

ENP EQUIVALENT CLASSES OF GRAPHS

ABSTRACT. Let $G(V, E)$ be a simple graph of order n with vertex set V and edge set E . Let (u, v) denotes an unordered vertex pair of distinct vertices of G . For a vertex $u \in G$, let $N(u)$ be the set of all vertices of G which are adjacent to u in G . Then for $0 \leq i \leq n - 1$, the i -equi neighbor set of G is defined as: $N_e(G, i) = \{(u, v) : u, v \in V, u \neq v \text{ and } |N(u)| = |N(v)| = i\}$. The equi-neighbor polynomial $N_e[G; x]$ of G is defined as $N_e[G; x] = \sum_{i=0}^{(n-1)} |N_e(G, i)|x^i$. Two graphs G and H are said to be ENP- equivalent if and only if $N_e[G; x] = N_e[H; x]$. A graph H is said to be ENP-unique if H is ENP-equivalent to itself. A graph G is said to be ENP-cardinal unique if there does not exist a graph H with the same number of vertices as G such that H is equivalent to G . This paper identifies several ENP-unique graphs, explores ENP-equivalent graph classes, and characterizes specific ENP-ardinal unique graph classes.

[2010]05C31, 05C39 i - equi neighbor set, equi neighbor polynomial.

1. INTRODUCTION

Graph theory provides powerful tools for modeling relationships, and within this field, graph polynomials are essential for characterizing structural properties. This paper focuses on the equi-neighbor polynomial (ENP), $N_e[G; x]$, which is used to classify graphs based on the distribution of vertices that share the same degree. Let $G(V, E)$ be a simple graph of order n (number of vertices). For any vertex $u \in V$, let $N(u)$ be the set of its adjacent vertices (its neighbors). The core concept is the i -equi neighbor set of G , denoted $N_e(G, i)$, which is defined for $0 \leq i \leq n - 1$ as the set of unordered pairs of distinct vertices (u, v) where both vertices have the same degree i :

$$(1.1) \quad N_e(G, i) = \{(u, v) : u, v \in V, u \neq v \text{ and } |N(u)| = |N(v)| = i\}$$

$$(1.2) \quad = \left\{ (u, v) \in \binom{V}{2} : |N(u)| = |N(v)| = i \right\},$$

where $\binom{V}{2}$ denotes the set of all unordered pairs of vertices of G . The equi neighbor polynomial $N_e[G; x]$ is then defined as the generating function where the coefficient of x^i is the cardinality of the i - equineighbor set[1]. That is,

$$N_e[G; x] = \sum_{i=0}^{(n-1)} |N_e(G, i)|x^i.$$

Based on this polynomial, we define graph equivalence relations and uniqueness properties: Two graphs G and H are ENP-equivalent (denoted $G \overset{\mathcal{N}_E}{\sim} H$) if and only if their equi-neighbor polynomials are identical, that is, $N_e[G; x] = N_e[H; x]$. Obviously, the relation $\overset{\mathcal{N}_E}{\sim}$ is an equivalence relation on the class \mathcal{G} of all finite simple graphs. The set of all graphs ENP-equivalent to a graph G is denoted as $[G]_{\mathcal{N}_E}$ and is defined as $[G]_{\mathcal{N}_E} = \{H \in \mathcal{G} : N_e[H; x] = N_e[G; x]\}$. A graph H is said to be ENP-unique if $[H]_{\mathcal{N}_E} = \{H\}$.

Observe that isomorphic graphs have same equi neighbour polynomial, that is, if $H \cong G$, then H is ENP - equivalent to G . But non isomorphic graphs may have same equi neighbor polynomial. For instane, the graphs shown in the figure are ENP -equivalent.

Graph polynomials are crucial tools in graph theory. In this work, we focus on the equi-neighbor polynomial(ENP). Let $G(V, E)$ be a simple graph of order n . Let (u, v) denotes an unordered vertex pair of distant vertices of G . The i -equi neighbour set of G is defined as $N_e(G, i) = \{(u, v) : u, v \in V, u \neq v \text{ and } |N(u)| = |N(v)| = i\}$, for $0 \leq i \leq n - 1$. The polynomial $N_e[G; x] = \sum_{i=0}^{(n-1)} |N_e(G, i)|x^i$ is defined as the equi neighbour polynomial of G . We say that two graphs G and H are ENP -equivalent ($G \stackrel{\mathcal{N}_E}{\sim} H$) if and only if $N_e[G; x] = N_e[H; x]$.

Obviously, the relation $\stackrel{\mathcal{N}_E}{\sim}$ is an equivalence relation on the class \mathcal{G} of all simple finite graphs. The set of all graphs ENP -equivalent to a graph G is denoted as $[G]_{\mathcal{N}_E}$ and is defined as $[G]_{\mathcal{N}_E} = \{H \in \mathcal{G} : N_e[H; x] = N_e[G; x]\}$. A graph H is said to be ENP -unique if $[H]_{\mathcal{N}_E} = \{H\}$. Observe that isomorphic graphs have the same equi neighbour polynomial, that is, if $H \cong G$, then H is ENP - equivalent to G . But non-isomorphic graphs may have the same equi-neighbor polynomial. For instance, the graphs shown in the figure 1 have the same equi-neighbor polynomial $3x^2 + x$. Hence G and H are ENP -equivalent. In this

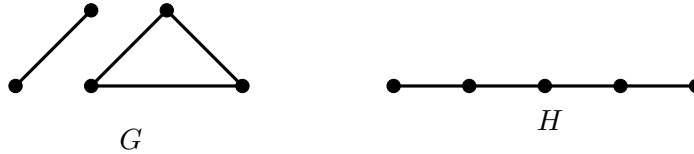


FIGURE 1. Two ENP -equivalent graphs G and H

paper, we aim to identify and characterize various graph classes based on these definitions, establishing fundamental theorems relating the ENP of a graph to that of its complement and unions, and determining which well-known graph families possess ENP -unique or ENP -cardinal unique properties. we identify some ENP - unique graphs, some ENP -equivalent graph classes, and some ENP -cardinal unique graph classes.

2. PROPERTIES OF THE EQUI-NEIGHBOR POLYNOMIAL

We begin by establishing properties of the ENP, particularly its relationship with graph complements and components.

Theorem 1. *Let G be a graph with n vertices and let \overline{G} denotes the complement of G . Then $N_e[\overline{G}; x] = x^{n-1}N_e[G; \frac{1}{x}]$.*

Proof. Observe that the sum of the degrees of a vertex in G and \overline{G} equals $n - 1$. Let u and v be two distinct verties in $V(G)$ such that $|N(u)| = |N(v)| = i$ in G . Then u has $n - 1 - i$ neighbors in \overline{G} which are the non-adjacent vertices of u in G . The same is true for v . Hence, the pair (u, v) has $(n - 1 - i)$ -equi-neighbors in \overline{G} and there are $|N_e(G, i)|$ such pairs. Thus,

$$N_e[\overline{G}; x] = \sum_{i=0}^{n-1} |N_e(G, i)|x^{n-1-i} = x^{n-1}N_e\left[G; \frac{1}{x}\right].$$

This completes the proof. □

Theorem 2. *Let G be a graph with n vertices ($n \geq 3$). Then $\overline{G} \in [G]_{\mathcal{N}_E}$ iff $n(G, i) = n(G, n - 1 - i)$; $n(G, i) \geq 2$; $0 \leq i \leq n - 1$, where $n(G, i)$ represents the number of vertices of G with i neighbors.*

Proof. Assume that $\overline{G} \in [G]_{\mathcal{N}_E}$. From the definition of ENP it follows that $N_e[\overline{G}; x] = N_e[G; x]$. Therefore, $|N(G, i)| = |N_e(\overline{G}, i)|$ for $0 \leq i \leq n - 1$. This tells us that $n(G, i) = n(\overline{G}, i)$; for $n(G, i) \geq 2$; $0 \leq i \leq n - 1$. Observe that the necessary part of the proof follows from the fact that the sum of the degree of a vertex in G and in \overline{G} equals $(n - 1)$. Conversely, assume that $n(G, i) = n(G, n - 1 - i)$; $n(G, i) \geq 2$; $0 \leq i \leq n - 1$. This means that $n(G, i) = n(\overline{G}, i)$; $n(G, i) \geq 2$; $0 \leq i \leq n - 1$. It follows that $N_e[G; x] = N_e[\overline{G}; x]$. This completes the proof. \square

Lemma 3. *If G_1 and G_2 are two components of a graph G with n and m vertices respectively, satisfying $n \geq m$, then*

$$N_e[G; x] = N_e[G_1; x] + N_e[G_2; x] + \sum_{i=0}^{m-1} n(G_1, i)n(G_2, i)x^i,$$

where $n(G_k, i)$ represents the number of vertices of G_k with i neighbors, $k = 1, 2$.

Theorem 4. *For a graph G , $G + K_1 \in [G]_{\mathcal{N}_E}$ if and only if G has no isolated vertex.*

Proof. Assume that $G + K_1 \in [G]_{\mathcal{N}_E}$. This means that $N_e[G + K_1; x] = N_e[G; x]$. Note that

$$N_e[G + K_1; x] = N_e[G; x] + N_e[K_1; x] + n(G, 0) \text{ (lemma 3)}$$

This implies that $N_e[G + K_1; x] = N_e[G; x] + n(G, 0) \Rightarrow n(G, 0) = 0$. Conversely, assume that G has no isolated vertex. This means that $n(G, 0) = 0$. Now, from lemma 3 we have

$$N_e[G + K_1; x] = N_e[G; x] + N_e[K_1; x] = N_e[G; x]$$

Hence $G + K_1 \in [G]_{\mathcal{N}_E}$. This completes the proof. \square

Theorem 5. *Let G be a graph other than the null graph with n isolated vertices. Let H be any graph with m isolated vertices and $H \in [G]_{\mathcal{N}_E}$. Then the following results are satisfied.*

- (i) *If $n = 0$ or 1 , then $m = 0$ or 1 .*
- (ii) *If $n \geq 2$, then $m = n$.*

Proof. Here, we consider two cases:

Case(i): Let $n = 0$ or 1 . Then $|N_e(G, 0)| = 0$. Since $H \in [G]_{\mathcal{N}_E}$, then we have $|N_e(H, 0)| = 0$. Therefore, the number of isolated vertices in H equals either 0 or 1 .

Case(ii): Let $n \geq 2$. Since $H \in [G]_{\mathcal{N}_E}$, then $|N_e(H, 0)| = \binom{n}{2}$. Hence, the result follows from the property 7 of $N_e[G; x]$ [1].

This completes the proof. \square

3. ENP-UNIQUE AND ENP-CARDINAL UNIQUE GRAPHS

We now characterize specific families of graphs based on their ENP uniqueness properties.

Lemma 6. *A sequence S such that d_1, d_2, \dots, d_n of non negative integers with $d_1 \geq d_2 \geq \dots \geq d_n$ and $n \geq 2$, $d_1 \geq 1$ is graphical if and only if the sequence $d_2 - 1, d_3 - 1, \dots, d_{d_1+1}, d_{d_1+2}, \dots, d_n$ is graphical.*

Theorem 7. *The null graph is the only graph that is ENP unique.*

Proof. Let G be a null graph with n vertices. Then $N_e[G; x] = \binom{n}{2}$. Let H be any graph with m vertices such that $H \in [G]_{\mathcal{N}_E}$. Then obviously $m \geq n$. Also $N_e[H; x] = \binom{n}{2}$. It follows that the number of isolated vertices in H is equal to n . Let $V(H) = \{u_1, u_2, \dots, u_n, u_{n+1}, \dots, u_m\}$; where $|N(u_i)| = 0$ for $i = 1, 2, \dots, n$; $|N(u_i)| = p_i$ for

$i = n + 1, n + 2, \dots, m$. Note that $0 < p_i < m - 1$ and $p_i \neq p_j$ for $i, j \in \{n + 1, n + 2, \dots, m\}$; $i \neq j$. Let H_1 be the subgraph of H induced by u_1, u_2, \dots, u_n and H_2 be the subgraph of H induced by u_{n+1}, \dots, u_m . Then $H = H_1 + H_2$. Now the degree sequence of the graph H_2 is $p_{n+1}, p_{n+2}, \dots, p_m$; where $0 < p_i < m - 1$ and $p_i \neq p_j$ for $i, j \in \{n + 1, n + 2, \dots, m\}$; $i \neq j$. But this sequence is not graphical, by lemma 6. This means that H can contain only the n isolated vertices u_1, u_2, \dots, u_n . Thus $H \cong G$. This completes the proof. \square

3.1. Cycle graphs C_n .

Lemma 8. *For a cycle graph C_n , we have*

$$N_e[C_n; x] = \binom{n}{2} x^2, n \geq 3.$$

Theorem 9. *The cycle graph C_n is ENP cardinal unique for $n = 3, 4, 5$.*

Proof. Let H be any graph with n vertices such that $H \in [C_n]_{\mathcal{N}_E}$. Then by the lemma 8, $N_e[H; x] = \binom{n}{2} x^2$; $n = 3, 4, 5$. This means that all the n vertices in H have two neighbors. Then obviously $H \cong C_n$. This completes the proof. \square

Theorem 10. *The cycle graph C_n is not ENP cardinal unique for $n \geq 6$.*

Proof. Let H be the graph with n vertices; $n \geq 6$ consists of precisely a vertex disjoint m -cycle; $3 \leq m \leq n - 3$, and $(n - m)$ -cycle. Then C_n and H have the same degree sequence. Hence $H \in [C_n]_{\mathcal{N}_E}$. Observe that C_n and H are not isomorphic.

This completes the proof. \square

3.2. Complete graphs K_n .

Lemma 11. *For a complete graph K_n , we have*

$$N_e[K_n; x] = \binom{n}{2} x^{n-1}; n \geq 2.$$

Theorem 12. *For $n \geq 1$, the Complete graph K_n is ENP-cardinal unique.*

Proof. Let H be any graph with n vertices such that $H \in [K_n]_{\mathcal{N}_E}$; $n \geq 1$. Then by lemma 11 $N_e[H; x] = \binom{n}{2} x^{n-1}$. This means that all the n vertices in H have $(n - 1)$ neighbors. Then obviously $H \cong K_n$. This completes the proof. \square

3.3. The Star graph $K_{1,n}$.

Lemma 13. *For a star graph $K_{1,n}$, $N_e[K_{1,n}; x] = \begin{cases} \binom{n}{2} x, & n \neq 1 \\ x, & n = 1. \end{cases}$*

Theorem 14. *The star graph $K_{1,n}$; $n \geq 1$ is ENP-cardinal unique only if n is odd.*

Proof. Here, we consider the following two cases.

Case(i): Assume that n is odd. Let H be any graph with $n + 1$ vertices such that $H \in [K_{1,n}]_{\mathcal{N}_E}$; $n \geq 1$. Then by lemma 13 we have $N_e[H; x] = \binom{n}{2} x$. This means that the n vertices of H have only one neighbor. Then $H \cong K_{1,n}$.

Case(ii): Assume that n is even. Let H be a graph on $n + 1$ vertices that consists of precisely vertex disjoint, $\frac{n}{2}$ path graphs of length two and an isolated vertex. Then obviously $N_e[H; x] = \binom{n}{2} x$. Hence $H \in [K_{1,n}]_{\mathcal{N}_E}$. But $K_{1,n}$ and H are not isomorphic.

This completes the proof. \square

3.4. The Complete Bipartite graph $K_{m,n}$.

Lemma 15. *For a complete bipartite graph $K_{m,n}$; where $m, n \geq 2$, we have the following.*

$$N_e[K_{m,n}; x] = \begin{cases} \binom{m}{2}x^n + \binom{n}{2}x^m, & m \neq n, \\ n(2n-1)x^n, & m = n. \end{cases}$$

Theorem 16. *The complete bipartite graph $K_{m,n}$ is not ENP- cardinal unique; for every $m, n > 2$.*

Proof. Without loss of generality assume that $m \leq n$. First we assume that $m < n$.

Case(i): Assume that both m and n are odd.

Subcase(i): Let $m-2 > n-m$. Here we construct a graph H on $m+n$ vertices as follows. Let $V(H) = \{u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n\}$. Draw the edges of H as follows. Join u_1 with v_2 and v_i ; $i \in \{1, 2, \dots, n\}$; $i \neq 3$. Join u_j with u_{j+1} and v_i ; $i \in \{1, 2, \dots, n\}$; $i \neq j, n-(j-2)$ for $j = 2, 3, 4, 5, \dots, m-1$. Join u_m with v_i ; $i \in \{1, 2, \dots, n\}$; $i \neq m$. As a result of this the number of neighbors of u_i ; $n(u_i)$ equals n ; for $i = 1, 2, \dots, m$.

Also $n(v_1) = m, n(v_3) = m-2$ and

$$n(v_i) = \begin{cases} m-1, & i = 2, 4, 5, 6, \dots, n-m+2 \\ m-2, & i = n-m+3, n-m+4, \dots, m \\ m-1, & i = m+1, m+2, \dots, n. \end{cases}$$

It follows that the number of v_i 's with $(m-1)$ neighbors equals $2(n-m)$ and let us rename those v_i 's as $w_1, w_2, \dots, w_{2(n-m)}$. The number of v_i 's with $(m-2)$ neighbors equals $2m-n-1$ and let us rename those v_i 's as $l_1, l_2, \dots, l_{2m-n-1}$. Now complete $E(H)$ by including the edges $w_1l_1, l_1l_2, l_2l_3, \dots, l_{2m-n-2}l_{2m-n-1}, l_{2m-n-1}w_{2(n-m)}$ and $w_2w_3, w_4w_5, \dots, w_{2n-2m-2}w_{2n-2m-1}$. Thus the graph H has the same degree sequence as that of $K_{m,n}$. But H is not isomorphic to $K_{m,n}$, since H has the odd cycle $u_1u_2v_1u_1$.

Subcase(ii): Let $m-2 < n-m$.

Here we construct a graph H_1 on $m+n$ vertices as follows. Let $V(H_1) = \{a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n\}$. Draw the edges of H_1 as follows. Join a_1 with b_2 and b_i ; $i \in \{1, 2, \dots, n\}$; $i \neq 3$. Join a_j with a_{j+1} and b_i ; $i \in \{1, 2, \dots, n\}$; $i \neq j, n-(j-2)$ for $j = 2, 3, 4, 5, \dots, m-1$. Join a_m with v_i ; $i \in \{1, 2, \dots, n\}$; $i \neq m$. As a result of this the number of neighbors of a_i ; $n(a_i)$ equals n ; for $i = 1, 2, \dots, m$.

Also $n(b_1) = m, n(b_3) = m-2$ and

$$n(b_i) = \begin{cases} m-1, & i = 2, 4, 5, 6, \dots, m \\ m, & i = m+1, m+2, \dots, n-m+2 \\ m-1, & i = n-m+3, n-m+4, \dots, n. \end{cases}$$

It follows that the number of b_i 's with $(m-1)$ neighbors equals $2m-4$ and let us rename those b_i 's as $p_1, p_2, \dots, p_{2m-4}$. The number of b_i 's with $(m-2)$ neighbors equals one which is b_3 . Now complete $E(H_1)$ by including the edges $p_1b_3, b_3p_{2m-4}, p_2p_3, p_4p_5, \dots, p_{2m-6}p_{2m-5}$. Thus the graph H_1 has the same degree sequence as that of $K_{m,n}$. But H_1 is not isomorphic to $K_{m,n}$, since H_1 has the odd cycle $a_1a_2b_1a_1$.

Case(ii): Assume that both m and n are even.

Subcase(i): Let $m-2 < n-m$. The discussion under this case coincides with the subcase(ii) of case(i).

Subcase(ii): Let $m-2 > n-m$. Here we construct a graph F on $m+n$ vertices as follows. Let $V(F) = \{q_1, q_2, \dots, q_m, r_1, r_2, \dots, r_n\}$. Draw the edges of F as follows. Join q_1 with q_2 and r_i ; $i \in \{1, 2, \dots, n\}$; $i \neq 2$. Join q_j with q_{j+1} and r_i ; $i \in \{1, 2, \dots, n\}$;

$i \neq j - 1, j$ for $j = 2, 3, 4, 5, \dots, m - 1$. Join q_m with $r_i; i \in \{1, 2, \dots, n\}, i \neq m$. As a result of this the number of neighbors of $q_i; n(q_i)$ equals n ; for $i = 1, 2, \dots, m$ and $n(r_1) = m - 1, n(r_2) = m - 3$ and

$$n(r_i) = \begin{cases} m - 2, & i = 3, 4, 5, \dots, m - 2 \\ m - 1, & i = m - 1, m \\ m, & i = m + 1, m + 2, \dots, n. \end{cases}$$

It follows that the number of r_i 's with $(m - 1)$ neighbors equals 3 and let us rename those r_i 's as s_1, s_2 and s_3 . The number of r_i 's with $(m - 2)$ neighbors equals $m - 4$ and let us rename those r_i 's as t_1, t_2, \dots, t_{m-4} . Also the number of r_i 's with $(m - 3)$ neighbors equals one which is r_2 . Now complete $E(F)$ by including the edges $s_1r_2, r_2t_1, t_1t_2, t_2t_3, \dots, t_{m-5}t_{m-4}, t_{m-4}r_2$ and s_2s_3 . Thus the graph F has the same degree sequence as that of $K_{m,n}$. But F is not isomorphic to $K_{m,n}$, since F has the odd cycle $q_1q_2q_3\dots q_m r_1 q_1$.

Subcase(iii): Let $m - 2 = n - m$. Here we construct a graph H^* on $m + n$ vertices as follows. Let $V(H^*) = \{x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n\}$. Draw the edges of H^* as follows. Join x_1 with x_2 and $y_i; i \in \{1, 2, \dots, n\}; i \neq 3$. Join x_j with x_{j+1} and $y_i; i \in \{1, 2, \dots, n\}; i \neq j, n - (j - 2)$ for $j = 2, 3, 4, 5, \dots, m - 1$. Join x_m with $y_i; i \in \{1, 2, \dots, n\}; i \neq m$. As a result of this the number of neighbors of $x_i; n(x_i)$ equals n ; for $i = 1, 2, \dots, m$.

Then $n(y_1) = m, n(y_3) = m - 2$ and $n(y_i) = m - 1; i = 2, 4, 5, \dots, n$.

It follows that the number of y_i 's with $(m - 2)$ neighbors equals one, which is y_3 . The number of y_i 's with $(m - 1)$ neighbors equals $n - 2$ and let us rename those y_i 's as k_1, k_2, \dots, k_{n-2} . Now complete $E(H^*)$ by including the edges $k_1y_3, y_3k_{n-2}; k_2k_3, k_4k_5, \dots, k_{n-4}k_{n-3}$. Thus the graph H^* has the same degree sequence as that of $K_{m,n}$. But H^* is not isomorphic to $K_{m,n}$, since H^* has the odd cycle $x_1x_2y_1x_1$.

Case(iii): Assume that m is odd and n is even.

Subcase(i): Let $m - 2 > n - m$. The discussion under this case coincides with the subcase(ii) of case(ii).

Subcase(ii): Let $m - 2 < n - m$. The discussion under this case coincides with the subcase(ii) of case(i).

Subcase(iii): Let $m - 2 = n - m$. The discussion under this case coincides with the subcase(iii) of case(ii).

Case(iv): Assume that m is even and n is odd.

Subcase(i): Let $m - 2 > n - m$. The discussion under this case coincides with the subcase(i) of case(i).

Subcase(ii): Let $m - 2 < n - m$. The discussion under this case coincides with the subcase(ii) of case(ii).

Now we assume that $m = n$. Here we construct a graph G with $2n$ vertices as follows. Let $V(G) = V_1 \cup V_2$; where $V_1 = \{c_1, c_2, \dots, c_n\}; V_2 = \{d_1, d_2, \dots, d_n\}$. Draw complete graphs with the n vertices in V_1 and V_2 . Then complete $E(G)$ by drawing the edges $c_1d_1, c_2d_2, \dots, c_nd_n$. Thus the graph G has the same degree sequence as that of $K_{m,n}$. But G is not isomorphic to $K_{m,n}$, since G has the odd cycle $c_1c_2c_3c_1$. This completes the proof. □

3.5. The friendship graph F_n . A friendship graph F_n is the one-point union of n copies of the cycle C_3 .

Lemma 17. For $n \geq 2$, we have $N_e[F_n; x] = n(2n - 1)x^2 [1]$.

Theorem 18. Let F_n be the Friendship graph with P vertices. Then F_n is not ENP-cardinal unique; $n \geq 2$.

Proof. Since F_n is the Friendship graph, the number of vertices, $P = 2n + 1$. From lemma 17, we have $N_e[F_n; x] = n(2n - 1)x^2$. Let H be the graph on $2n + 1$ vertices that consists precisely of n numbers of path graph of length two and an isolated vertex. Hence $N_e[H; x] = N_e[F_n; x]$. But F_n and H is not isomorphic. This completes the proof. \square

Theorem 19. *The null graph on n vertices is ENP– cardinal unique; $n \geq 1$.*

Proof. Let G be a null graph with n vertices. Then $N_e[G; x] = \binom{n}{2}$. Let H be any graph with n vertices such that $H \in [G]_{\mathcal{N}_E}$. Then $N_e[H; x] = \binom{n}{2}$. It follows that the number of isolated vertices in H equals n . Hence $H \cong N_n$. This completes the proof. \square

4. EQUIVALENCE OF GRAPH OPERATIONS

This section investigates the ENP equivalence of graphs resulting from specific operations or structures.

Theorem 20. *If G is ENP– cardinal unique, then $G + N_n$ is also ENP– cardinal unique; where N_n is the null graph on n vertices; $n \geq 1$.*

Proof. Let the number of vertices in G be m .

Case(i): Assume that $n(G, 0) = 0$. From lemma 3, we have

$$(4.1) \quad N_e[G + N_n; x] = N_e[G; x] + N_e[N_n; x] = N_e[G; x] + \binom{n}{2}$$

Let H be any graph on $m + n$ vertices such that $H \in [G + N_n]_{\mathcal{N}_E}$. Then the number of isolated vertices in H is n , by Theorem 5. Now

$$(4.2) \quad N_e[H; x] = N_e[G + N_n; x] = N_e[G; x] + \binom{n}{2}, \text{ by lemma 3}$$

Let $V(H) = X \cup Y$; where $X = \{u \in V(H) : |N(u)| \geq 1\}$ and $Y = \{v_1, v_2, \dots, v_n\}$; where $v_i \in V(H)$ and $|N(v_i)| = 0$; $i = 1, 2, \dots, n$. Let H^* be a subgraph of H induced by X . Then obviously $H \cong H^* + N_n$. Now

$$(4.3) \quad N_e[H; x] = N_e[H^*; x] + N_e[N_n; x]; \text{ by lemma 3}$$

$$(4.4) \quad = N_e[H^*; x] + \binom{n}{2}$$

From (2) and (3) it follows that $N_e[G; x] = N_e[H^*; x]$. Also note that $|V(G)| = |V(H^*)| = m$. Since G is ENP– cardinal unique; $H^* \cong G$. Hence $H \cong G + N_n$.

Therefore $G + N_n$ is ENP– cardinal unique.

Case(ii): Assume that $n(G, 0) \geq 1$. Let $n(G, 0) = p$; $p \geq 1$. Also let $V(G) = A \cup B$; where $A = \{u \in V(H) : |N(u)| \geq 1\}$ and $B = \{w_1, w_2, \dots, w_p\}$; where $w_i \in V(G)$ and $|N(w_i)| = 0$; $i = 1, 2, \dots, p$. Let G^* be a subgraph of G induced by A . Then obviously $G \cong G^* + N_p$. Now

$$(4.5) \quad \begin{aligned} N_e[G; x] &= N_e[G^*; x] + N_e[N_p; x]; \text{ by lemma 3} \\ &= N_e[G^*; x] + \binom{p}{2} \end{aligned}$$

Let K be any graph on $m + n$ vertices such that $K \in [G + N_n]_{\mathcal{N}_E}$. Then the number of isolated vertices in K is $p + n$, by Theorem 5. Now

$$\begin{aligned}
 N_e[K; x] &= N_e[G + N_n; x] \\
 &= N_e[G^* + N_p + N_n; x]; \text{ since } G \cong G^* + N_p \\
 &= N_e[G^* + N_{p+n}; x] \\
 &= N_e[G^*; x] + N_e[N_{p+n}; x]; \text{ by lemma 3} \\
 (4.6) \quad &= N_e[G^*; x] + \binom{p+n}{2}
 \end{aligned}$$

Let $V(K) = A^* \cup B^*$ where $A^* = \{u \in V(K) : |N(u)| \geq 1\}$ and $B^* = \{m_1, m_2, \dots, m_{p+n}\}$; where $m_i \in V(K)$ and $|N(m_i)| = 0$; $i = 1, 2, \dots, p + n$. Let K^* be the subgraph of K induced by A^* . Then obviously $K = K^* + N_{p+n}$. Now

$$\begin{aligned}
 N_e[K; x] &= N_e[K^* + N_{p+n}; x] \\
 &= N_e[K^*; x] + N_e[N_{p+n}; x]; \text{ by lemma 3} \\
 (4.7) \quad &= N_e[K^*; x] + \binom{p+n}{2}
 \end{aligned}$$

From (5) and (6), we have $N_e[G^*; x] = N_e[K^*; x]$. Note that $|V(G^*)| = |V(K^*)|$. Also G^* is ENP– cardinal unique; by case(i). Therefore

$$\begin{aligned}
 G^* \cong K^* &\Rightarrow G^* + N_{p+n} \cong K^* + N_{p+n} \\
 &\Rightarrow G^* + N_p + N_n \cong K^* + N_{p+n} \\
 &\Rightarrow G + N_n \cong K.
 \end{aligned}$$

This completes the proof. □

4.1. Path graphs P_n .

Lemma 21. *For a path graph P_n , we have*

$$N_e[P_n; x] = x + \binom{n-2}{2}x^2; n \geq 2.$$

Theorem 22. *The path graph P_n ; for $n = 2, 4$ is ENP– cardinal unique.*

Proof. Let H be any graph with n vertices such that $H \in [P_n]_{\mathcal{N}_E}$; $n = 2, 4$. Then by lemma 21, $N_e[H; x] = x + \binom{n-2}{2}x^2$; $n = 2, 4$.

Case(i): Let $n = 2$. Then $N_e[H; x] = x$. This means that both the vertices in H are of degree 1. Then $H \cong P_2$.

Case(ii): Let $n = 4$. Then $N_e[H; x] = x + x^2$. This means that the two vertices in H are of degree 1 and the remaining two vertices in H are of degree 2. Then obviously $H \cong P_4$.

This ompletes the proof. □

Theorem 23. *The path graph P_n ; $n \in \{3, 5, 6, 7, 8, \dots\}$ is not ENP– cardinal unique.*

Proof. Let us consider the following two cases.

Case(i): Let $n = 3$. Then lemma 21, $N_e[P_3; x] = x$. Let H be a graph on 3 vertices that consists precisely of a path graph of length two and an isolated vertex. Then $N_e[H; x] = x$. But obviously H is not isomorphic to P_3 .

Case(ii): Let $n \geq 5$. From lemma 21., we have $N_e[P_n; x] = x + \binom{n-2}{2}x^2$; $n \geq 5$. Let H be a graph on n vertices that consists precisely of a path graph of length two and a cycle graph on $n - 2$ vertices; $n \geq 5$. Then

$$\begin{aligned}
 N_e[H; x] &= N_e[P_2 + C_{n-2}; x] \\
 &= N_e[P_2; x] + N_e[C_{n-2}; x]; \text{ by lemma 3} \\
 &= x + \binom{n-2}{2} x^2; \text{ by lemma 21, by lemma 8} \\
 &= N_e[P_n; x].
 \end{aligned}$$

But H is not isomorphic P_n . This completes the proof. □

Theorem 24. *If G_1 and G_2 are ENP- cardinal unique $G_1 + G_2$ need not be ENP- cardinal unique.*

Proof. Let $G_1 = K_4$ and $G_2 = C_3$. G_1 is ENP- cardinal unique by theorem 12 and G_2 is ENP- cardinal unique by theorem 9. Now consider the graph $G_1 + G_2$.

$$\begin{aligned}
 N_e[G_1 + G_2; x] &= N_e[K_4 + C_3; x] \\
 &= N_e[K_4; x] + N_e[C_3; x]; \text{ by lemma 3} \\
 &= 6x^3 + 3x^2; \text{ by lemma 11, by lemma 8}
 \end{aligned}$$

Let G be the graph on 7 vertices as shown in the figure. Then $N_e[G; x] = 6x^3 + 3x^2$. But clearly G is not isomorphic to $G_1 + G_2$. This completes the proof.

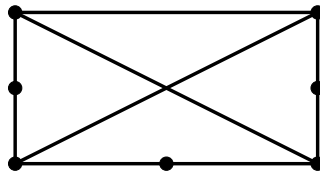


FIGURE 2. The graph G

4.2. The Lollipop graph $L_{m,n}$. The lollipop graph $L_{m,n}$ is a graph obtained by joining a complete graph K_m to a path P_n with a bridge. □

Lemma 25. *For a Lollipop graph $L_{m,n}$,*

$$N_e[L_{m,n}; x] = \begin{cases} \binom{m-1}{2} x^{m-1} + \binom{n-1}{2} x^2, & \text{if } m \geq 4 \\ \binom{n-1}{2} x^2 + x, & \text{if } m = 1 \\ \binom{n}{2} x^2 + x, & \text{if } m = 2 \\ \binom{n+1}{2} x^2, & \text{if } m = 3. \end{cases}$$

Theorem 26. $L_{n,1} \in [K_{n-1}]_{\mathcal{N}_E}$

Proof. The result follows from lemma 25 and lemma 11. □

4.3. The Wheel graph W_n . A Wheel graph $W_n, n > 3$ is obtained by taking the join of the cycle C_{n-1} and K_1 .

Lemma 27.

$$N_e[W_n; x] = \begin{cases} \binom{n-1}{2} x^3, & \text{if } n \neq 4 \\ 6x^3, & \text{if } n = 4. \end{cases}$$

Theorem 28. $[F_n]_{\mathcal{N}_E}$ contains both C_{2n} and W_{2n+1} .

Proof. The result follows from lemma 8 and lemma 27. □

A Tadpole $T_{(n,l)}$ is a graph obtained by attaching a path P_l to one of the vertices of the cycle C_n by a bridge.

Lemma 29. For a Tadpole graph $T_{(n,l)}$ with $n > 2$, we have the following.

$$N_e[T_{(n,l)}; x] = \begin{cases} N_e[C_n; x] + N_e[P_l; x] + n(l-2)x^2 - x, & \text{if } l > 1 \\ N_e[C_n; x] + N_e[P_l; x] - (n-1)x^2, & \text{if } l = 1. \end{cases}$$

Theorem 30. $T_{n,l} \in [C_{n+l-2}]_{\mathcal{N}_E}$.

Proof. The result follows from lemma 29 and lemma 8. □

Theorem 31. $[L_{1,n}]_{\mathcal{N}_E}$ contains P_{n+1} .

Proof. The result follows from lemma 25 and lemma 21. □

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