we show that F is injective. Let $x, y \in H$ such that $F(x) = F(y)$. Consider the $(m+1)$ -tuple $(x, \dots, x, y) \in H^{m+1}$. Then

$$d_h(F(x), \dots, F(x), F(y)) = 0.$$

Using the given condition, we obtain

$$0 \geq \varphi(d_h(x, \dots, x, y)).$$

Since $\varphi(t) > 0$ for all $t > 0$ and $\varphi(0) = 0$, it follows that

$$d_h(x, \dots, x, y) = 0,$$

and hence $x = y$. Therefore, F is injective.

Since F is also surjective, it follows that F is bijective. Let $G = F^{-1} : H \rightarrow H$ denote the inverse mapping.

Next, we show that G satisfies a contractive-type condition. Let $(v_1, \dots, v_{m+1}) \in H^{m+1}$ and define $u_i = G(v_i)$ for $i = 1, \dots, m+1$. Then $F(u_i) = v_i$, and hence

$$d_h(v_1, \dots, v_{m+1}) = d_h(F(u_1), \dots, F(u_{m+1})) \geq \varphi(d_h(u_1, \dots, u_{m+1})).$$

This implies

$$d_h(G(v_1), \dots, G(v_{m+1})) \leq \psi(d_h(v_1, \dots, v_{m+1})),$$

where the function $\psi : [0, \infty) \rightarrow [0, \infty)$ is defined by

$$\psi(t) = \sup\{s \geq 0 : \varphi(s) \leq t\}.$$

Since φ is non-decreasing and $\varphi(t) > t$ for all $t > 0$, it follows that $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$. Now, fix $x_0 \in H$ and define a sequence (x_n) by

$$x_{n+1} = G(x_n), \quad n \geq 0.$$

Set

$$a_n = d_h(x_n, x_{n+1}, \dots, x_{n+m}).$$

Using the above inequality, we obtain

$$a_{n+1} \leq \psi(a_n).$$

Since $\psi(t) < t$ for all $t > 0$, it follows that (a_n) is a non-increasing sequence of nonnegative real numbers. Hence, there exists $L \geq 0$ such that $a_n \rightarrow L$.

Passing to the limit in the inequality $a_{n+1} \leq \psi(a_n)$, we obtain

$$L \leq \psi(L).$$

If $L > 0$, then $\psi(L) < L$, which is a contradiction. Hence $L = 0$. Therefore,

$$a_n \rightarrow 0,$$

which implies that (x_n) is a hemi-Cauchy sequence.

Since (H, d_h) is h -complete, there exists $x^* \in H$ such that

$$x_n \rightarrow x^*.$$

Next, we show that x^* is a fixed point of F . From $x_{n+1} = G(x_n)$ and the contractive-type inequality, it follows that G is sequentially continuous. Hence,

$$x_{n+1} = G(x_n) \rightarrow G(x^*).$$

But also $x_{n+1} \rightarrow x^*$, and therefore

$$G(x^*) = x^*.$$

Applying F to both sides, we obtain

$$F(x^*) = x^*.$$

Finally, we prove uniqueness. Suppose that $x^*, y^* \in H$ are fixed points of F . Then

$$d_h(x^*, y^*, \dots) = d_h(F(x^*), F(y^*), \dots) \geq \varphi(d_h(x^*, y^*, \dots)).$$

If $d_h(x^*, y^*, \dots) > 0$, then $\varphi(d_h(x^*, y^*, \dots)) > d_h(x^*, y^*, \dots)$, which is a contradiction. Hence,

$$d_h(x^*, y^*, \dots) = 0,$$

and therefore $x^* = y^*$. This proves the uniqueness of the fixed point. \square

Remark 4.7. *This theorem extends classical expansive mapping results by replacing linear expansion conditions with nonlinear control functions, thereby significantly broadening the scope of fixed point theory in hemi-metric spaces.*

5 Examples

In this section, we illustrate the validity of the main results with concrete examples.

Example 5.1. Let $H = \mathbb{R}$ and define the hemi-metric of order m by

$$d_h(x_1, \dots, x_{m+1}) = \max_{1 \leq i, j \leq m+1} |x_i - x_j|.$$

Define $F : H \rightarrow H$ by $F(x) = 2x$.

Then, for any $(x_1, \dots, x_{m+1}) \in H^{m+1}$,

$$d_h(F(x_1), \dots, F(x_{m+1})) = \max_{1 \leq i, j \leq m+1} |2x_i - 2x_j| = 2 d_h(x_1, \dots, x_{m+1}).$$

Thus, F is expansive with constant $a = 2 > 1$.

Clearly, F is bijective. Solving $F(x) = x$, we obtain $x = 0$.

Hence, all the hypotheses of Theorem 4.2 are satisfied, and $x^* = 0$ is the unique fixed point.

Example 5.2. Let $H = \mathbb{R}$ equipped with the hemi-metric

$$d_h(x_1, \dots, x_{m+1}) = \max_{1 \leq i, j \leq m+1} |x_i - x_j|,$$

and let $K = [0, \infty)$. Define $F : K \rightarrow K$ by $F(x) = 2x$.

Then, for any $(x_1, \dots, x_{m+1}) \in K^{m+1}$,

$$d_h(F(x_1), \dots, F(x_{m+1})) = 2 d_h(x_1, \dots, x_{m+1}),$$

so F is expansive on K with constant $a = 2 > 1$.

Clearly, $F(K) \subset K$, and F is surjective on K . Since (\mathbb{R}, d_h) is h -complete and K is closed in \mathbb{R} , it follows that (K, d_h) is h -complete.

Solving $F(x) = x$, we obtain $x = 0 \in K$.

Thus, all the hypotheses of Theorem 4.4 are satisfied, and $x^* = 0$ is the unique fixed point in K .

Example 5.3. Let $H = \mathbb{R}$ equipped with the hemi-metric

$$d_h(x_1, \dots, x_{m+1}) = \max_{1 \leq i, j \leq m+1} |x_i - x_j|,$$

and define $F : H \rightarrow H$ by

$$F(x) = 2x + 1.$$

Then, for any $(x_1, \dots, x_{m+1}) \in H^{m+1}$,

$$d_h(F(x_1), \dots, F(x_{m+1})) = \max_{1 \leq i, j \leq m+1} |2x_i + 1 - (2x_j + 1)| = 2 d_h(x_1, \dots, x_{m+1}).$$

Thus, F is expansive with constant $a = 2 > 1$.

Clearly, F is bijective. Solving $F(x) = x$, we obtain $x = -1$.

Hence, all the hypotheses of Theorem 4.2 are satisfied, and $x^* = -1$ is the unique fixed point.

Example 5.4. Let $H = \mathbb{R}$ equipped with the hemi-metric

$$d_h(x_1, \dots, x_{m+1}) = \max_{1 \leq i, j \leq m+1} |x_i - x_j|,$$

and define $F : H \rightarrow H$ by $F(x) = 2x$. Let

$$\varphi(t) = \frac{3}{2}t.$$

Then $\varphi(t) > t$ for all $t > 0$, φ is continuous and non-decreasing, and $\varphi(0) = 0$.

For any $(x_1, \dots, x_{m+1}) \in H^{m+1}$,

$$d_h(F(x_1), \dots, F(x_{m+1})) = 2d_h(x_1, \dots, x_{m+1}) \geq \varphi(d_h(x_1, \dots, x_{m+1})).$$

Thus, all the hypotheses of Theorem 4.6 are satisfied. Solving $F(x) = x$, we obtain $x = 0$.

Therefore, $x^* = 0$ is the unique fixed point.

Remark 5.5. The above examples demonstrate the applicability of the main results to linear mappings under different settings, including global, local, and generalized expansivity conditions. They illustrate how hemi-metric structures naturally extend classical fixed point results to multi-point distance frameworks.

6 Application to Integral Equations

We apply the obtained fixed point results to establish the existence and uniqueness of solutions for a class of linear integral equations.

Theorem 6.1. Let $H = C([0, 1], \mathbb{R})$ be the space of continuous functions on $[0, 1]$, endowed with the hemi-metric

$$d_h(f_1, \dots, f_{m+1}) = \max_{1 \leq i, j \leq m+1} \|f_i - f_j\|_\infty,$$

where $\|f\|_\infty = \sup_{t \in [0, 1]} |f(t)|$. Then (H, d_h) is an h -complete hemi-metric space.

Let $K : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ be a continuous kernel and define the operator $F : H \rightarrow H$ by

$$(Ff)(t) = \int_0^1 K(t, s)f(s) ds.$$

Assume that there exists a constant $\lambda > 1$ such that

$$\|Ff - Fg\|_\infty \geq \lambda \|f - g\|_\infty \quad \text{for all } f, g \in H,$$

and that F is surjective.

Then the integral equation

$$f(t) = \int_0^1 K(t, s)f(s) ds$$

admits a unique solution in H .

Proof. The space $C([0, 1], \mathbb{R})$ is complete with respect to the supremum norm. Since the hemi-metric d_h is defined in terms of the supremum norm, it follows that (H, d_h) is h -complete.

For any $f_1, \dots, f_{m+1} \in H$, we have

$$d_h(F(f_1), \dots, F(f_{m+1})) = \max_{1 \leq i, j \leq m+1} \|F(f_i) - F(f_j)\|_\infty.$$

By the given assumption,

$$\|F(f_i) - F(f_j)\|_\infty \geq \lambda \|f_i - f_j\|_\infty \quad \text{for all } i, j.$$

Taking the maximum over all i, j , we obtain

$$d_h(F(f_1), \dots, F(f_{m+1})) \geq \lambda d_h(f_1, \dots, f_{m+1}).$$

Thus, F is an expansive mapping with constant $\lambda > 1$ on (H, d_h) . Since F is surjective and (H, d_h) is h -complete, all the hypotheses of Theorem 4.2 are satisfied. Therefore, F admits a unique fixed point $f^* \in H$.

By the definition of a fixed point, f^* satisfies

$$f^*(t) = \int_0^1 K(t, s) f^*(s) ds \quad \text{for all } t \in [0, 1],$$

which shows that f^* is the unique solution of the given integral equation. \square

Remark 6.2. *The assumptions of expansivity and surjectivity on the operator F are strong and may not hold for general integral operators. However, such conditions may hold for specially constructed operators or restricted classes of kernels, and the result illustrates the applicability of the developed theory.*

7 Conclusion

In this paper, we have investigated fixed point results for expansive mappings in the setting of hemi-metric spaces, which generalize classical metric structures by incorporating $(m + 1)$ -point distance functions. The main contributions include the establishment of global and local fixed point theorems under expansive conditions, together with their extension via control functions within this generalized framework.

The results obtained extend and unify several classical theorems in fixed point theory, particularly those concerning expansive mappings, by adapting them to hemi-metric spaces. The use of the inverse mapping technique provides an effective alternative to traditional contraction-based methods and offers a broader perspective for the analysis of nonlinear problems in generalized metric structures.

Furthermore, illustrative examples are presented to demonstrate the applicability of the theoretical results, and an application to integral equations is developed to highlight the relevance of the proposed approach in functional analysis.

Future research may focus on extending these results to more general settings such as ordered hemi-metric spaces, multivalued mappings, and nonlinear integral or differential equations. In addition, the study of stability, data dependence, and iterative approximation methods in hemi-metric spaces constitutes a promising direction for further investigation.

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