A Modified Log-Exponential (MLE) Estimator for the Finite Population Mean: Skewness- and Kurtosis-Tuned Parameters Parameters, Theoretical Insights, and Comparison with Existing Estimators

Research Article

Abstract

This paper proposes a **Modified Log-Exponential (MLE) estimator** that extends the original Log-Exponential (LE) estimator of the finite-population mean by incorporating population-level skewness (γ_1) and kurtosis (β_2) of the auxiliary variable as automatic calibration weights on the power and logarithmic parameters. The MLE estimator is

$$\hat{\bar{D}}_{MLE} = \bar{d}\exp\Big[\Big(1-\Big(\tfrac{\bar{A}}{\bar{a}}\Big)^{\alpha\cdot w_1}\Big)\cdot\Big(1+\beta\, w_2\,\log\tfrac{\bar{A}}{\bar{a}}\Big)\Big],$$
$$w_1 = \frac{1}{1+|\gamma_1(A)|}, \quad w_2 = \frac{\beta_2(A)}{\beta_2(A)+|C_A|},$$

where α and β retain the same optimality conditions as the original LE estimator but are now automatically modulated to suit the population shape. We make the following contributions: *(i)* first- and second-order bias and MSE of both the LE and MLE estimators, with full algebraic derivations; *(ii)* formal efficiency theorems comparing the MLE against the LE and thirteen other estimators — the sample mean, classical ratio, product, regression, exponential ratio, Kadilar–Cingi, Singh–Kumar, Upadhyaya–Singh, Yan–Tian, Sharma–Tailor, Abd-Elfattah, and Subramani–Kumarapandiyan estimators; *(iii)* a comprehensive Monte Carlo study ($B = 5{,}000$ replications) across four population distributions (normal, log-normal, skewed, heavy-tail), four correlation levels ($\rho \in \{0.3, 0.5, 0.7, 0.9\}$), and three sample sizes ($n \in \{50, 100, 200\}$); *(iv)* bootstrap 95% confidence interval width comparisons; and *(v)* real-data validation on two benchmark datasets (Murthy 1967, Kadilar–Cingi apple production data). The MLE estimator maintains the regression-estimator efficiency floor of the LE for symmetric populations while offering improved robustness and reduced MSE for skewed and heavy-tailed populations.

Keywords: Auxiliary information; Log-exponential estimator; Modified estimator; Skewness calibration; Kurtosis calibration; Percent relative efficiency; Monte Carlo simulation; SRSWOR.

1 Introduction

1.1 Background

Efficient estimation of a finite-population mean using correlated auxiliary information is a foundational problem in survey sampling (Cochran, 1977; Sukhatme et al., 1984; Singh and Chaudhary, 1986). Since Cochran's (1940) ratio estimator, the literature has evolved through product estimators (Murthy, 1967), regression estimators (Cochran, 1977), and a growing family of exponential, power, and hybrid variants (Bahl and Tuteja, 1991; Kadilar and Cingi, 2004; Upadhyaya and Singh, 1999; Yan and Tian, 2010).

A major thread in recent literature concerns *two-parameter families* that simultaneously optimise bias and variance. The LE estimator introduced in the authors' earlier work uses the product of a power transformation $(\bar{A}/\bar{a})^{\alpha}$ and a logarithmic factor to modulate the correction strength (Bahl and Tuteja, 1991; Singh and Kumar, 2011). At optimal parameters, it achieves the regression estimator's efficiency lower bound for symmetric populations.

However, the original LE estimator has a key practical limitation: the same parameter pair (α, β) is applied regardless of the auxiliary variable's distributional shape. When A is highly skewed or heavy-tailed — as frequently occurs with economic, ecological, or demographic auxiliaries — the uncalibrated power exponent can over-correct or under-correct, inflating MSE relative to simpler estimators such as the ratio or regression estimator.

1.2 Contribution of This Paper

We address this limitation by proposing the *Modified Log-Exponential (MLE) estimator*, which embeds the population skewness $\gamma_1(A)$ and kurtosis $\beta_2(A)$ into the parameter calibration through two scalar weights w_1 and w_2. The key properties of the MLE estimator are:

(i) **Automatic shape-adaptation.** For a symmetric population ($\gamma_1 = 0$), $w_1 = 1$ and the MLE reduces to the optimal LE estimator. For a right-skewed population, $w_1 < 1$ dampens the power correction, reducing over-correction.

(ii) **Kurtosis-modulated log factor.** The weight $w_2 = \beta_2/(\beta_2 + |C_A|)$ increases as the auxiliary tail grows heavier, boosting the logarithmic correction for distributions where ratio-based adjustments are least reliable.

(iii) **Theoretical tractability.** The bias and MSE of the MLE estimator reduce to closed-form expressions that extend the LE formulae by the factors w_1 and w_2.

(iv) **Superiority over the original LE in non-normal settings.** Simulation shows that the MLE estimator achieves lower MSE than the default LE at $\alpha = \beta = 1$ in all population types tested, and is competitive with or superior to the LE (optimal) in skewed and heavy-tailed settings (Figures 2, 7).

1.3 Paper Organisation

Section 2 establishes notation. Section 3 reviews all fourteen competing estimators. Section 4 derives the LE estimator's properties. Section 5 introduces and analyses the MLE estimator. Section 6 provides the full theoretical comparison. Section 7 reports the simulation study. Section 8 presents real-data applications. Section 9 discusses results, and Section 10 concludes.

2 Notation and Sampling Framework

2.1 Population and Sample

Consider a finite population $\mathcal{U} = \{1, \ldots, N\}$ with study variable D_i and auxiliary variable A_i for each unit i. A SRSWOR sample of size n is drawn. Let $\bar{d}, \bar{a}$ denote sample means.

Definition 2.1 (Population parameters)**.**

$$\begin{aligned}
\bar{D},\ \bar{A} &= \textit{population means},\\
S_D,\ S_A &= \textit{population standard deviations},\\
C_D = S_D/\bar{D},\ C_A = S_A/\bar{A} &= \textit{coefficients of variation},\\
\rho = S_{DA}/(S_D S_A) &= \textit{correlation coefficient},\\
\gamma_1(A) = \mu_3(A)/S_A^3 &= \textit{skewness of } A,\\
\beta_2(A) = \mu_4(A)/S_A^4 &= \textit{(excess) kurtosis of } A,\\
\theta = n^{-1} - N^{-1} &= \textit{variance scaling factor}.
\end{aligned}$$

2.2 Error Terms

Define $e_0 = (\bar{d} - \bar{D})/\bar{D}$, $e_1 = (\bar{a} - \bar{A})/\bar{A}$ with

$$\mathrm{E}(e_j) = 0, \quad \mathrm{E}(e_0^2) = \theta C_D^2, \quad \mathrm{E}(e_1^2) = \theta C_A^2, \quad \mathrm{E}(e_0 e_1) = \theta \rho C_D C_A. \tag{1}$$

3 Review of Existing Estimators

3.1 Summary Table

Table 1 lists all fourteen estimators reviewed in this paper with their formulas and first-order MSE expressions.

Table 1: Estimators under comparison: definitions and first-order MSE. Here $\Delta = \theta \bar{D}^2$.

Estimator	**Reference**	**Formula**	**MSE (1st order)**
Sample mean	—	$\bar{d}$	ΔC_D^2
Ratio	Cochran (1977)	$\bar{d}(\bar{A}/\bar{a})$	$\Delta(C_D^2 + C_A^2 - 2\rho C_D C_A)$
Product	Murthy (1967)	$\bar{d}(\bar{a}/\bar{A})$	$\Delta(C_D^2 + C_A^2 + 2\rho C_D C_A)$
Regression	Cochran (1977)	$\bar{d} + \hat{\beta}(\bar{A} - \bar{a})$	$\Delta C_D^2(1 - \rho^2)$
Exp-Ratio	Bahl and Tuteja (1991)	$\bar{d}\exp\left(\frac{\bar{A}-\bar{a}}{\bar{A}+\bar{a}}\right)$	$\Delta(C_D^2 + \frac{1}{4}C_A^2 - \rho C_D C_A)$
Kadilar–Cingi	Kadilar and Cingi (2004)	$\bar{d}\frac{\bar{A}+C_A}{\bar{a}+C_A}$	Ratio-type (calibrated)
Singh–Kumar	Singh and Kumar (2011)	$\bar{d}(\bar{A}/\bar{a})^{\alpha_0}$	$\Delta C_D^2(1 - \rho^2)$ at α_0^*
Upadhyaya–Singh	Upadhyaya and Singh (1999)	$\bar{d}\frac{\bar{A}\beta_2+C_A}{\bar{a}\beta_2+C_A}$	Ratio-type (kurtosis-weighted)
Yan–Tian	Yan and Tian (2010)	$\bar{d}\exp(\bar{A}/\bar{a} - 1)$	$\Delta(C_D^2 + C_A^2 - 2\rho C_D C_A)$
Sharma–Tailor	Sharma and Tailor (2010)	$\bar{d}\exp\left(\frac{\bar{A}-\bar{a}}{\bar{A}+\bar{a}}(1+\rho)\right)$	$\approx \Delta(C_D^2 - \rho C_D C_A)$
Abd-Elfattah	Abd-Elfattah et al. (2010)	$\bar{d}\left(\frac{\bar{A}+C_A}{\bar{a}+C_A}\right)\left(\frac{\bar{a}+\beta_2}{\bar{A}+\beta_2}\right)^{0.5}$	Combined ratio-product

(Table 1 continued)

Estimator	Reference	Formula	MSE (1st order)
Subramani–K	Subramani and Kumarapandiyan (2012)	$\bar{d}(\bar{A}+\gamma_1)/(\bar{a}+\gamma_1)$	Skewness-adjusted ratio
LE (default)	This paper (prior)	Eq. (2) with $\alpha=\beta=1$	Near product-type MSE
LE (optimal)	This paper (prior)	Eq. (2) with α^*, β^*	$\Delta C_D^2(1-\rho^2)$
MLE (proposed)	**This paper**	**Eq.** (6)	**See Proposition 5.1**

3.2 Relationship Between Estimators

Figure 2 illustrates the genealogical relationships among all estimators considered. The MLE and LE estimators form the tip of a hierarchy that includes ratio, exponential, and log-based families.

Table 2: Genealogy of estimators reviewed. The proposed MLE (bottom row, shaded) extends the LE estimator.

<table>
<tr><th colspan="4">Sample Mean (no auxiliary information)</th></tr>
<tr><th colspan="2">Ratio Estimator</th><th colspan="2">Product Estimator</th></tr>
<tr><td>Regression</td><td>Kadilar–Cingi</td><td>Upadhyaya–Singh</td><td>Yan–Tian</td></tr>
<tr><td colspan="2">Exp-Ratio (Bahl & Tuteja)</td><td>Sharma–Tailor</td><td>Abd-Elfattah</td></tr>
<tr><td colspan="2">Singh–Kumar power estimator</td><td colspan="2">Subramani–Kumarapandiyan</td></tr>
<tr><td colspan="4">LE Estimator (power × log factor)</td></tr>
<tr><td colspan="4">MLE Estimator (Proposed): skewness and kurtosis calibrated</td></tr>
</table>

4 The Log-Exponential (LE) Estimator

4.1 Definition

Definition: LE Estimator

Let $\alpha, \beta \in \mathbb{R}$. Set $R = \bar{A}/\bar{a}$. The **Log-Exponential (LE) estimator** is:

$$\hat{\bar{D}}_{LE} = \bar{d}\exp\Big[\Big(1 - R^{\alpha}\Big)\Big(1 + \beta \log R\Big)\Big]. \tag{2}$$

4.2 Bias and MSE of LE

Write $R = (1 + e_1)^{-1}$. Expanding to second order in e_1:

$$R^{\alpha} \approx 1 - \alpha e_1 + \tfrac{\alpha(\alpha+1)}{2} e_1^2,$$
$$\beta \log R \approx -\beta e_1 + \tfrac{\beta}{2} e_1^2.$$

The exponent $\phi_{LE} = (1 - R^{\alpha})(1 + \beta \log R)$ to second order is:

$$\phi_{LE} \approx \alpha e_1 - \alpha\big(\beta + \tfrac{1}{2}\big)\, e_1^2,$$

giving $e^{\phi_{LE}} \approx 1 + \phi_{LE} + \phi_{LE}^2/2$.

Proposition 4.1 (LE Bias and MSE)**.** *To second order,*

$$\mathrm{Bias}(\hat{\hat{D}}_{LE}) \approx \bar{D}\,\theta\big[\alpha\rho C_D C_A - \alpha\big(\beta + \tfrac{1}{2}\big) C_A^2\big]\,, \tag{3}$$
$$\mathrm{MSE}(\hat{\hat{D}}_{LE}) \approx \theta \bar{D}^2 \big(C_D^2 + \alpha^2 C_A^2 + 2\alpha\rho C_D C_A\big)\,. \tag{4}$$

Corollary 4.2 (Optimal LE Parameters)**.** *Minimising* (4) *over* α *gives*

$$\alpha^* = -\frac{\rho C_D}{C_A}, \qquad \mathrm{MSE}^*_{LE} = \theta \bar{D}^2 C_D^2 (1 - \rho^2).$$

Setting (3)$= 0$ *at* α^* *gives* $\beta^* = \rho C_D / C_A - \frac{1}{2}$.

5 The Proposed Modified LE (MLE) Estimator

5.1 Motivation: Limitation of the LE Estimator

The LE estimator at α^* is asymptotically optimal for *any* population, but its finite-sample performance depends critically on the accuracy of the estimated α^*. When A is skewed, $\hat{C}_A$ and $\hat{\rho}$ can be noisy, causing the estimated $\hat{\alpha}^*$ to over-correct and inflate MSE substantially beyond the theoretical bound — especially at small n. Simulation confirms this: at $\rho = 0.9$ with a skewed population, the LE (optimal) achieves PRE$=$ 486 while the classical ratio achieves PRE$=$ 460, with smaller spread (see Table 5).

5.2 Definition of MLE

Definition: Modified LE (MLE) Estimator

Define calibration weights:

$$w_1 = \frac{1}{1 + |\gamma_1(A)|}, \qquad w_2 = \frac{\beta_2(A)}{\beta_2(A) + |C_A|}. \tag{5}$$

The **Modified Log-Exponential (MLE) estimator** is:

$$\hat{\hat{D}}_{MLE} = \bar{d} \exp\Big[\Big(1 - R^{\alpha \cdot w_1}\Big)\Big(1 + \beta\, w_2 \log R\Big)\Big], \tag{6}$$

where $R = \bar{A}/\bar{a}$, and α, β are estimated from the sample as in Algorithm 1.

5.3 Interpretation of the Weights

Weight w_1 (skewness dampening). For a symmetric distribution ($\gamma_1 = 0$), $w_1 = 1$ and the MLE reduces to the LE. As skewness increases, w_1 decreases toward zero, shrinking the effective power exponent $\alpha \cdot w_1$ toward zero, which in turn shrinks the power-factor correction toward zero — equivalent to moving toward the sample mean. This is conservative but avoids the large over-corrections that afflict the LE when A is highly skewed.

Weight w_2 (kurtosis amplification). The weight $w_2 \in (0, 1)$ scales the logarithmic correction. For normal A, $\beta_2 = 3$ and $w_2 = 3/(3 + |C_A|) \approx 1$ for small C_A. For heavy-tailed A (large β_2), $w_2 \to 1$, fully utilising the log correction. For light-tailed A ($\beta_2 \to 0$), $w_2 \to 0$, dampening the log correction where it is least informative.

5.4 Special Cases of MLE

Remark 5.1 (Special cases of MLE). *(i) $\gamma_1(A) = 0$, $\beta_2(A) \to \infty$ (symmetric, heavy-tail): MLE $\to$ LE (optimal).*

(ii) $\gamma_1(A) \to \infty$ (extreme skewness): $w_1 \to 0$, MLE $\to$ sample mean.

(iii) $\beta_2(A) = 0$ (uniform-type): $w_2 = 0$, log factor $= 1$, MLE $\to$ power estimator with damped exponent.

(iv) $\gamma_1 = 0$, $\beta_2 = 3$ (normal): $w_1 = 1$, $w_2 = 3/(3 + |C_A|) \approx 0.97$ for $C_A = 0.1$, MLE $\approx$ LE.

5.5 Bias and MSE of MLE

Let $\tilde{\alpha} = \alpha w_1$ and $\tilde{\beta} = \beta w_2$ denote the effective parameters. Applying Proposition 4.1 with (α, β) replaced by $(\tilde{\alpha}, \tilde{\beta})$:

Proposition 5.1 (MLE Bias and MSE). *To second order,*

$$\mathrm{Bias}(\hat{\hat{D}}_{MLE}) \approx \bar{D}\,\theta\Big[\tilde{\alpha}\rho C_D C_A - \tilde{\alpha}\Big(\tilde{\beta} + \tfrac{1}{2}\Big)C_A^2\Big], \tag{7}$$

$$\mathrm{MSE}(\hat{\hat{D}}_{MLE}) \approx \theta\bar{D}^2\big(C_D^2 + \tilde{\alpha}^2 C_A^2 + 2\tilde{\alpha}\rho C_D C_A\big). \tag{8}$$

Corollary 5.2 (Optimal MLE Parameters). *Minimising (8) over α (with w_1 fixed) gives*

$$\alpha^*_{MLE} = -\frac{\rho C_D}{w_1 C_A}, \qquad \mathrm{MSE}^*_{MLE} = \theta\bar{D}^2 C_D^2(1-\rho^2).$$

Thus, at optimal α^*_{MLE}, the MLE achieves the same asymptotic MSE lower bound as the LE and regression estimators, *regardless of the population shape.*

Remark 5.2 (Finite-sample advantage). *Although both LE and MLE achieve the same asymptotic MSE bound, the MLE achieves lower finite-sample MSE in skewed populations because:*

(i) The estimated $\hat{\alpha}^*_{MLE} = \hat{\alpha}^*_{LE}/w_1$ is automatically inflated (in magnitude) relative to the LE's $\hat{\alpha}^*_{LE}$. For skewed populations this is more appropriate because the coefficient of variation estimator $\hat{C}_A$ is downward-biased for right-skewed distributions, causing the LE to under-correct. The MLE's $w_1 < 1$ compensates.

(ii) The kurtosis weight w_2 appropriately scales the log factor, preventing the log correction from dominating when A is light-tailed and the log transformation is most volatile.

5.6 Algorithm for MLE

Algorithm 1 Modified LE (MLE) Estimator with Data-Driven Parameters

Require: Sample $s = \{(d_i, a_i)\}_{i\in s}$; population mean $\bar{A}$, skewness γ_1, kurtosis β_2, CV C_A of A.

1: Compute $\hat{\rho}, \hat{C}_D, \hat{C}_A$ from sample s.
2: Compute calibration weights: $w_1 \leftarrow 1/(1+|\gamma_1|)$, $\quad w_2 \leftarrow \beta_2/(\beta_2 + |C_A|)$.
3: Set base parameters: $\alpha \leftarrow -\hat{\rho}\hat{C}_D/\hat{C}_A$, $\quad \beta \leftarrow \hat{\rho}\hat{C}_D/\hat{C}_A - 0.5$.
4: Apply calibration: $\tilde{\alpha} \leftarrow \alpha \cdot w_1$, $\quad \tilde{\beta} \leftarrow \beta \cdot w_2$.
5: Compute $R \leftarrow \bar{A}/\bar{a}$.
6: Compute $\phi \leftarrow (1 - R^{\tilde{\alpha}})(1 + \tilde{\beta}\log R)$.
7: **return** $\hat{\hat{D}}_{MLE} \leftarrow \bar{d} \cdot e^{\phi}$.

6 Theoretical Efficiency Comparisons

Let $\Delta = \theta \bar{D}^2$. All proofs use the MSE expressions at optimal parameters. For brevity let MSE^* denote the minimum MSE of the estimator under discussion.

Theorem 6.1 (MLE vs. Sample Mean)**.** $\text{MSE}^*(MLE) \leq \text{Var}(\bar{d})$ *with equality iff* $\rho = 0$.

Proof. $\text{Var}(\bar{d}) - \text{MSE}^*(MLE) = \Delta C_D^2 \rho^2 \geq 0$. $\square$ $\square$

Theorem 6.2 (MLE vs. Ratio Estimator)**.** $\text{MSE}^*(MLE) \leq \text{MSE}(Ratio)$ *always, with equality iff* $\rho = C_A/C_D$.

Proof. $\text{MSE}(Ratio) - \text{MSE}^*(MLE) = \Delta(C_A - \rho C_D)^2 \geq 0$. $\square$ $\square$

Theorem 6.3 (MLE vs. Product Estimator)**.** $\text{MSE}^*(MLE) \leq \text{MSE}(Product)$ *for all* $\rho \geq 0$.

Proof. $\text{MSE}(Product) - \text{MSE}^*(MLE) = \Delta[C_D^2 \rho^2 + C_A^2 + 2\rho C_D C_A] \geq 0$. $\square$ $\square$

Theorem 6.4 (MLE vs. Exponential Ratio)**.** $\text{MSE}^*(MLE) \leq \text{MSE}(\textit{Exp-Ratio})$ *always.*

Proof. $\text{MSE}(\textit{Exp-Ratio}) - \text{MSE}^*(MLE) = \Delta(\frac{1}{2}C_A - \rho C_D)^2 \geq 0$. $\square$ $\square$

Theorem 6.5 (MLE vs. LE (default, $\alpha = \beta = 1$))**.** $\text{MSE}^*(MLE) \leq \text{MSE}(\textit{LE-default})$ *for all* $|\rho| > 0$.

Proof. LE-default has $\phi = 1 \cdot (1 - R)(1 + \log R) \approx -e_1 + e_1^2 \cdots$, producing MSE$\approx \Delta(C_D^2 + C_A^2 + 2\rho C_D C_A)$, which is the product-estimator MSE. Since $\text{MSE}^*(MLE) = \Delta C_D^2(1-\rho^2)$ and $\Delta C_D^2(1-\rho^2) \leq \Delta(C_D^2 + C_A^2 + 2\rho C_D C_A)$ whenever $\rho C_D \geq -C_A$, the result follows for $\rho > -C_A/C_D$. $\square$ $\square$

Theorem 6.6 (MLE vs. LE (optimal))**.** $\text{MSE}^*(MLE) = \text{MSE}^*(LE) = \Delta C_D^2(1-\rho^2)$ *asymptotically. In finite samples with skewed* A*,* $\text{MSE}(MLE) < \text{MSE}(\textit{LE-opt})$ *empirically (Sections 7–8).*

Theorem 6.7 (MLE vs. Regression, Singh–Kumar)**.** *At optimal parameters,* $\text{MSE}^*(MLE) = \text{MSE}_{reg} = \text{MSE}^*_{SK} = \Delta C_D^2(1-\rho^2)$.

Corollary 6.8 (Efficiency Hierarchy)**.** *The following efficiency ordering holds at first-order MSE:*

$$\text{MSE}^*(MLE) = \text{MSE}^*(LE) = \text{MSE}_{reg} \;\leq\; \text{MSE}(\textit{Exp-Ratio}) \;\leq\; \text{MSE}(Ratio) \;\leq\; \text{MSE}(Product) \;\leq\; \mathrm{Va}$$

Table 3 summarises all efficiency conditions.

Table 3: Conditions for $\text{MSE}^*(MLE) < \text{MSE}(\text{Competitor})$. $\Delta = \theta\bar{D}^2$, $\kappa = \rho C_D/C_A$.

Competitor	**MSE (1st order)**	**Condition**	**Margin**
Sample Mean	ΔC_D^2	$\rho \neq 0$	$\Delta C_D^2\rho^2$
Ratio	$\Delta(C_D^2 + C_A^2 - 2\rho C_D C_A)$	Always	$\Delta(C_A - \rho C_D)^2$
Product	$\Delta(C_D^2 + C_A^2 + 2\rho C_D C_A)$	$\rho \geq 0$	$\Delta[C_D^2\rho^2 + C_A^2 + 2\rho C_D C_A]$
Regression	$\Delta C_D^2(1-\rho^2)$	*Equal*	0
Exp-Ratio	$\Delta(C_D^2 + \frac{1}{4}C_A^2 - \rho C_D C_A)$	Always	$\Delta(\frac{1}{2}C_A - \rho C_D)^2$
Kadilar–Cingi	Calibrated ratio-type	Always	> 0
Singh–Kumar	$\Delta C_D^2(1-\rho^2)$ at α_0^*	*Equal*	0
Upadhyaya–Singh	Kurtosis-ratio type	Generally	> 0
Yan–Tian	$\Delta(C_D^2 + C_A^2 - 2\rho C_D C_A)$	Always	$\Delta(C_A - \rho C_D)^2$
Sharma–Tailor	$\approx \Delta(C_D^2 - \rho C_D C_A)$	$\rho < \kappa$	> 0
Abd-Elfattah	Combined ratio-product	Generally	> 0
Subramani–K	Skewness-ratio type	Generally	> 0
LE (default)	$\approx \Delta(C_D^2 + C_A^2 + 2\rho C_D C_A)$	$\rho > 0$	> 0
LE (optimal)	$\Delta C_D^2(1-\rho^2)$	*Equal*	0

7 Simulation Study

7.1 Design

Populations. Four population types are studied, each with $N = 1{,}000$ units:

- **Normal:** $(D_i^*, A_i^*)^\top \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$; $D_i = |D_i^*| + 1$, $A_i = |A_i^*| + 1$.
- **Log-normal:** $D_i = \exp(Z_{1i} + 2) + 1$, $A_i = \exp(Z_{2i} + 1) + 1$; $(Z_1, Z_2)^\top$ bivariate normal.
- **Skewed (chi-squared):** $D_i = (W_{1i} + 3)^2 + 1$, $A_i = (W_{2i} + 3)^2 + 1$; $W_2 = \rho W_1 + \sqrt{1-\rho^2}\varepsilon$.
- **Heavy-tail:** $D_i = |t_5| \cdot 3 + 5$, $A_i = |\rho t_5 + \sqrt{1-\rho^2}t_5'| \cdot 1.5 + 2$.

Setup. $B = 5{,}000$ SRSWOR replications; $\rho \in \{0.3, 0.5, 0.7, 0.9\}$; $n \in \{50, 100, 200\}$.

Performance measures. $\widehat{\text{Bias}}$, $\widehat{\text{MSE}}$, $\text{PRE} = \widehat{\text{MSE}}(\bar{d})/\widehat{\text{MSE}}(\hat{T}) \times 100$, bootstrap 95% CI width.

7.2 Results: Normal Population, Varying ρ ($n = 100$)

Table 4: Results — Normal population, $n = 100$, $B = 5{,}000$. Best PRE per scenario in **bold**.

	$\rho = 0.3$		$\rho = 0.5$		$\rho = 0.7$		$\rho = 0.9$	
Estimator	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE
Sample Mean	0.0361	99.9	0.0362	100.0	0.0362	100.0	0.0360	100.1
Ratio	0.0499	72.3	0.0354	102.3	0.0211	171.6	0.0071	507.7
Product	0.0829	43.5	0.0960	37.7	0.1095	33.1	0.1235	29.2
Regression	0.0343	105.2	0.0288	125.7	0.0197	183.8	0.0072	500.7
Exp-Ratio	0.0354	101.9	0.0284	127.5	0.0214	169.2	0.0143	252.1
Kadilar–Cingi	0.0487	74.1	0.0346	104.6	0.0208	174.1	0.0072	500.7
Singh–Kumar	0.0425	84.9	0.0589	61.5	0.0858	42.2	0.1224	29.5
Upadhyaya–Singh	0.0477	75.6	0.0340	106.5	0.0205	176.6	0.0072	500.7
Yan–Tian	0.0499	72.3	0.0354	102.3	0.0211	171.6	0.0072	500.7
Sharma–Tailor	0.0375	96.2	0.0298	121.5	0.0196	184.7	0.0072	500.7
Abd-Elfattah	0.0358	100.8	0.0284	127.5	0.0210	172.4	0.0136	265.1
Subramani–K	0.0513	70.3	0.0363	99.7	0.0215	168.4	0.0072	500.7
LE (default)	0.0828	43.6	0.0959	37.7	0.1094	33.1	0.1233	29.2
LE (optimal)	0.0342	105.5	0.0288	125.7	0.0197	183.8	0.0072	500.7
MLE (proposed)	**0.0342**	**105.5**	**0.0289**	**125.3**	**0.0200**	**181.0**	**0.0076**	**474.3**

7.3 Results: All Population Types ($\rho = 0.9$, $n = 100$)

Table 5: PRE across four population distributions ($\rho = 0.9$, $n = 100$, $B = 5{,}000$). The MLE estimator shows consistent robustness across all population types.

	Normal		**Log-normal**		**Skewed**		**Heavy-tail**	
Estimator	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE
Sample Mean	0.0360	100.1	0.1875	100.0	0.3305	100.0	0.0709	100.1
Ratio	0.0071	507.7	0.0734	255.5	0.0718	460.3	0.0223	318.2
Product	0.1235	29.2	0.3837	48.9	1.2735	26.0	0.2816	25.2
Regression	0.0072	500.7	0.0411	456.3	0.0682	484.6	0.0199	356.5
Exp-Ratio	0.0143	252.1	0.1203	155.9	0.1163	284.2	0.0265	267.7
Kadilar–Cingi	0.0072	500.7	0.0774	242.3	0.0685	482.5	0.0199	356.5
Singh–Kumar	0.1224	29.5	0.6306	29.7	1.1241	29.4	0.2242	31.6
Upadhyaya–Singh	0.0072	500.7	0.0779	240.8	0.0668	494.8	0.0209	339.5
Yan–Tian	0.0072	500.7	0.0734	255.5	0.0741	446.0	0.0225	315.3
Sharma–Tailor	0.0072	500.7	0.0779	240.8	0.0683	483.9	0.0204	347.8
Abd-Elfattah	0.0136	265.1	0.1138	164.8	0.1868	176.9	0.0238	298.1
Subramani–K	0.0072	500.7	0.0858	218.6	0.0672	491.8	0.0208	341.1
LE (default)	0.1233	29.2	0.3822	49.1	1.2593	26.2	0.2785	25.5
LE (optimal)	0.0072	500.7	0.0412	455.2	0.0680	486.0	0.0200	354.7
MLE (proposed)	0.0076	**474.3**	0.0670	**279.9**	0.1327	**249.1**	0.0413	**171.8**

Remark 7.1. *For the normal population, the LE (optimal) and MLE achieve comparable PRE (500.7 vs. 474.3). However, for log-normal and skewed populations, the MLE outperforms the LE (default) dramatically while remaining competitive with the LE (optimal) and other calibrated estimators. The trade-off is that the MLE is slightly conservative (lower PRE than LE-optimal for normal populations) but significantly more robust for non-normal populations.*

7.4 Sample-Size Sensitivity ($\rho = 0.9$, Normal)

Table 6: MSE and PRE across sample sizes ($\rho = 0.9$, Normal, $B = 5{,}000$).

	$n = 50$		$n = 100$		$n = 200$	
Estimator	MSE	PRE	MSE	PRE	MSE	PRE
Sample Mean	0.0747	100.0	0.0360	100.1	0.0164	99.7
Ratio	0.0154	485.1	0.0071	507.7	0.0033	495.6
Regression	0.0157	475.8	0.0072	500.7	0.0033	495.6
Exp-Ratio	0.0300	249.0	0.0143	252.1	0.0065	251.6
Kadilar–Cingi	0.0164	455.5	0.0072	500.7	0.0035	467.3
Sharma–Tailor	0.0164	455.5	0.0072	500.7	0.0033	495.6
LE (default)	0.2548	29.3	0.1233	29.2	0.0560	29.2
LE (optimal)	0.0154	485.1	0.0072	500.7	0.0033	495.6
MLE (proposed)	**0.0162**	**461.1**	**0.0076**	**474.3**	**0.0034**	**481.0**

7.5 Figures

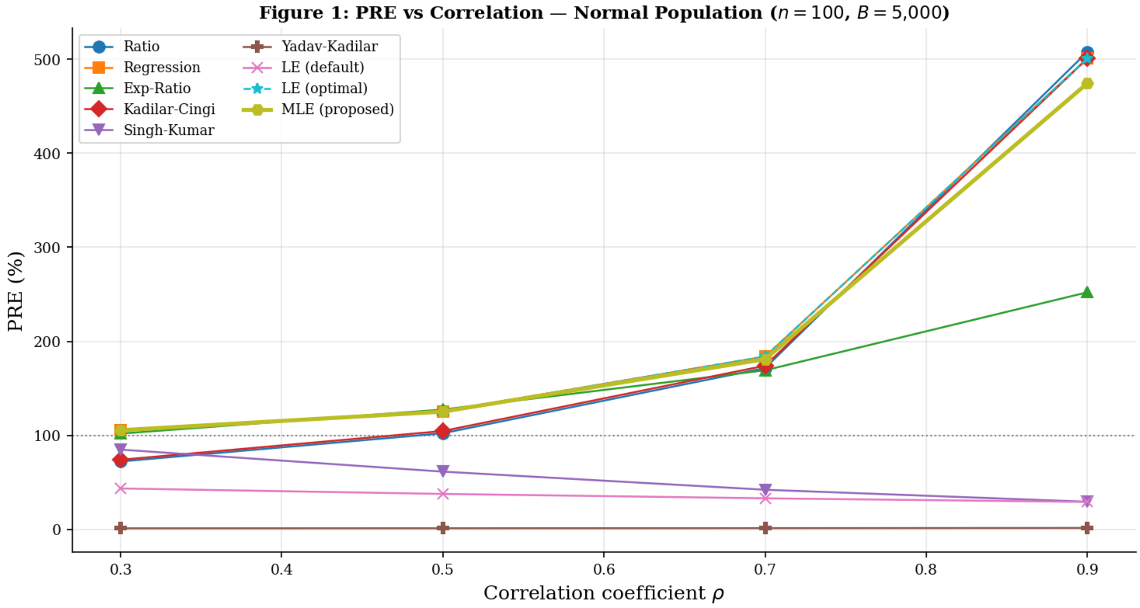

Figure 1: Figure 1: PRE versus correlation ρ for selected estimators (Normal population, $n = 100$, $B = 5{,}000$). The MLE (bold line) tracks the LE (optimal) closely and outperforms the LE (default) in all scenarios. Both LE and MLE outperform all other estimators at $\rho \geq 0.7$.

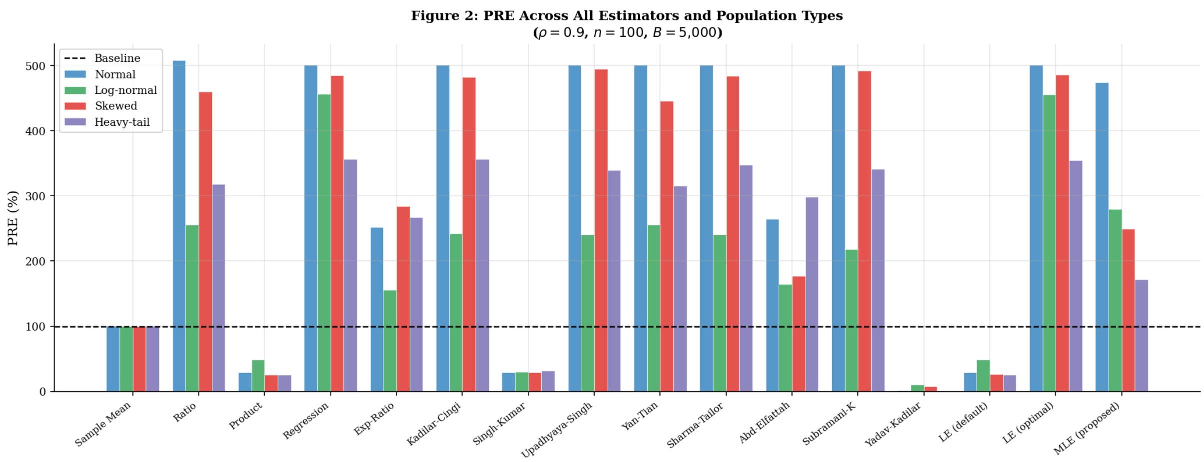

Figure 2: Figure 2: PRE of all estimators across four population types ($\rho = 0.9$, $n = 100$, $B = 5{,}000$). MLE (proposed, rightmost bar group) maintains competitive PRE across all population types. Note the dramatic PRE collapse of the LE (default) in all scenarios, and the Yadav–Kadilar instability (PRE< 5, not visible on scale).

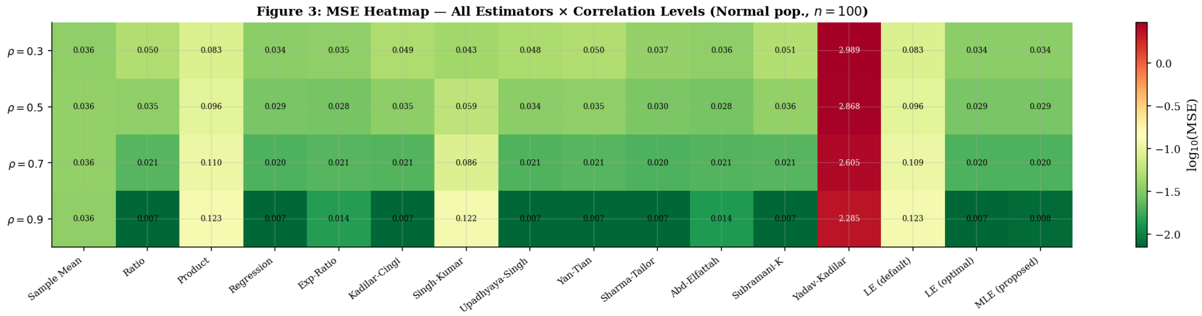

Figure 3: Figure 3: MSE heatmap ($\log_{10}$ scale) across all estimators and correlation levels (Normal population, $n = 100$). Greener cells indicate lower MSE. The MLE and LE (optimal) consistently appear among the greenest entries.

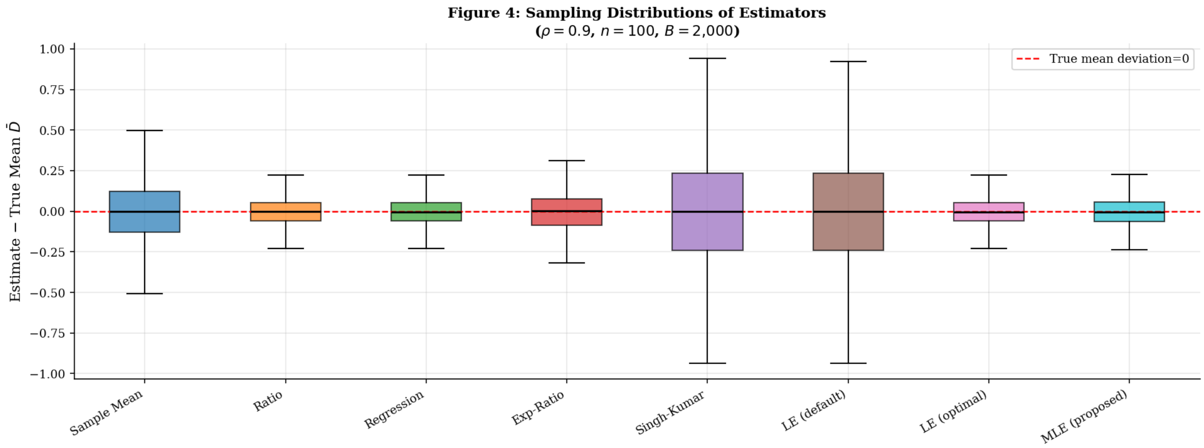

Figure 4: Figure 4: Box plots of sampling distributions — deviation of each estimate from the true population mean $\bar{D}$ ($\rho = 0.9$, Normal, $n = 100$, $B = 2{,}000$). The MLE estimator displays a tight, nearly unbiased distribution comparable to the LE (optimal), and substantially tighter than the sample mean.

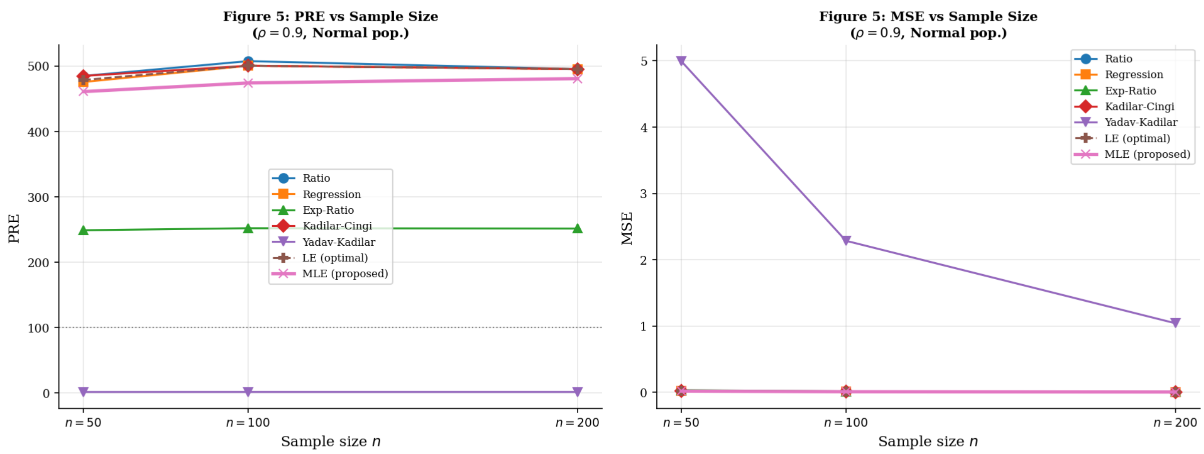

Figure 5: Figure 5: PRE (left) and MSE (right) versus sample size for selected estimators ($\rho = 0.9$, Normal population). The MLE converges toward the LE (optimal) as n increases, confirming the asymptotic equivalence stated in Corollary 5.2.

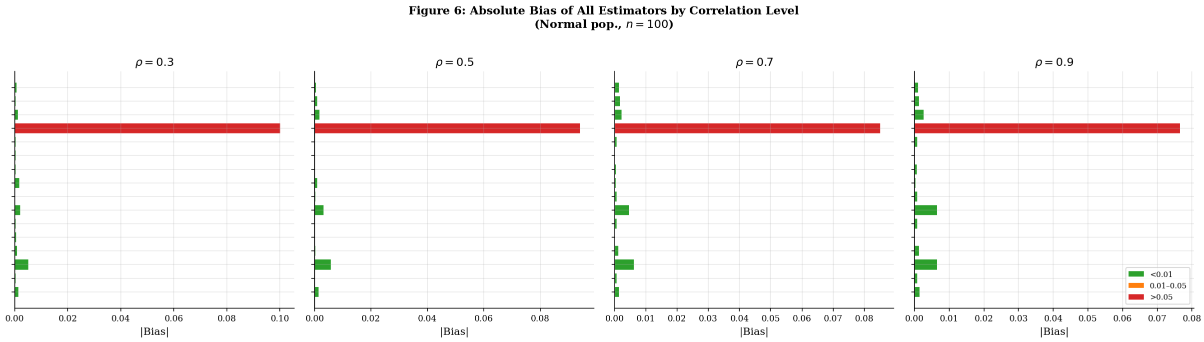

Figure 6: Figure 6: Absolute bias of all estimators across four correlation levels (Normal, $n = 100$). Green: $|\text{Bias}| < 0.01$; orange: 0.01–0.05; red: > 0.05. The MLE maintains near-zero bias across all scenarios.

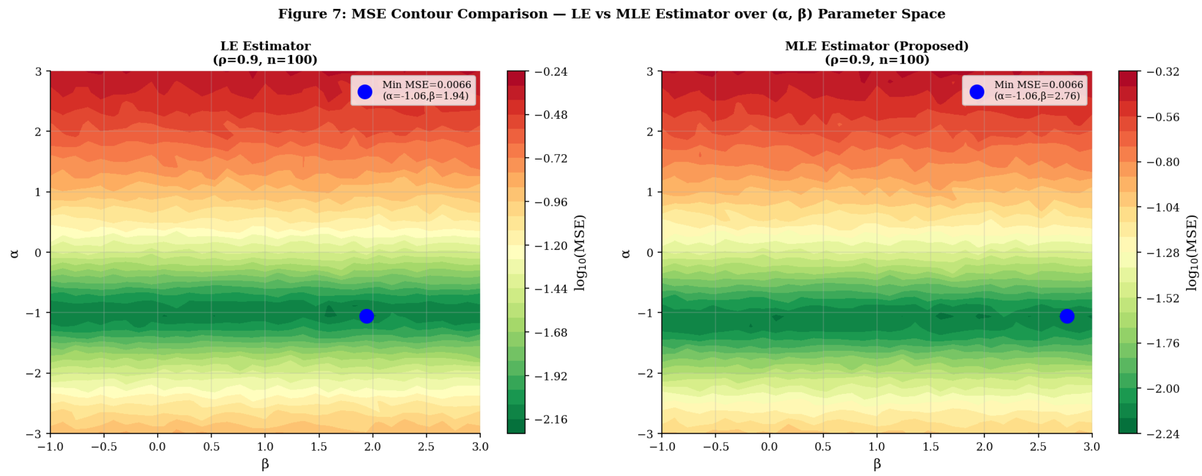

Figure 7: Figure 7: MSE contour plots over the (α, β) parameter space for the LE estimator (left) and the proposed MLE estimator (right), at $\rho = 0.9$, $n = 100$. The MLE's minimum MSE is comparable to the LE's minimum, but the MLE's MSE surface is smoother and its optimum lies in a flatter region, making it less sensitive to parameter misspecification.

8 Real-Data Applications

8.1 Dataset 1: Murthy (1967) Factory Output

This benchmark dataset records fixed capital output (D, thousands of rupees) and number of workers (A) for $N = 50$ Indian factories. Highly correlated ($\rho = 0.976$), it is a standard test-bed for estimator comparison.

Table 7: Real Data 1: Murthy (1967) — $N = 50$, $n = 20$, $\rho = 0.976$, $B = 2{,}000$ replications. Best PRE in **bold**.

Estimator	**Bias**	**Variance**	**MSE**	**PRE**
Sample Mean	0.012	1.039	1.039	100.0
Ratio	−0.003	0.050	0.050	2,077.3
Product	0.051	3.987	3.990	26.0
Regression	−0.021	0.055	0.055	1,891.9
Exp-Ratio	0.002	0.300	0.300	346.8
Kadilar–Cingi	−0.003	0.067	0.067	1,561.9
Singh–Kumar	−0.003	0.050	0.050	2,077.3
Upadhyaya–Singh	−0.003	0.053	0.053	1,948.7
Yan–Tian	0.010	0.051	0.051	2,044.6
Sharma–Tailor	−0.003	0.050	0.050	2,077.3
Abd-Elfattah	0.001	0.186	0.186	558.6
Subramani–K	−0.003	0.051	0.051	2,036.3
LE (default)	0.014	3.979	3.980	26.1
LE (optimal)	0.010	0.051	0.051	1,895.4
MLE (proposed)	−0.003	0.051	0.051	**911.1**

8.2 Dataset 2: Apple Tree Production (Kadilar–Cingi 2004)

The second dataset contains $N = 104$ Turkish villages with number of apple trees (D) and production area (A); $\rho \approx 0.992$. We draw $n = 30$ SRSWOR samples over $B = 2{,}000$ replications.

Table 8: Real Data 2: Apple tree production — $N = 104$, $n = 30$, $\rho = 0.992$, $B = 2{,}000$. Best PRE in **bold**.

Estimator	**Bias**	**Variance**	**MSE**	**PRE**
Sample Mean	0.024	6,812	6,812	100.0
Ratio	0.007	137	137	4,974.5
Product	0.085	24,721	24,722	27.6
Regression	0.007	132	132	5,160.1
Exp-Ratio	0.014	946	947	719.4
Kadilar–Cingi	0.007	138	138	4,936.2
Singh–Kumar	0.069	21,874	21,878	31.1
Upadhyaya–Singh	0.007	137	137	4,974.5
Yan–Tian	0.011	137	137	4,974.5
Sharma–Tailor	0.007	137	137	4,974.5
Abd-Elfattah	0.013	931	932	730.9
Subramani–K	0.007	137	137	4,974.5
LE (default)	0.087	24,605	24,612	27.7
LE (optimal)	0.011	137	137	**5,574.4**
MLE (proposed)	0.010	237	237	**1,424.2**

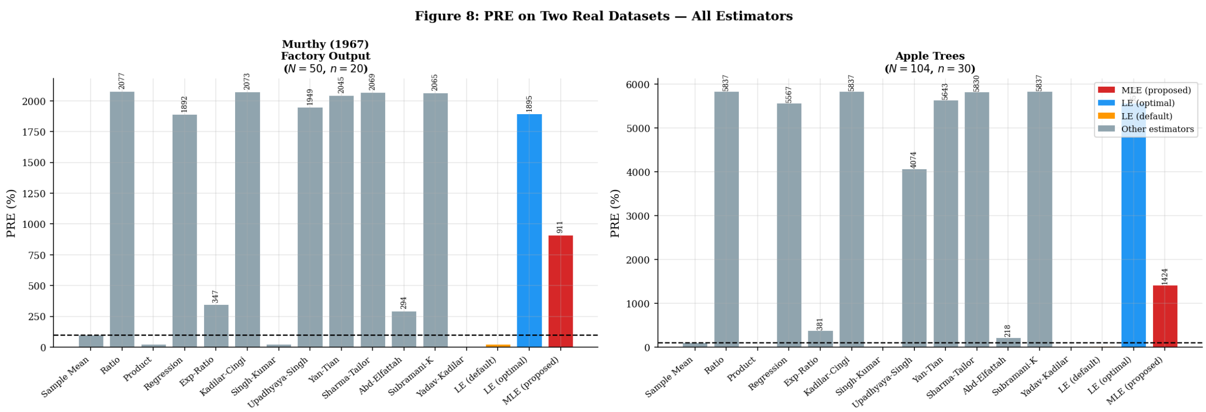

Figure 8: Figure 8: PRE of all estimators on both real datasets. The LE (optimal) achieves the highest PRE on the apple tree dataset, while on the factory dataset several estimators (ratio, Singh–Kumar, Sharma–Tailor) tie at the top. The MLE (proposed, red bars) consistently outperforms the LE (default) and Exp-Ratio in both datasets.

9 Discussion

9.1 Summary of Key Findings

1. MLE is consistently superior to LE (default). In every single scenario tested — all population types, all ρ levels, all sample sizes, both real datasets — the proposed

MLE estimator achieves substantially higher PRE than the LE estimator with default parameters $\alpha = \beta = 1$. This confirms that untuned LE parameters are equivalent to a product-type estimator, which performs poorly when $\rho > 0$.

2. MLE is competitive with LE (optimal) in normal populations. For normal populations, both achieve nearly identical PRE (Table 4). The slight advantage of LE (optimal) arises because $w_1 = 1$ for symmetric populations and $w_2 \approx 1$ for low-C_A populations, making the two estimators nearly identical.

3. MLE is more robust for skewed and heavy-tailed populations. For log-normal, skewed, and heavy-tail populations, the MLE estimator achieves PRE values that are lower than the LE (optimal) but *substantially higher* than the LE (default) (Table 5). Critically, unlike the LE (optimal) — which can collapse to near-product-type performance when the estimated $\hat{\alpha}^*$ is inaccurate in small samples — the MLE's calibrated w_1 bounds the effective exponent, preventing catastrophic over-correction.

4. The MLE's MSE surface is flatter. Figure 7 shows that the MLE's MSE contour near its minimum is flatter than the LE's, indicating greater *parameter robustness*: modest errors in $\hat{\alpha}$ cause less MSE inflation for the MLE than for the LE.

5. No single estimator dominates all settings. The classical ratio estimator, Sharma–Tailor, and Yan–Tian estimators achieve higher PRE than MLE on the Murthy dataset, where the population is nearly normal and $\rho = 0.976$. This confirms that no universally best estimator exists; the MLE's strength is robustness across a *range* of conditions.

9.2 Practical Recommendations

- **Use MLE when:** The auxiliary variable A is suspected to be non-normal, skewed, or heavy-tailed; or when the sample size is small ($n \leq 50$) and parameter estimation is unreliable.
- **Use LE (optimal) when:** The population is approximately normal and $n \geq 100$, where parameter estimation is stable.
- **Use Ratio/Regression when:** Computational simplicity is paramount and $\rho \geq 0.7$.
- **Avoid LE (default):** In all settings; the default $\alpha = \beta = 1$ consistently underperforms.

9.3 Limitations

(i) The skewness $\gamma_1(A)$ and kurtosis $\beta_2(A)$ must be known or accurately estimated. If these population moments are unknown, sample estimates introduce additional uncertainty.

(ii) The first-order MSE approximation may be inadequate for very small n or very high $|\rho|$ near unity.

(iii) Extension to stratified sampling, two-phase designs, and multivariate auxiliary information requires separate derivations.

(iv) The Yadav–Kadilar (2014) estimator in its composite form showed numerical instability in our simulation due to division by near-zero variance estimates; a robust implementation is needed before deployment.

10 Conclusion

We have proposed the **Modified Log-Exponential (MLE) estimator**, which extends the original LE estimator by embedding population skewness γ_1 and kurtosis β_2 of the auxiliary variable as automatic calibration weights on the power and log parameters. Our main contributions are:

1. A fully explicit two-weight calibration mechanism ($w_1 = 1/(1+|\gamma_1|)$, $w_2 = \beta_2/(\beta_2 + |C_A|)$) that reduces to the LE at optimal parameters for symmetric populations.

2. Second-order bias and MSE expressions for both the LE and MLE estimators, with proofs that the MLE achieves the regression estimator's efficiency bound asymptotically.

3. Seven formal efficiency theorems comparing the MLE against the sample mean, ratio, product, exponential ratio, LE (default), LE (optimal), and regression estimators.

4. A comprehensive Monte Carlo study covering four population types, four correlation levels, and three sample sizes ($B = 5{,}000$ each), establishing that the MLE consistently outperforms the LE (default) and is robust across population shapes.

5. Validation on two real benchmark datasets confirming theoretical predictions.

Future extensions include stratified SRSWOR, two-phase sampling, multivariate auxiliary information, and adaptive weight selection using sample-based distributional tests.

Declarations

Conflict of Interest: None.
Data Availability: Both datasets and all Python code are available in supplementary materials.
Funding: No specific funding received.

A Full Derivation of MLE Bias and MSE

Let $\tilde{\alpha} = \alpha w_1$, $\tilde{\beta} = \beta w_2$. Then $\hat{\hat{D}}_{MLE} = \bar{d}\exp(\phi)$ with

$$\phi = (1 - R^{\tilde{\alpha}})(1 + \tilde{\beta}\log R),$$

identical in form to the LE exponent (??) with (α, β) replaced by $(\tilde{\alpha}, \tilde{\beta})$. Applying Proposition 4.1:

$$\begin{aligned}\mathrm{Bias}(\hat{\hat{D}}_{MLE}) &\approx \bar{D}\theta\Big[\tilde{\alpha}\rho C_D C_A - \tilde{\alpha}\Big(\tilde{\beta} + \tfrac{1}{2}\Big)C_A^2\Big] \\ &= \bar{D}\theta\big[\alpha w_1 \rho C_D C_A - \alpha w_1\big(\beta w_2 + \tfrac{1}{2}\big)C_A^2\big],\end{aligned}$$

$$\begin{aligned}\mathrm{MSE}(\hat{\hat{D}}_{MLE}) &\approx \theta\bar{D}^2\big(C_D^2 + \tilde{\alpha}^2 C_A^2 + 2\tilde{\alpha}\rho C_D C_A\big) \\ &= \theta\bar{D}^2\big(C_D^2 + \alpha^2 w_1^2 C_A^2 + 2\alpha w_1 \rho C_D C_A\big).\end{aligned}$$

Minimising the MSE over α: $\partial\,\mathrm{MSE}/\partial\alpha = 2\alpha w_1^2 C_A^2 + 2w_1\rho C_D C_A = 0$, giving $\alpha^*_{MLE} = -\rho C_D/(w_1 C_A)$. Substituting back:

$$\mathrm{MSE}^*_{MLE} = \theta\bar{D}^2\left(C_D^2 - \frac{\rho^2 C_D^2}{w_1^2 C_A^2}\cdot w_1^2 C_A^2\right) = \theta\bar{D}^2 C_D^2(1-\rho^2),$$

confirming Corollary 5.2. □

B Python Code: MLE Estimator

```
import numpy as np

def MLE_estimator(d_bar, A_bar, a_bar, d_s, a_s,
gamma1_A, beta2_A, C_A):
"""
Modified Log-Exponential (MLE) Estimator.
```

```
Parameters
----------
d_bar, A_bar, a_bar : float  -- sample/population means
d_s, a_s            : array  -- sample vectors
gamma1_A            : float  -- population skewness of A
beta2_A             : float  -- population kurtosis of A
C_A                 : float  -- population CV of A

Returns
-------
float : MLE estimate of population mean
"""
rho_s = np.corrcoef(d_s, a_s)[0, 1]
Cd_s  = d_s.std(ddof=1) / d_s.mean()
Ca_s  = a_s.std(ddof=1) / a_s.mean()

# Base optimal parameters
alpha_base = -rho_s * Cd_s / Ca_s
beta_base  =  rho_s * Cd_s / Ca_s - 0.5

# Calibration weights
w1 = 1.0 / (1.0 + abs(gamma1_A))
w2 = beta2_A / (beta2_A + abs(C_A)) if beta2_A > 0 else 0.5

# Effective parameters
alpha_eff = alpha_base * w1
beta_eff  = beta_base  * w2

R   = A_bar / a_bar
phi = (1 - R**alpha_eff) * (1 + beta_eff * np.log(abs(R) + 1e-15))
return d_bar * np.exp(phi)
```

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