Spectral Analysis of Compressed Zero Divisor Graphs over $\prod_{k=1}^{n} \mathbb{Z}_{p_k}$ for $2 \leq n \leq 5$, where each p_k is a prime

Original Research Article

Abstract

This paper investigates the spectral characteristics and energy parameters of compressed zero-divisor graphs corresponding to product rings of the form $\prod_{k=1}^{n} \mathbb{Z}_{p_k}$ for $2 \leq n \leq 5$, where each p_k is a prime. The eigenvalue spectrum,determinant,trace,spectral radius,and energy indices of the Adjacency, Laplacian, and Seidel matrices are computed and compared.
For $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$, all three Adjacency, Laplacian, and Seidel energies were equal to 2, reflecting spectral symmetry in the simplest case. As the product expanded, a notable escalation in the energies was observed.
The spectral radius ρ showed a parallel growth, indicating increasing graph complexity with higher product. Across all product rings, the Laplacian and Seidel energies consistently exceeded the adjacency energy, showing that larger ring structures lead to greater spectral variation and stronger graph connectivity. These findings provide a unified perspective on structural connectivity, regularity, and algebraic symmetry within compressive graphs of product rings and serve as a foundation for further research involving larger annihilator classes and other ring classes.

Keywords: Compressed Zero-Divisor Graph; Product Rings; Adjacency Matrix; Laplacian Matrix; Seidal Matrix; Spectral Properties.

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1 Introduction

The study of algebraic structures through graph-theoretic approaches has emerged as a significant area of modern mathematical research. One such construction that bridges ring theory and graph theory is the zero-divisor graph, originally introduced to visualize the relationships between zero-divisors in a commutative ring with unity. The concept of zero-divisor graphs for commutative rings was first introduced by I. Beck(1). Mulay (5) illustrated a method for constructing a compressed zero-divisor graph $\Gamma_c(R)$ from the ring R by merging zero divisors that share the same annihilator ideal into a single equivalence class. In recent years, researchers(6; 7; 8; 9) have extended this concept to the compressed zero-divisor graph, which simplifies the classical zero-divisor graph by identifying elements with identical annihilators. The compressed zero-divisor graphs corresponding to product rings illustrate the interactions among annihilator classes and reveal complicated connectivity properties that depend on the number of factors. As the number of factors increases, the complexity of the graphs grows exponentially, giving rise to various spectral properties. Understanding these patterns through matrix representations provides valuable insight into both the algebraic and graph-theoretic aspects of the structure. The present study focuses on the adjacency, Laplacian, and Seidel matrices associated with the compressed zero-divisor graphs of

$$\prod_{k=1}^{n} \mathbb{Z}_{p_k}$$

for $2 \leq n \leq 5$. By analyzing their eigenvalue spectra, we computed and compared the graph energies corresponding to each matrix type. This approach not only characterizes the spectral distribution of the graphs but also highlights the influence of ring composition on graph energy.

The spectral characteristics of the adjacency, Laplacian, and Seidel matrices defined on the graphs remain a fascinating and challenging area of research. Since determining all the roots of a characteristic polynomial becomes increasingly difficult as the matrix order increases, we examine the adjacency spectrum of compressed zero-divisor graphs of product rings built over $\mathbb{Z}_{p_k}$, where each p_k is prime, for several values $2 \leq n \leq 5$. The concept of zero-divisor graphs and their adjacency eigenvalues were first examined by Young M. (13), while subsequent studies have focused on the Laplacian and Seidel spectral properties in (12; 14; 15; 16; 17; 18; 19). The notion of graph energy was first formalized by Gutman in his seminal work 'The Energy of a Graph' (11). Initially, the idea received limited attention from mathematicians. However, in the following decades, the study of graph energy expanded significantly, leading to various extensions and modifications. In 2006, Gutman and Zhou introduced the concept of Laplacian energy, defined as the sum of the absolute deviations of Laplacian eigenvalues from their average value (14).

Furthermore, the comparative study of these matrices offers a unified perspective on structural connectivity, regularity, and algebraic symmetry within compressed zero-divisor graphs. The results obtained in this work are expected to contribute to the ongoing development of spectral graph theory in algebraic contexts and serve as a foundation for further research involving larger product rings and other ring classes.

2 Preliminaries

Let Γ_c be the compressed zero-divisor graph associated with a commutative ring. In this section, we present the definitions of various matrices and related spectral indices that will be used throughout the paper.

2.1 Adjacency Matrix

The adjacency matrix of Γ_c, denoted by $A(\Gamma_c) = [a_{ij}]$, is defined as

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}$$

This matrix encodes the edge connections between the vertices of Γ_c.

2.2 Degree Matrix

The degree matrix of Γ_c, denoted by $D(\Gamma_c)$, is a diagonal matrix whose entries are given by

$$d_{ii} = \deg(v_i),$$

where $\deg(v_i)$ denotes the degree of the vertex v_i.

2.3 Laplacian Matrix

For the compressed zero-divisor graph, the Laplacian matrix is given by

$$L(\Gamma_c) = D(\Gamma_c) - A(\Gamma_c).$$

The Laplacian matrix captures both the adjacency and degree information of the graph and plays a significant role in various spectral characterizations.

2.4 Seidel Matrix

The Seidel matrix of Γ_c is defined as

$$S(\Gamma_c) = J - I - 2A(\Gamma_c),$$

where J is the all-ones matrix and I is the identity matrix. The Seidel matrix provides an alternative representation of the graph structure, emphasizing both adjacency and non-adjacency relations.

These matrices, adjacency, Laplacian, and Seidel, capture distinct struc- tural properties of the graph and are fundamental in spectral graph theory.

2.5 Adjacency Energy

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the adjacency matrix $A(\Gamma_c)$. The adjacency energy of the compressed zero-divisor graph is defined as

$$E_A(\Gamma_c) = \sum_{i=1}^{n} |\lambda_i|$$

2.6 Laplacian Energy

If $\mu_1, \mu_2, \ldots, \mu_m$ are the eigenvalues of the Laplacian matrix $L(\Gamma_c)$, then the Laplacian energy is given by

$$E_L(\Gamma_c) = \sum_{i=1}^{m} \left| \mu_i - \frac{2|E|}{m} \right|,$$

where $|E|$ denotes the number of edges in Γ_c.

2.7 Siedel Energy

For the Seidel matrix $S(\Gamma_c)$ with eigenvalues $\sigma_1, \sigma_2, \ldots, \sigma_m$, the Seidel energy is defined as

$$E_S(\Gamma_c) = \sum_{i=1}^{m} |\sigma_i|.$$

These energy indices measure the overall dispersion of eigenvalues and serve as important spectral invariants. They establish connections between the algebraic structure of the underlying ring and the spectral characteristics of its compressed zero-divisor graph.

3 Main results and observations

In the main results, the primary findings are obtained from the study of the adjacency, Laplacian, and Seidel energies of compressed zero-divisor graphs associated with product rings.

The results highlight the spectral characteristics of the corresponding compressed zero-divisor graphs. The following figures illustrate the compressed zero-divisor graphs of the considered product rings

$$R = \prod_{k=1}^{n} \mathbb{Z}_{p_k}, \quad \text{where } 2 \leq n \leq 5.$$

3.1 Theorem

Let $R = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$ with p_1, p_2 primes. For the compressed zero-divisor graph $\Gamma_c(R)$ (the graph consisting of two vertices joined by a single edge) the adjacency, Laplacian and Seidel energies are equal and satisfy

$$E_A(\Gamma_c(R)) = E_L(\Gamma_c(R)) = E_S(\Gamma_c(R)) = 2.$$

Proof:
For case $n = 2$,

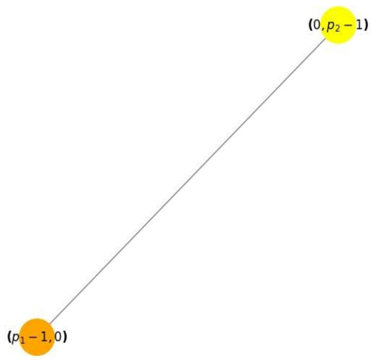

Figure 1: $\Gamma_c(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$

The adjacency matrix is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Its characteristic polynomial is $\det(A - \lambda I) = \lambda^2 - 1$, so the eigenvalues of A are 1 and -1. Hence the adjacency energy, the sum of absolute eigenvalues, equals

$$E_A = |1| + |-1| = 2.$$

The degree matrix is $D = \text{diag}(1, 1)$, therefore the Laplacian matrix equals

$$L = D - A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Computing $\det(L-\mu I)$ yields $\mu(\mu-2)$, so the Laplacian eigenvalues are 0 and 2. With $m=2$ vertices and $|E|=1$ edge, we have $\frac{2|E|}{m}=1$, and by the Laplacian energy formula.

$$E_L = \big|0-1\big| + \big|2-1\big| = 1+1 = 2.$$

Finally, the Seidel matrix (with J the all-one matrix and I the identity) is

$$S = J - I - 2A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

whose eigenvalues are $\sigma_1 = 1$ and $\sigma_2 = -1$. Thus, the Seidel energy is equal to

$$E_S = |1| + |-1| = 2.$$

Since all three energies evaluate to 2, the claim follows.

3.2 Remark

From the spectral study of the compressed zero-divisor graph $\Gamma_c(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$, the following properties of its associated matrices adjacency (A), Laplacian (L), and Seidel (S) are obtained:

1. **Adjacency matrix:** The spectral radius of A is $\rho(A) = 1$ the trace of A is $\mathrm{tr}(A) = 0$, and the determinant of A is $\det(A) = -1$.
2. **Laplacian matrix:** The spectral radius of L is $\rho(L) = 2$ the trace of L is $\mathrm{tr}(L) = 2$, and the determinant of L is $\det(L) = 0$.
3. **Seidel matrix:** The Seidel spectral radius of S is $\rho(S) = 1$, the trace of S is $\mathrm{tr}(S) = 0$, and the determinant of S is $\det(S) = -1$.

Thus, from the above data, we conclude that for the ring $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$, the compressed zero-divisor graph is K_2, consisting of two connected vertices. Hence, it represents the simplest non-trivial connected graph structure, which is regular, symmetric, and minimally complex. Because of this simple and perfectly balanced structure, the adjacency energy, the Laplacian Energy, and the Seidel energy are all equal.

3.3 Theorem

Let

$$R = \prod_{k=1}^{3} \mathbb{Z}_{p_k} = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3},$$

where $p_1, p_2,$ and p_3 are prime numbers. Then, for the compressed zero-divisor graph $\Gamma_c(R)$, the adjacency energy is

$$E_A(\Gamma_c(R)) = 7.30054.$$

Proof: For case $n = 3$,

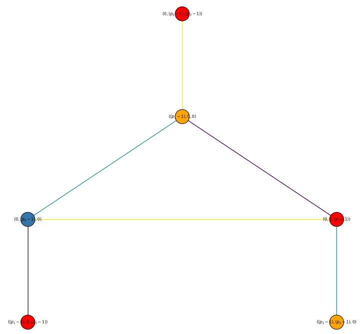

Figure 2: $\Gamma_c(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3})$

The adjacency matrix of the compressed zero–divisor graph $\Gamma_c(R)$ is given by

$$A(\Gamma_c(R)) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{6\times 6}.$$

Since $A(\Gamma_c(R))$ is a real symmetric matrix, all of its eigenvalues are real. The characteristic polynomial of A is obtained as

$$\chi_A(\lambda) = \det(A - \lambda I) = \lambda^6 - 6\lambda^4 - 2\lambda^3 + 6\lambda^2 - 1.$$

Factoring gives

$$\chi_A(\lambda) = (\lambda^2 - 2\lambda - 1)(\lambda^2 + \lambda - 1)^2.$$

the eigenvalues of the matrix are computed by using Python Programming

$$\lambda_1 = 1 + \sqrt{2}, \quad \lambda_2 = 1 - \sqrt{2}, \quad \lambda_{3,4} = \frac{-1-\sqrt{5}}{2}, \quad \lambda_{5,6} = \frac{-1+\sqrt{5}}{2}.$$

Numerically, these are

$$\lambda_1 = 2.41421, \quad \lambda_2 = -0.41421, \quad \lambda_{3,4} = -1.61803, \quad \lambda_{5,6} = 0.61803.$$

Hence, the adjacency energy is

$$\begin{aligned} E_A(\Gamma_c(R)) &= \sum_{i=1}^{6} |\lambda_i| \\ &= |2.41421| + |-0.41421| + 2|-1.61803| + 2|0.61803| \\ &= 7.30054. \end{aligned}$$

Therefore, the adjacency energy of the compressed zero–divisor graph $\Gamma_c(R)$ is 7.30054.

3.4 Theorem

Let

$$R = \prod_{k=1}^{3} \mathbb{Z}_{p_k} = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3},$$

where p_1, p_2, and p_3 are prime numbers. Then, for the compressed zero–divisor graph $\Gamma_c(R)$, the Laplacian energy is

$$E_L(\Gamma_c(R)) = 9.21112.$$

Proof: Let $A(\Gamma_c(R))$ denote the adjacency matrix of $\Gamma_c(R)$ and let D be the diagonal matrix of vertex degrees. Then the Laplacian matrix is given by

$$L = D - A.$$

For the graph $\Gamma_c(R)$, the Laplacian matrix is

$$L(\Gamma_c(R)) = \begin{pmatrix} 3 & -1 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{6\times 6}.$$

The Laplacian matrix is also symmetric, hence all its eigenvalues are real. The eigenvalues of $L(\Gamma_c(R))$ are computed by using python programming

$$\mu_i = \{\, 0,\ 2,\ 4.30278,\ 4.30278,\ 0.69722,\ 0.69722 \,\}.$$

The average degree of the graph is

$$\frac{2|E|}{m} = 2,$$

where $|E|$ and m denote the number of edges and vertices respectively. By definition, the Laplacian energy is

$$E_L(\Gamma_c(R)) = \sum_{i=1}^{6} |\mu_i - \tfrac{2|E|}{m}|.$$

Substituting the values, we obtain

$$E_L(\Gamma_c(R)) = |0 - 2| + |2 - 2| + 2|4.30278 - 2| + 2|0.69722 - 2| = 9.21112.$$

Hence, the Laplacian energy of the compressed zero–divisor graph $\Gamma_c(R)$ is 9.21112.

3.5 Theorem

Let

$$R = \prod_{k=1}^{3} \mathbb{Z}_{p_k} = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3},$$

where p_1, p_2, and p_3 are prime numbers. Then, for the compressed zero–divisor graph $\Gamma_c(R)$, the Seidel energy is

$$E_S(\Gamma_c(R)) = 13.41642.$$

Proof: By definition, the Seidel matrix is given by

$$S = J - I - 2A,$$

where J is the all-ones matrix, I is the identity matrix, and A is the adjacency matrix of $\Gamma_c(R)$. Thus, the Seidel matrix of $\Gamma_c(R)$ is

$$S(\Gamma_c(R)) = \begin{pmatrix} 0 & -1 & -1 & 1 & 1 & -1 \\ -1 & 0 & -1 & 1 & -1 & 1 \\ -1 & -1 & 0 & -1 & 1 & 1 \\ 1 & 1 & -1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}_{6\times 6}.$$

The characteristic polynomial of S is computed as

$$\chi_S(\sigma) = \det(S - \sigma I) = (\sigma^2 - 5)^3 = \sigma^6 - 15\sigma^4 + 75\sigma^2 - 125.$$

Hence, the eigenvalues of S are computed by using python programming

$$\sigma_1 = \sigma_2 = \sigma_3 = -\sqrt{5} = -2.23607, \quad \sigma_4 = \sigma_5 = \sigma_6 = \sqrt{5} = 2.23607.$$

Therefore, the Seidel energy of $\Gamma_c(R)$ is

$$E_S(\Gamma_c(R)) = \sum_{i=1}^{6} |\sigma_i| = 6\sqrt{5} = 13.41642.$$

This completes the proof.

3.6 Remark

For the compressed zero-divisor graph $\Gamma_c(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3})$, where p_1, p_2, p_3 are distinct prime numbers, the following spectral properties of the associated matrices are obtained:

1. **Adjacency matrix:** The spectral radius of A is $\rho(A) = 2.41421$, the trace of A is $\operatorname{tr}(A) = 0$, and the determinant of A is $\det(A) = -1$.
2. **Laplacian matrix:** The spectral radius of L is $\rho(L) = 4.30278$, the trace of L is $\operatorname{tr}(L) = 12$, and the determinant of L is $\det(L) = 0$.
3. **Seidel matrix:** The Seidel spectral radius of S is $\rho(S) = 5$, the trace of S is $\operatorname{tr}(S) = 0$, and the determinant of S is $\det(S) = -125$.

Thus, for the ring $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3}$, the compressed zero-divisor graph contains 6 vertices and 6 edges. The graph is connected, moderately symmetric, and partially regular, making it more complex than the two-vertex case K_2. It has exactly one connected component, with moderate irregularity in vertex degrees. Overall, this structure reflects increased complexity and stronger interconnectedness among vertices compared to the simpler case. These higher energy values indicate stronger global connectivity and greater structural complexity compared to the simpler case.

3.7 Theorem

Let

$$R = \prod_{k=1}^{4} \mathbb{Z}_{p_k} = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4},$$

where p_1, p_2, p_3, and p_4 are prime numbers. Then, for the compressed zero–divisor graph $\Gamma_c(R)$, the adjacency energy is given by

$$E_A(\Gamma_c(R)) = 20.$$

Proof: For case $n = 4$,

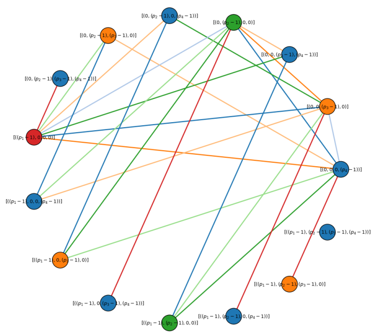

Figure 3: $\Gamma_c(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4})$

The adjacency matrix of the compressed zero–divisor graph $\Gamma_c(R)$ is

$$A(\Gamma_c(R)) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{14\times 14}$$

Clearly, $A(\Gamma_c(R))$ is a real symmetric matrix, hence all its eigenvalues are real.
The characteristic polynomial of A is obtained as

$$\chi_A(\lambda) = \det(A - \lambda I) = (\lambda + 1)(\lambda - 1)^5(\lambda^2 - 5\lambda + 1)(\lambda^2 + 3\lambda + 1)^3.$$

From this factorization, the spectrum of A (with multiplicities) was computed using Python programming.

$$\{\, 1^{(5)},\ -1,\ 4.79129,\ 0.20871,\ (-2.61803)^{(3)},\ (-0.38197)^{(3)} \,\}.$$

Therefore, the adjacency energy of $\Gamma_c(R)$ is

$$\begin{aligned} E_A(\Gamma_c(R)) &= \sum_i |\lambda_i| \\ &= 5|1| + |-1| + |4.79129| + |0.20871| + 3|-2.61803| + 3|-0.38197| \\ &= 20. \end{aligned}$$

Hence, $E_A(\Gamma_c(R)) = 20$, which completes the proof.

3.8 Theorem

Let

$$R = \prod_{k=1}^{4} \mathbb{Z}_{p_k} = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4},$$

where p_1, p_2, p_3, and p_4 are prime numbers. Then, for the compressed zero–divisor graph $\Gamma_c(R)$, the Laplacian energy is given by

$$E_L(\Gamma_c(R)) = 34.50292.$$

Proof: By definition, the Laplacian matrix is given by

$$L = D - A,$$

where A is the adjacency matrix and D is the degree matrix.

The degree matrix is

$$D = \mathrm{diag}\{7, 7, 7, 7, 3, 3, 3, 3, 3, 3, 1, 1, 1, 1\}.$$

Thus, the Laplacian matrix L of order 14×14 is given by

$$L - \mu I = \begin{pmatrix}
7-\mu & -1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & -1 \\
-1 & 7-\mu & -1 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
-1 & -1 & 7-\mu & -1 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
-1 & -1 & -1 & 7-\mu & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 3-\mu & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 3-\mu & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 & 3-\mu & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & -1 & 3-\mu & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 3-\mu & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 3-\mu & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\mu & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\mu & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\mu & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\mu
\end{pmatrix}_{14\times 14}.$$

The characteristic polynomial of L is computed as

$$\begin{aligned}
\chi_L(\mu) = \det(L - \mu I) = \mu\big(&\mu^{13} - 50\mu^{12} + 1098\mu^{11} - 13980\mu^{10} + 114843\mu^{9} - 641394\mu^{8} \\
&+ 2501194\mu^{7} - 6886184\mu^{6} + 13372929\mu^{5} - 18087270\mu^{4} \\
&+ 16569816\mu^{3} - 9751248\mu^{2} + 3312400\mu - 492128\big).
\end{aligned}$$

The eigenvalues of L obtained using Python programming are:

$$0^{(1)}, \quad 2^{(2)}, \quad 1.20871^{(1)}, \quad 5.79129^{(1)},$$
$$0.84661^{(3)}, \quad 8.56976^{(3)}, \quad 3.58363^{(3)}.$$

The Laplacian energy of the graph $\Gamma_c(R)$ is given by

$$E_L = \sum_{i=1}^{14} \left| \mu_i - \frac{2|E|}{m} \right|.$$

For the given graph,

$$E_L(\Gamma_c(R)) = \sum_{i=1}^{14} \left| \mu_i - \frac{2 \times 25}{14} \right| = 34.50292.$$

Hence, the result follows.

3.9 Theorem

Let $R = \prod_{k=1}^{4} \mathbb{Z}_{p_k} = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4}$, where p_1, p_2, p_3, and p_4 are prime numbers. Then, for the compressed zero–divisor graph $\Gamma_c(R)$, the Seidel energy is given by

$$E_S(\Gamma_c(R)) = 41.58194.$$

Proof: By definition, the Seidel matrix of a graph is given by

$$S = J - I - 2A,$$

where J is the all-ones matrix, I is the identity matrix, and A is the adjacency matrix of $\Gamma_c(R)$.

The Seidel matrix $S(\Gamma_c(R))$ is given by

$$S = \begin{pmatrix}
0 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\
-1 & 0 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\
-1 & -1 & 0 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\
-1 & -1 & -1 & 0 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 0 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & 0 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & 1 & 0 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 1 & 1 & -1 & 1 & 1 & -1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
-1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}_{14 \times 14}$$

The characteristic polynomial of S is computed as

$$\begin{aligned}
\chi_S(\sigma) = \det(S - \sigma I) = \sigma^{14} &- 91\sigma^{12} - 184\sigma^{11} + 2661\sigma^{10} + 9496\sigma^9 - 24775\sigma^8 \\
&- 158256\sigma^7 - 93149\sigma^6 + 753456\sigma^5 - 18087270\sigma^4 \\
&+ 1812879\sigma^4 + 1636200\sigma^3 + 653751\sigma^2 + 118584\sigma + 8019.
\end{aligned}$$

Hence, the eigenvalues of S, computed using Python programming, are:

$$\begin{aligned}
&8.08276\ (1), \quad -1\ (1), \quad -4.08276\ (1), \quad 4.23607\ (3), \\
&-3\ (4), \quad -0.23607\ (3), \quad 3\ (1).
\end{aligned}$$

The Seidel energy of the graph $\Gamma_c(R)$ is defined as

$$E_S(\Gamma_c(R)) = \sum_{i=1}^{14} |\sigma_i|,$$

where σ_i are the eigenvalues of the Seidel matrix S.

Thus,

$$E_S(\Gamma_c(R)) = \sum_{i=1}^{14} |\sigma_i| = 41.58194.$$

Hence, the result follows directly from computation.

3.10 Remark

Let

$$R = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4},$$

where p_1, p_2, p_3, p_4 are primes, and let $\Gamma_c(R)$ denote the corresponding compressed zero-divisor graph. Then the following spectral properties hold for the adjacency, Laplacian, and Seidel matrices of $\Gamma_c(R)$.

(a) **Adjacency matrix A:** The spectral radius is $\rho(A) = 4.79129$. The trace and determinant are

$$\operatorname{tr}(A) = 0, \qquad \det(A) = -1.$$

(b) **Laplacian matrix L:** Let $D = \operatorname{diag}\{7, 7, 7, 7, 3, 3, 3, 3, 3, 3, 1, 1, 1, 1\}$ be the degree matrix so that $L = D - A$. Then

$$\operatorname{tr}(L) = 50, \qquad \det(L) = 0,$$

and the spectral radius is $\rho(L) = 8.56976$.

(c) **Seidel matrix S:** The Seidel matrix is $S = J - I - 2A$. We have

$$\operatorname{tr}(S) = 0, \qquad \det(S) = 8019,$$

and the Seidel spectral radius is $\rho(S) = 8.08276$.

For the ring $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4}$, the compressed zero-divisor graph contains 14 vertices and 25 edges. The graph is connected, moderately symmetric, and partially regular, making it structurally more complex than the three-factor case. It has exactly one connected component, with moderate to high irregularity in vertex degrees. Overall, the graph exhibits increased complexity, partial symmetry, and stronger global interconnectedness among vertices. The larger number of vertices and edges amplifies the structural intricacy compared to lower-factor cases.

3.11 Theorem

Let

$$R = \prod_{k=1}^{5} \mathbb{Z}_{p_k} = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4} \times \mathbb{Z}_{p_5},$$

where p_1, p_2, p_3, p_4, and p_5 are prime numbers. Then, for the compressed zero–divisor graph $\Gamma_c(R)$, the adjacency energy is given by

$$E_A(\Gamma_c(R)) = 49.44144.$$

Proof: For case $n = 5$,

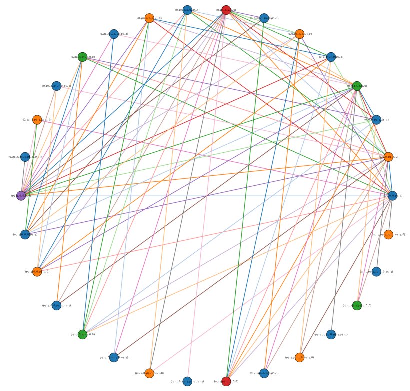

Figure 4: $\Gamma_c(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4} \times \mathbb{Z}_{p_5})$

The adjacency matrix of the compressed zero–divisor graph $\Gamma_c(R)$ is denoted by $A(\Gamma_c(R))$ and has the form

$$A(\Gamma_c(R)) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & \cdots & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & \cdots & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{30 \times 30}$$

Clearly, A is a real symmetric matrix, hence all its eigenvalues are real. The characteristic polynomial of A is

$$\chi_A(\lambda) = \left(\lambda^2 - \lambda - 1\right)^9 \left(\lambda^2 + 4\lambda - 1\right)^4 \left(\lambda^4 - 7\lambda^3 - 16\lambda^2 + 7\lambda + 1\right)$$

The eigenvalues of $A(\Gamma_c(R))$ were calculated by using Python programming and are listed below together with their multiplicities.

$$-4.23607\ (4), \quad -2.09973\ (1), \quad -0.61803\ (9), \quad -0.11444\ (1),$$
$$0.23607\ (4), \quad 0.47625\ (1), \quad 1.61801\ (9), \quad 8.73792\ (1).$$

Hence, the adjacency energy is

$$E_A(\Gamma_c(R)) = \sum_{i=1}^{30} |\lambda_i| = 49.44144.$$

3.12 Theorem

Let $R = \prod_{k=1}^{5} \mathbb{Z}_{p_k} = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4} \times \mathbb{Z}_{p_5}$, where p_1, p_2, p_3, p_4, and p_5 are prime numbers. Then, for the compressed zero–divisor graph $\Gamma_c(R)$, the Laplacian energy is given by

$$E_L(\Gamma_c(R)) = 125.18464.$$

Proof: By definition, the Laplacian matrix is $L = D - A$, where A is the adjacency matrix and D is the degree matrix.

$$D = \text{diag}\{15, 15, 15, 15, 15, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 1, 1, 1, 1, 1\}.$$

Then the Laplacian matrix is

$$L = \begin{pmatrix}
15 & -1 & -1 & -1 & -1 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & -1 \\
-1 & 15 & -1 & -1 & -1 & 0 & \cdots & \cdots & 0 & 0 & 0 & -1 & 0 \\
-1 & -1 & 15 & -1 & -1 & -1 & \cdots & \cdots & 0 & 0 & -1 & 0 & 0 \\
-1 & -1 & -1 & 15 & -1 & -1 & \cdots & \cdots & 0 & -1 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & 15 & -1 & \cdots & \cdots & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & -1 & 7 & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & -1 & 0 & 0 & 0 & -1 & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & \cdots & \cdots & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & \cdots & \cdots & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & 1
\end{pmatrix}_{30\times 30}$$

The characteristic polynomial $\chi_L(\mu) = \det(L - \mu I)$ expands to

$$\mu(\mu^2 - 9\mu + 17)^5(\mu^3 - 19\mu^2 + 79\mu - 66)(\mu^4 - 29\mu^3 + 239\mu^2 - 586\mu + 360)^4$$

The eigenvalues of L are computed using Python programming.

$$0(1),\ 0.92604(4),\ 1.11872(1),\ 2.6835(4),\ 2.69722(5),$$
$$4.3647(1),\ 6.30278(5),\ 8.65768(4),\ 13.51658(1),\ 16.73278(4).$$

The Laplacian energy is given by

$$E_L(\Gamma_c(R)) = \sum_{i=1}^{30} \left| \mu_i - \frac{2|E|}{m} \right| = \sum_{i=1}^{30} \left| \mu_i - \frac{2 \times 90}{30} \right| = 125.18464.$$

Hence, the result follows directly from computation.

3.13 Theorem

Let $R = \prod_{k=1}^{5} \mathbb{Z}_{p_k} = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4} \times \mathbb{Z}_{p_5}$, where p_1, p_2, p_3, p_4, and p_5 are prime numbers. Then, for the compressed zero–divisor graph $\Gamma_c(R)$, the Seidel energy is given by

$$E_S(\Gamma_c(R)) = 105.22954.$$

Proof: By definition, the Seidel matrix is $S = J - I - 2A$, where J is the all–ones matrix and I is the identity matrix. Thus,

$$S(\Gamma_c(R)) = \begin{pmatrix} 0 & -1 & -1 & -1 & -1 & 1 & \cdots & \cdots & 1 & 1 & 1 & 1 & -1 \\ -1 & 0 & -1 & -1 & -1 & 1 & \cdots & \cdots & 1 & 1 & 1 & -1 & 1 \\ -1 & -1 & 0 & -1 & -1 & -1 & \cdots & \cdots & 1 & 1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 0 & -1 & -1 & \cdots & \cdots & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & -1 & \cdots & \cdots & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 0 & \cdots & \cdots & 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & 1 & 1 & 1 & -1 & \cdots & \cdots & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & \cdots & \cdots & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & \cdots & \cdots & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & \cdots & \cdots & 1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 1 & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & 1 & 0 \end{pmatrix}_{30\times 30}$$

The characteristic polynomial $\chi_S(\sigma) = \det(S - \sigma I)$ simplifies to

$$(\sigma^2 - 6\sigma - 11)^4(\sigma^2 + 4\sigma - 1)^9(\sigma^4 - 12\sigma^3 - 166\sigma^2 - 228\sigma - 59)$$

The eigenvalues are calculated using python programming

$$-7.07486(1),\ -4.23607(9),\ -1.47214(4),\ -1.18517(1),$$
$$-0.34155(1),\ 0.23607(9),\ 7.47214(4),\ 20.60158(1).$$

Hence, the Seidel energy of $\Gamma_c(R)$ is

$$E_S(\Gamma_c(R)) = \sum_{i=1}^{30} |\sigma_i| = 105.22954.$$

3.14 Remark

Let

$$R = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4} \times \mathbb{Z}_{p_5},$$

where p_1, p_2, p_3, p_4, p_5 are primes, and let $\Gamma_c(R)$ denote the compressed zero-divisor graph. Then the following spectral properties hold for the adjacency matrix A, the Laplacian matrix L, and the Seidel matrix S of $\Gamma_c(R)$.

(a) **Adjacency matrix A:** The spectral radius is

$$\rho(A) = 8.73792,$$

and the trace and determinant satisfy

$$\mathrm{tr}(A) = 0, \qquad \det(A) = -1.$$

(b) **Laplacian matrix L:** Let

$$D = \mathrm{diag}\{15, 15, 15, 15, 15, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 1, 1, 1, 1, 1\},$$

so that $L = D - A$. Then

$$\mathrm{tr}(L) = 180, \qquad \det(L) = 0,$$

and the computed Laplacian spectral radius is

$$\rho(L) = 16.73278.$$

(c) **Seidel matrix** S**:** The Seidel matrix $S = J - I - 2A$ satisfies

$$\operatorname{tr}(S) = 0, \qquad \det(S) = 863819,$$

and the computed Seidel spectral radius is

$$\rho(S) = 20.60158.$$

For the ring $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4} \times \mathbb{Z}_{p_5}$, the compressed zero-divisor graph contains 30 vertices and 90 edges. The graph is connected, moderately symmetric, and partially regular, making it structurally much more complex than the four-factor case. It has exactly one connected component, with moderate to high irregularity in vertex degrees. Overall, the graph demonstrates a significant increase in structural complexity, while symmetry decreases as more prime factors are added. The growing number of vertices and edges amplifies the graph's connectivity and richness, showing how the structure becomes increasingly intricate and less symmetric with each additional factor.

4 Conclusion

The study of compressed zero-divisor graphs of rings $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$ reveals a clear progression in structural complexity as the number of prime factors increases. For the simplest case of two factors, the graph is K_2, a perfectly balanced, regular, and symmetric structure. Adding a third factor increases the number of vertices and edges, introduces moderate irregularity, and reduces symmetry, while still maintaining a single connected component.

With four and five prime factors, the graphs grow significantly larger, with 14 and 30 vertices respectively, exhibiting partial symmetry, higher irregularity, and stronger global interconnectedness. These observations indicate that each additional prime factor amplifies the graph's complexity and connectivity, while gradually diminishing regularity and symmetry. Overall, the analysis demonstrates a consistent pattern: as the ring becomes richer in prime components, its compressed zero-divisor graph becomes increasingly intricate and interconnected.

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