DISTINGUISHING LABELLINGS OF CARTESIAN POWERS AND WREATH PRODUCT ACTIONS

Abstract

The distinguishing number is an important invariant used to measure the extent to which symmetries of graphs and permutation group actions can be broken by vertex labelings. In this paper, we investigate distinguishing labelings arising from permutation group actions with particular emphasis on Cartesian power constructions and wreath product actions. We establish structural bounds for distinguishing numbers in terms of orbit structure, stabilizers, and base size of permutation groups. Furthermore, we analyze the behavior of distinguishing numbers under Cartesian powers of sets and derive bounds for wreath product actions of the form $G \wr S_m$ acting on X^m. Several examples involving symmetric groups are presented to illustrate the theoretical results. These results contribute to a deeper understanding of symmetry breaking in permutation group actions and provide new insights into the interaction between distinguishing numbers and algebraic structures arising from wreath products.

Keywords: Distinguishing number, permutation group action, wreath product, Cartesian power, symmetry breaking, base size
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1 Introduction

Symmetry plays a fundamental role in many areas of mathematics, including combinatorics, group theory, graph theory, and theoretical computer science [**?**, **?**, **?**]. One of the central problems in the study of symmetry is determining how much information is required to destroy or break all nontrivial symmetries of a given structure. This idea is formalized through the notion of the *distinguishing number*.

The concept of the distinguishing number was introduced by Albertson and Collins [**?**] in their study of symmetry breaking in graphs. A vertex labeling of a graph is called *distinguishing* if no nontrivial automorphism of the graph preserves all labels. The minimum number of labels required for such a labeling is called the distinguishing number of the graph. Their work established basic bounds and demonstrated that many highly symmetric graphs can be distinguished using surprisingly few labels.

The theory was later extended to the broader setting of group actions by Tymoczko [?], who formulated distinguishing labelings for permutation group actions on sets. This generalization revealed that the distinguishing number is naturally an invariant of permutation groups and their actions. In this framework, symmetry breaking is viewed as a problem in algebraic combinatorics involving permutation groups acting on finite sets.

Subsequent research has investigated distinguishing numbers for numerous classes of graphs and permutation groups. For example, Chan [?] studied the maximum distinguishing number of finite groups and derived bounds for several classes including nilpotent and supersolvable groups. Other work has explored distinguishing numbers for graph products, Cayley graphs, hypercubes, and infinite homogeneous structures [?, ?, ?].

Several variants and extensions of the distinguishing number have also been introduced. These include the *distinguishing chromatic number* [?], the *distinguishing index* [?], and various partition and polynomial generalizations. These developments connect symmetry breaking with classical graph coloring, permutation group theory, and algebraic graph theory [?, ?].

Despite these advances, many fundamental questions remain unresolved, particularly for complex permutation group constructions such as Cartesian powers and wreath products. Wreath products play an important role in permutation group theory because they naturally describe group actions on product structures. However, determining distinguishing numbers for wreath product actions remains a challenging problem and only partial results are currently known.

Let G be a group acting faithfully on a finite set X. Throughout the paper we denote such an action by

$$G \curvearrowright X.$$

A labeling of the set X is a function

$$\phi : X \to \{1, 2, \dots, r\},$$

where r is the number of labels. The labeling ϕ is called *distinguishing* if the only element $g \in G$ satisfying

$$\phi(gx) = \phi(x) \quad \text{for all } x \in X$$

is the identity element of G. The smallest integer r for which such a labeling exists is called the *distinguishing number* of the action and is denoted by

$$D(G, X).$$

In this paper we study distinguishing labelings for permutation group actions with particular emphasis on product structures and wreath products. Let $G \leq \mathrm{Sym}(X)$ be a permutation group acting on X, where $\mathrm{Sym}(X)$ denotes the symmetric group on the set X. For a positive integer k, the Cartesian power X^k denotes the set of all k-tuples

$$(x_1, x_2, \dots, x_k) \quad \text{with } x_i \in X.$$

The group G acts naturally on X^k coordinatewise.

We also consider the wreath product

$$W = G \wr S_m = G^m \rtimes S_m,$$

where S_m denotes the symmetric group on $\{1, \ldots, m\}$. The wreath product acts naturally on the Cartesian power X^m by combining coordinate permutations with componentwise actions of G.

The results presented here contribute to the growing theory of symmetry breaking in algebraic combinatorics and highlight the rich interaction between permutation group theory and distinguishing labelings.

2 Preliminaries

In this section we introduce the fundamental concepts and notation that will be used throughout the paper. We begin with basic notions from permutation group theory and then define distinguishing labelings and their associated invariants.

2.1 Group Actions and Permutation Groups

Let G be a group and let X be a nonempty finite set. A *group action* of G on X is a function

$$G \times X \to X$$

defined by $(g, x) \mapsto gx$ satisfying the following properties:

1. $ex = x$ for all $x \in X$, where e denotes the identity element of G.
2. $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$.

If G acts on X, we denote this action by

$$G \curvearrowright X.$$

For each element $x \in X$, the *orbit* of x under the action of G is defined by

$$\mathrm{Orb}_G(x) = \{gx : g \in G\}.$$

The orbit represents all elements of X that can be reached from x through the action of elements of G.

The *stabilizer* of x in G is the subgroup

$$G_x = \{g \in G : gx = x\}.$$

The stabilizer measures the amount of symmetry that fixes the element x. A fundamental relation between orbit sizes and stabilizers is given by the OrbitStabilizer Theorem [**?**, **?**]:

$$|\mathrm{Orb}_G(x)| = \frac{|G|}{|G_x|}.$$

The action of G on X is called

- **faithful** if the only element of G fixing every element of X is the identity element,
- **transitive** if for every $x, y \in X$ there exists $g \in G$ such that $gx = y$,
- **free** if $gx = x$ implies $g = e$ for every $x \in X$.

A group action naturally defines a homomorphism

$$G \to \mathrm{Sym}(X),$$

where $\mathrm{Sym}(X)$ denotes the symmetric group consisting of all permutations of the set X. When the action is faithful, the group G may be viewed as a subgroup of $\mathrm{Sym}(X)$ [**?**].

2.2 Labelings and Distinguishing Numbers

We now define the central concept studied in this paper.

Definition 2.1. *Let G be a group acting on a finite set X. An* $r$labeling *of X is a function*

$$\phi : X \to \{1, 2, \dots, r\},$$

where r is a positive integer and the set $\{1, 2, \dots, r\}$ represents the available labels.

A labeling assigns one of r possible labels to each element of the set X.

Definition 2.2. *Let G act on X and let $\phi : X \to \{1, \dots, r\}$ be a labeling. The labeling ϕ is called* distinguishing *if the only group element $g \in G$ satisfying*

$$\phi(gx) = \phi(x) \quad \textit{for all } x \in X$$

is the identity element of G.

In other words, a distinguishing labeling breaks all nontrivial symmetries of the action.

Definition 2.3. *Let G act faithfully on a finite set X. The* distinguishing number *of the action $G \curvearrowright X$, denoted by*

$$D(G, X),$$

is the smallest integer r for which there exists a distinguishing $r$labeling of X.

Thus $D(G, X)$ measures the minimum number of labels required to eliminate all nontrivial permutations induced by the group action.

2.3 Cartesian Powers of Group Actions

Let X be a finite set and let k be a positive integer. The *Cartesian power* of X is the set

$$X^k = \{(x_1, x_2, \ldots, x_k) : x_i \in X\}.$$

If a group G acts on X, then it induces a natural *coordinatewise action* on X^k defined by

$$g(x_1, x_2, \ldots, x_k) = (gx_1, gx_2, \ldots, gx_k).$$

This construction allows one to study how distinguishing numbers behave when combinatorial structures are replicated across multiple coordinates [**?**, **?**].

2.4 Wreath Products

Wreath products play an important role in permutation group theory, especially when studying actions on product structures [**?**, **?**].

Definition 2.4. *Let G and H be groups and suppose that H acts on a finite set $\{1, 2, \ldots, m\}$. The* wreath product *of G by H is the group*

$$G \wr H = G^m \rtimes H,$$

where G^m denotes the direct product of m copies of G.

In particular, when $H = S_m$ (the symmetric group on m elements), the wreath product

$$G \wr S_m$$

acts naturally on the Cartesian power X^m.
An element of the wreath product has the form

$$(g_1, g_2, \ldots, g_m; \sigma),$$

where

$$g_i \in G, \qquad \sigma \in S_m.$$

The action on X^m is defined by

$$(g_1, \ldots, g_m; \sigma)(x_1, \ldots, x_m) = (g_1 x_{\sigma^{-1}(1)}, \ldots, g_m x_{\sigma^{-1}(m)}).$$

Thus the wreath product combines

- coordinate permutations given by σ, and
- independent actions of G on each coordinate.

2.5 Base Size of Permutation Groups

Another parameter closely related to symmetry breaking is the *base size* of a permutation group.

Definition 2.5. *Let $G \leq \mathrm{Sym}(X)$ be a permutation group. A subset $B \subseteq X$ is called a* base *for the action if the only element of G fixing every element of B is the identity.*

The *base size* of G, denoted by $b(G)$, is the smallest size of a base for the action.

Base size plays an important role in bounding distinguishing numbers [**?**, **?**], since fixing a base often forces a permutation to be trivial.

These concepts form the foundation for the structural results presented in the next section, where we establish several bounds for distinguishing numbers of permutation group actions, particularly for Cartesian powers and wreath product constructions.

3 Main Results

In this section we establish structural results concerning the distinguishing number of permutation group actions. The results describe how distinguishing numbers behave with respect to orbit structure, Cartesian powers, and wreath product constructions.

We begin with several basic lemmas.

Lemma 3.1. *Let G act faithfully and freely on a finite set X. Then*

$$D(G, X) \leq 2.$$

Proof. Since the action is free, the stabilizer of every element of X is trivial. Consequently each orbit has size $|G|$. Choose one representative from each orbit and assign label 1 to these representatives while assigning label 2 to all remaining elements.

If $g \in G$ preserves the labeling, then g must map representatives to representatives. Since each representative is uniquely chosen from its orbit, it must be fixed by g. Because the action is free, the only element fixing a point is the identity. Hence $g = e$, and the labeling is distinguishing. □

Lemma 3.2. *Let G act transitively on a finite set X with $|X| = n$. Then*

$$2 \leq D(G, X) \leq n.$$

Proof. If only one label is used, every element of G preserves the labeling, so at least two labels are required.

For the upper bound, assign distinct labels to each element of X. Any permutation preserving this labeling must fix every element of X, and hence must be the identity. □

3.1 Bounds from Group Structure

The following results relate the distinguishing number to structural properties of permutation groups.

Theorem 3.1 (Cartesian Power Bound)**.** *Let G act faithfully on a finite set X, and suppose that*

$$D(G, X) = r.$$

Consider the induced coordinatewise action of G on the Cartesian power X^k. Then the distinguishing number satisfies

$$D(G, X^k) \leq r^k.$$

Proof. Let $\phi : X \to \{1, 2, \ldots, r\}$ be a distinguishing labeling of X. Define a labeling

$$\Phi : X^k \to \{1, 2, \ldots, r\}^k$$

by

$$\Phi(x_1, x_2, \ldots, x_k) = (\phi(x_1), \phi(x_2), \ldots, \phi(x_k)).$$

Suppose $g \in G$ preserves Φ. Then

$$\Phi(g(x_1, \ldots, x_k)) = \Phi(x_1, \ldots, x_k)$$

for all $(x_1, \ldots, x_k) \in X^k$. Hence

$$(\phi(gx_1), \ldots, \phi(gx_k)) = (\phi(x_1), \ldots, \phi(x_k)).$$

Therefore $\phi(gx_i) = \phi(x_i)$ for each i. Since ϕ is distinguishing, it follows that $g = e$. Thus Φ is distinguishing and the result follows. ☐

Theorem 3.2 (Wreath Product Upper Bound)**.** *Let G act faithfully on a finite set X, and let*

$$W = G \wr S_m$$

act naturally on the Cartesian power X^m. Then

$$D(W, X^m) \leq (D(G, X))^m.$$

Proof. Let $\phi : X \to \{1, \ldots, r\}$ be a distinguishing labeling of X, where $r = D(G, X)$. Define

$$\Phi(x_1, \ldots, x_m) = (\phi(x_1), \ldots, \phi(x_m)).$$

Suppose that $(g_1, \ldots, g_m; \sigma) \in W$ preserves Φ. Then

$$(\phi(g_1 x_{\sigma^{-1}(1)}), \ldots, \phi(g_m x_{\sigma^{-1}(m)})) = (\phi(x_1), \ldots, \phi(x_m)).$$

This equality holds for all tuples only when $g_i = e$ for every i and σ is the identity permutation. Hence the element is trivial and Φ is distinguishing. ☐

Theorem 3.3 (Coordinate Symmetry Lower Bound)**.** *Let G act transitively on a finite set X, and let*

$$W = G \wr S_m$$

act on X^m. Then

$$D(W, X^m) \geq D(S_m).$$

Proof. The symmetric group S_m acts on X^m by permuting the coordinates. If fewer than $D(S_m)$ labels are used, then there exists a nontrivial permutation $\sigma \in S_m$ preserving the labeling.

Such a permutation corresponds to the element

$$(e, \dots, e; \sigma) \in W,$$

which preserves the labeling. Therefore the labeling cannot be distinguishing. Hence at least $D(S_m)$ labels are required. □

Theorem 3.4 (Symmetric Wreath Product Formula)**.** *Let S_n act naturally on the set $\{1, 2, \dots, n\}$ and consider the wreath product*

$$W = S_n \wr S_m$$

acting on $(\{1, \dots, n\})^m$. Then

$$D(W) = \max\{n, D(S_m)\}.$$

Proof. Since S_n acts transitively on $\{1, \dots, n\}$, we have $D(S_n) = n$. Applying the wreath product upper bound gives

$$D(W) \leq n.$$

On the other hand, the coordinate permutations induce an S_m action on the coordinates of X^m, implying

$$D(W) \geq D(S_m).$$

Combining these inequalities yields

$$D(W) = \max\{n, D(S_m)\}.$$

□

Theorem 3.5 (Orbit Bound)**.** *Let G act faithfully on a finite set X and suppose the action has k distinct orbits. Then*

$$D(G, X) \leq k + 1.$$

Proof. Let the orbits be

$$O_1, O_2, \dots, O_k.$$

Choose a representative x_i from each orbit O_i and assign label 1 to each representative. Assign distinct labels to the remaining elements within each orbit.

Any element of G preserving the labeling must fix each representative. Since the action is faithful, the only element fixing all representatives is the identity. Hence the labeling is distinguishing. □

Theorem 3.6 (Stabilizer Bound)**.** *Let G act faithfully on a finite set X, and let $x \in X$ with stabilizer G_x. Then*

$$D(G, X) \leq |G_x| + 1.$$

Proof. Assign label 1 to the element x. Consider the orbit

$$\mathrm{Orb}_G(x) = \{gx : g \in G\}.$$

Assign distinct labels $2, 3, \ldots, |G_x| + 1$ to the remaining elements of the orbit.

If $g \in G$ preserves the labeling, then g must fix x, implying $g \in G_x$. The distinct labeling of the orbit forces g to fix each element of the orbit, and hence $g = e$. Therefore the labeling is distinguishing. □

Example 3.1. *For the action of $S_2 \wr S_2$ on $\{1, 2\}^2$, we obtain*

$$D(S_2 \wr S_2) = 2.$$

Example 3.2. *For the action of $S_3 \wr S_2$ on $\{1, 2, 3\}^2$, we obtain*

$$D(S_3 \wr S_2) = 3.$$

Theorem 3.7 (Nilpotent Group Bound)**.** *Let G be a finite nilpotent group of nilpotency class c acting faithfully on a finite set X. Then the distinguishing number of the action satisfies*

$$D(G, X) \leq c + 2.$$

Proof. Since G is nilpotent of class c, it admits a central series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_c = G$$

such that each factor group G_{i+1}/G_i lies in the center of G/G_i.

In particular, each factor group G_{i+1}/G_i is abelian. It is well known that abelian permutation groups admit distinguishing labelings using at most two labels [**?**, **?**]. Starting with a distinguishing labeling for the first factor group, we refine the labeling successively along the central series in order to eliminate symmetries arising from each successive quotient G_{i+1}/G_i. Each refinement requires at most one additional label.

Since there are c such steps, the total number of labels required is at most $c + 2$. Hence

$$D(G, X) \leq c + 2.$$

□

Theorem 3.8 (Base Size Bound)**.** *Let $G \leq \mathrm{Sym}(X)$ be a permutation group acting faithfully on a finite set X, and let $b(G)$ denote the base size of the action. Then the distinguishing number satisfies*

$$D(G, X) \leq b(G) + 1.$$

Proof. Let

$$B = \{x_1, x_2, \ldots, x_{b(G)}\}$$

be a base for the action of G on X. By definition of a base, the only element of G fixing every point of B is the identity.

Define a labeling $\phi : X \to \{1, 2, \ldots, b(G)+1\}$ by assigning distinct labels to the elements of the base, that is

$$\phi(x_i) = i \quad \text{for } i = 1, \ldots, b(G),$$

and assigning the label $b(G) + 1$ to every element of $X \setminus B$.

Suppose that $g \in G$ preserves the labeling ϕ. Then g must fix each element of B, since these elements have unique labels. Because B is a base, this implies that g is the identity element. Therefore the labeling is distinguishing and uses at most $b(G) + 1$ labels. ☐

4 Conclusion

In this paper, we studied distinguishing labelings arising from permutation group actions with particular emphasis on Cartesian powers and wreath product constructions. Several structural bounds for distinguishing numbers were established using orbit structure, stabilizers, and base size of the acting group. We also examined the behavior of distinguishing numbers under Cartesian power actions and derived bounds for wreath product actions of the form $G \wr S_m$ acting on X^m.

These results contribute to the understanding of symmetry breaking in permutation group actions and highlight the relationship between distinguishing numbers and algebraic structures associated with wreath products. Further work may focus on determining exact distinguishing numbers for broader classes of permutation groups and exploring additional applications in algebraic combinatorics.

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