UNDER PEER REVIEW

On the spectra of principal ideal graph of completely simple semigroups

Abstract

Consider T to be a semigroup. The principal left(right) ideal graph $\mathbf{PiG}_{\mathscr{L}}$ ($\mathbf{PiG}_{\mathscr{R}}$) is defined by taking the elements of T as vertices, with two elements being adjacent whenever their principal left (right) ideals intersect. We focus on the case of completely simple semigroups and compute various graph energies, including adjacency energy (A_d-energy), signless Laplacian energy (S_l-energy), and Laplacian energy (L_p-energy). Further, the interrelations among these energies are established.

Keywords: Principal ideal; graph Graph energy; Completely simple semigroup

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1 Introduction

Graphs constructed from algebraic ideals have emerged as a powerful tool to study algebraic structures through combinatorial methods. In particular, principal ideal graphs of semigroups establish connections between the algebraic properties of the semigroup and the structural features of the associated graphs. For a given semigroup T, the associated principal left (right) ideal graph is defined by taking the elements of T as vertices and any two vertices, being different, are adjacent exactly when their principal left (right) ideals intersect. This framework was introduced and explored in depth by Indu and John Indu and John (2012b,a), who characterized principal ideal graphs for completely simple semigroups.

Within the theory of semigroups, completely simple semigroups serve as a key structural concept since they can be represented in a matrix form that involves a group G and index sets. The semigroup $\mathscr{M}(G, \varUpsilon, \varDelta, \varPsi)$, defined as the set $G \times \varUpsilon \times \varDelta$ equipped with the operation

$$(u, \iota, \delta)(v, j, \eta) = (u\varPsi_{\delta j} v, \iota, \eta),$$

where $\varPsi$ is an appropriate matrix over G, serves as a classical example of a completely simple semigroup Howie (1995). Understanding the structure of principal ideal graphs in this setting provides valuable insights into both algebraic and graph-theoretic aspects.

This paper extends previous investigations on graph energies Ramane et al. (2023); Tan and Wang (2009) of principal ideal graphs from special classes such as rectangular bands George et al. (2025) to the broader context of completely simple semigroups. We analyze spectral properties of the adjacency, Laplacian, and signless Laplacian matrices associated with these graphs Anderson Jr and Morley (1985); Ganie et al. (2018); Merris (1994); Zhou and Gutman (2007).

2 Preliminaries

Let T be a semigroup and $m \in T$. The *principal left ideal* generated by m is defined as

$$T^1m = \{tm : t \in T\} \cup \{m\}.$$

The corresponding principal right ideal is defined in an analogous manner. The *principal left ideal graph* $\mathbf{PiG}_{\mathscr{L}}$ of T is defined by taking the elements of T as vertices, with two elements m, n being adjacent whenever their principal left ideals T^1m and T^1n intersectIndu and John (2012b). Similarly, the *principal right ideal graph* $\mathbf{PiG}_{\mathscr{R}}$ is defined via the intersection of principal right ideals mT^1 and nT^1.

Key structural results for these graphs in the setting of completely simple semigroups, originally established by Indu and John Indu and John (2012a), state that the graph $\mathbf{PiG}_{\mathscr{L}}$ is disconnected with $|\varDelta|$ components, each of which forms a complete subgraph on $|G| \cdot |\varUpsilon|$ vertices. Similarly, the graph $\mathbf{PiG}_{\mathscr{R}}$ is disconnected with $|\varUpsilon|$ components, each a complete subgraph on $|G| \cdot |\varDelta|$ vertices.

The group G, index sets $\varUpsilon$, $\varDelta$, and matrix $\varPsi = (\varPsi_{\lambda i})$ over G define the completely simple semigroup

$$\mathscr{M}(G, \varUpsilon, \varDelta, \varPsi) = G \times \varUpsilon \times \varDelta$$

with the operation

$$(u, \iota, \delta)(v, j, \eta) = (u\varPsi_{\delta j}v, \iota, \eta),$$

For a detailed account on semigroup theory and graph theory, seeHowie (1995) and Beineke and Wilson (2004), respectively.

Proposition 2.1 (Indu and John (2012a)). *For $\mathscr{M}(G, \varUpsilon, \varDelta, \varPsi)$, the principal left ideal graph $\mathbf{PiG}_{\mathscr{L}}$ has exactly $|\varDelta|$ connected components, resulting to a complete graph on $|G| \cdot |\varUpsilon|$ vertices.*

Proposition 2.2 (Indu and John (2012a)). *For $\mathscr{M}(G, \varUpsilon, \varDelta, \varPsi)$, the principal right ideal graph $\mathbf{PiG}_{\mathscr{R}}$ has exactly $|\varUpsilon|$ connected components, resulting to a complete graph with $|G| \cdot |\varDelta|$ vertices.*

In this work, we refer to the principal ideal graph of a semigroup $T = \mathscr{M}(G, \varUpsilon, \varDelta, \varPsi)$ as PiG. Also $\alpha, \beta, and, \gamma$ denote the cardinality of $G, \varUpsilon, and, \varDelta$ respectively.

3 Spectrum of $\mathrm{PiG}_{\mathscr{L}}$ and $\mathrm{PiG}_{\mathscr{R}}$

This section provide an overview of the determinantal polynomials of different matrices derived from $\mathbf{PiG}_{\mathscr{L}}$ and $\mathbf{PiG}_{\mathscr{R}}$ of $\mathscr{M}(G, \varUpsilon, \varDelta, \varPsi)$.

The following Theorems provide the characterization of A_d energy of $\mathbf{PiG}_{\mathscr{L}}$ and $\mathbf{PiG}_{\mathscr{R}}$.

Theorem 3.1. *In a completely simple semigroup $\mathscr{M}(G, \varUpsilon, \varDelta, \varPsi)$, the A_d-energy $\varOmega_{A_d}(\mathbf{PiG}_{\mathscr{L}})$ of the principal left ideal graph is $2\gamma(\alpha\beta - 1)$.*

Proof. $\mathbf{PiG}_{\mathscr{L}}$ is disconnected, having $o(\varDelta) = \gamma$ components and each forming a complete graph with $o(G) \cdot o(\varUpsilon) = \alpha\beta$ vertices according to proposition 2.1. So the adjacency matrix $A_d(\mathbf{PiG}_{\mathscr{L}})$

is a block diagonal matrix with $o(\mathit{\Delta})$ diagonal blocks and each diagonal block is a $\alpha\beta \times \alpha\beta$ square matrix.

$$A_d(\mathbf{PiG}_{\mathscr{L}}) = \begin{bmatrix} \mathscr{J}_{\alpha\beta} - \mathscr{I}_{\alpha\beta} & \mathscr{O}_{\alpha\beta} & \mathscr{O}_{\alpha\beta} & \cdots & \mathscr{O}_{\alpha\beta} \\ \mathscr{O}_{\alpha\beta} & \mathscr{J}_{\alpha\beta} - \mathscr{I}_{\alpha\beta} & \mathscr{O}_{\alpha\beta} & \cdots & \mathscr{O}_{\alpha\beta} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathscr{O}_{\alpha\beta} & \mathscr{O}_{\alpha\beta} & \mathscr{O}_{\alpha\beta} & \cdots & \mathscr{J}_{\alpha\beta} - \mathscr{I}_{\alpha\beta} \end{bmatrix}_{\alpha\beta\gamma \times \alpha\beta\gamma}$$

where $\mathscr{J}_{\alpha\beta}$,$\mathscr{I}_{\alpha\beta}$, and $\mathscr{O}_{\alpha\beta}$ denote the all-ones,identity, and zero matrices. Each $\mathscr{J}_{\alpha\beta} - \mathscr{I}_{\alpha\beta}$ block is given by

$$\mathscr{J}_{\alpha\beta} - \mathscr{I}_{\alpha\beta} = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{\alpha\beta \times \alpha\beta}$$

The determinantal polynomial of each non-zero block $\mathscr{J}_{\alpha\beta} - \mathscr{I}_{\alpha\beta}$ is

$$[t - (\alpha\beta - 1)](t+1)^{\alpha\beta - 1}.$$

Since the matrix $A_d(\mathbf{PiG}_{\mathscr{L}})$ consists $o(\mathit{\Delta})$ identical blocks, the determinantal polynomial of $A_d(\mathbf{PiG}_{\mathscr{L}})$ is

$$[t - (\alpha\beta - 1)]^{\gamma}(t+1)^{\gamma(\alpha\beta - 1)}.$$

Hence A_d-proper values of $A_d(\mathbf{PiG}_{\mathscr{L}})$ are $\alpha\beta - 1$ and -1 of multiplicity γ and $\gamma(\alpha\beta - 1)$ respectively. Also, the A_d-energy is $\mathit{\Omega}_{A_d}(\mathbf{PiG}_{\mathscr{L}}) = 2\gamma(\alpha\beta - 1)$. ☐

Theorem 3.2. *In a completely simple semigroup $\mathscr{M}(G, \mathit{\Upsilon}, \mathit{\Delta}, \mathit{\Psi})$, the A_d-energy $\mathit{\Omega}_{A_d}(\mathbf{PiG}_{\mathscr{R}})$ of the principal right ideal graph is $2\beta(\alpha\gamma - 1)$.*

Proof. According to proposition 2.2, $\mathbf{PiG}_{\mathscr{R}}$ is a disconnected, having $o(\mathit{\Upsilon}) = \beta$ components and each component is a complete graph with $o(G) \cdot o(\mathit{\Delta}) = \alpha\gamma$ vertices. So, the adjacency matrix $A_d(\mathbf{PiG}_{\mathscr{R}})$ is a block diagonal matrix with $o(\mathit{\Upsilon})$ diagonal blocks and each block is a $\alpha\gamma \times \alpha\gamma$ square matrix.

$$A_d(\mathbf{PiG}_{\mathscr{R}}) = \begin{bmatrix} \mathscr{J}_{\alpha\gamma} - \mathscr{I}_{\alpha\gamma} & \mathscr{O}_{\alpha\gamma} & \mathscr{O}_{\alpha\gamma} & \cdots & \mathscr{O}_{\alpha\gamma} \\ \mathscr{O}_{\alpha\gamma} & \mathscr{J}_{\alpha\gamma} - \mathscr{I}_{\alpha\gamma} & \mathscr{O}_{\alpha\gamma} & \cdots & \mathscr{O}_{\alpha\gamma} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathscr{O}_{\alpha\gamma} & \mathscr{O}_{\alpha\gamma} & \mathscr{O}_{\alpha\gamma} & \cdots & \mathscr{J}_{\alpha\gamma} - \mathscr{I}_{\alpha\gamma} \end{bmatrix}_{\alpha\beta\gamma \times \alpha\beta\gamma}$$

where$\mathscr{J}_{\alpha\beta}$,$\mathscr{I}_{\alpha\beta}$, and $\mathscr{O}_{\alpha\beta}$ denote the all-ones,identity, and zero matrices. Each $\mathscr{J}_{\alpha\gamma} - \mathscr{I}_{\alpha\gamma}$ block is given by

$$\mathscr{J}_{\alpha\gamma} - \mathscr{I}_{\alpha\gamma} = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{\alpha\gamma \times \alpha\gamma}$$

The determinantal polynomial of each non-zero block $\mathscr{J}_{\alpha\gamma} - \mathscr{I}_{\alpha\gamma}$ is

$$[t - (\alpha\gamma - 1)](t+1)^{\alpha\gamma - 1}.$$

Since the matrix$A_d(\mathbf{PiG}_{\mathscr{R}})$ have $o(\mathit{\Upsilon})$ identical blocks, the determinantal polynomial of $A_d(\mathbf{PiG}_{\mathscr{R}})$ is

$$[t - (\alpha\gamma - 1)]^{\beta}(t+1)^{\beta(\alpha\gamma - 1)}.$$

Hence A_d-proper values of $A_d(\mathbf{PiG}_{\mathscr{R}})$ are $\alpha\gamma - 1$ and -1 of multiplicity β and $\beta(\alpha\gamma - 1)$ respectively. Also, the A_d-energy $\mathit{\Omega}_{A_d}(\mathbf{PiG}_{\mathscr{R}}) = 2\beta(\alpha\gamma - 1)$. ☐

The following corollaries are immediate consequences of Theorem 3.1 and Theorem 3.2 in which we give a characterization for the largest A_d-proper value of $\mathbf{PiG}_{\mathscr{L}}$ and $\mathbf{PiG}_{\mathscr{R}}$.

Corollary 3.3. *Suppose $\tau(\mathbf{PiG}_{\mathscr{L}})$ be the largest A_d-proper value of $\mathbf{PiG}_{\mathscr{L}}$. Then $\tau(\mathbf{PiG}_{\mathscr{L}}) \geq 0$ and $\tau(\mathbf{PiG}_{\mathscr{L}}) = o(G) \cdot o(\Upsilon) - 1$. Also the multiplicity of $\tau(\mathbf{PiG}_{\mathscr{L}}) = o(\Delta)$.*

Proof. By Theorem 3.1, $\tau(\mathbf{PiG}_{\mathscr{L}}) = o(G) \cdot o(\Upsilon) - 1$ and the multiplicity of $\tau(\mathbf{PiG}_{\mathscr{L}}) = o(\Delta)$. Since G and Υ are non-empty sets, $o(G) \geq 1$ and $o(\Upsilon) \geq 1$. Hence $\tau(\mathbf{PiG}_{\mathscr{L}}) \geq 0$. ☐

In a similar manner, we can establish the case of $\mathbf{PiG}_{\mathscr{R}}$; thus, the proof is omitted.

Corollary 3.4. *Let $\tau(\mathbf{PiG}_{\mathscr{R}})$ be the largest A_d-proper value of $\mathbf{PiG}_{\mathscr{R}}$. Then $\tau(\mathbf{PiG}_{\mathscr{R}}) \geq 0$ and $\tau(\mathbf{PiG}_{\mathscr{R}}) = o(G) \cdot o(\Delta) - 1$. Also the multiplicity of $\tau(\mathbf{PiG}_{\mathscr{R}}) = o(\Upsilon)$.* ☐

The Laplacian energy of a graph is defined as the sum of the absolute values of the Laplacian matrixAnderson Jr and Morley (1985). Proposition 2.1 and proposition 2.2 respectively suggest the structure of the Laplacian matrices of $\mathbf{PiG}_{\mathscr{L}}$ and $\mathbf{PiG}_{\mathscr{R}}$, which eventually lead to the following two theorems, in which we describe the L_p-energy of $\mathbf{PiG}_{\mathscr{L}}$ and $\mathbf{PiG}_{\mathscr{R}}$.

Theorem 3.5. *For $T = \mathscr{M}(G, \Upsilon, \Delta, \Psi)$, the L_p-energy, $\Omega_{L_p}(\mathbf{PiG}_{\mathscr{L}})$ of the principal left ideal graph is $\alpha\beta\gamma(\alpha\beta - 1)$.*

Proof. $\mathbf{PiG}_{\mathscr{L}}$ is disconnected, having $o(\Delta) = \gamma$ components and each forming a complete graph with $o(G) \cdot o(\Upsilon) = \alpha\beta$ vertices according to proposition 2.1. So the Laplacian matrix $L_p(\mathbf{PiG}_{\mathscr{L}})$ is a block diagonal matrix with $o(\Delta)$ diagonal blocks and each diagonal block is a $o(G) \cdot o(\Upsilon) \times o(G) \cdot o(\Upsilon)$ square matrix.

$$L_p(\mathbf{PiG}_{\mathscr{L}}) = \begin{bmatrix} (\alpha\beta)\mathscr{I}_{\alpha\beta} - \mathscr{J}_{\alpha\beta} & \mathcal{O}_{\alpha\beta} & \dots & \mathcal{O}_{\alpha\beta} \\ \mathcal{O}_{\alpha\beta} & (\alpha\beta)\mathscr{I}_{\alpha\beta} - \mathscr{J}_{\alpha\beta} & \dots & \mathcal{O}_{\alpha\beta} \\ \vdots & \vdots & & \vdots \\ \mathcal{O}_{\alpha\beta} & \mathcal{O}_{\alpha\beta} & \dots & (\alpha\beta)\mathscr{I}_{\alpha\beta} - \mathscr{J}_{\alpha\beta} \end{bmatrix}_{\alpha\beta\gamma \times \alpha\beta\gamma}$$

where

$$(\alpha\beta)\mathscr{I}_{\alpha\beta} - \mathscr{J}_{\alpha\beta} = \begin{bmatrix} \alpha\beta - 1 & -1 & -1 & \cdots & -1 \\ -1 & \alpha\beta - 1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & -1 & -1 & \dots & \alpha\beta - 1 \end{bmatrix}_{\alpha\beta \times \alpha\beta}$$

The determinantal polynomial of each nonzero block $(\alpha\beta)\mathscr{I}_{\alpha\beta} - \mathscr{J}_{\alpha\beta}$ is

$$t(t - \alpha\beta)^{\alpha\beta - 1}.$$

Since the matrix $L_p(\mathbf{PiG}_{\mathscr{L}})$ have $o(\Delta)$ identical blocks, the determinantal equation of $L_p(\mathbf{PiG}_{\mathscr{L}})$ is

$$t^{\gamma}(t - \alpha\beta)^{\gamma(\alpha\beta - 1)} = 0.$$

Hence L_p-proper values of $L_p(\mathbf{PiG}_{\mathscr{L}})$ are $\alpha\beta$ of multiplicity $\gamma(\alpha\beta - 1)$ and 0 of multiplicity γ; and the L_p-energy $\Omega_L(\mathbf{PiG}_{\mathscr{L}}) = \alpha\beta\gamma(\alpha\beta - 1)$. ☐

Theorem 3.6. *In a completely simple semigroup $\mathscr{M}(G, \Upsilon, \Delta, \Psi)$, the L_p-energy $\Omega_{L_p}(\mathbf{PiG}_{\mathscr{R}})$ of the principal right ideal graph is $\alpha\beta\gamma(\alpha\gamma - 1)$.*

Proof. According to 2.2, the Laplacian matrix $L_p(\mathbf{PiG}_{\mathscr{R}})$ is a block diagonal matrix with $o(\Upsilon)$ diagonal blocks and each diagonal block is a $o(G) \cdot o(\Delta) \times o(G) \cdot o(\Delta)$ square matrix.

$$L_p(\mathbf{PiG}_{\mathscr{R}}) = \begin{bmatrix} (\alpha\gamma)\mathscr{I}_{\alpha\gamma} - \mathscr{J}_{\alpha\gamma} & \mathscr{O}_{\alpha\gamma} & \cdots & \mathscr{O}_{\alpha\gamma} \\ \mathscr{O}_{\alpha\gamma} & (\alpha\gamma)\mathscr{I}_{\alpha\gamma} - \mathscr{J}_{\alpha\gamma} & \cdots & \mathscr{O}_{\alpha\gamma} \\ \vdots & \vdots & & \vdots \\ \mathscr{O}_{\alpha\gamma} & \mathscr{O}_{\alpha\gamma} & \cdots & (\alpha\gamma)\mathscr{I}_{\alpha\gamma} - \mathscr{J}_{\alpha\gamma} \end{bmatrix}_{\alpha\beta\gamma \times \alpha\beta\gamma}$$

where

$$(\alpha\gamma)\mathscr{I}_{\alpha\gamma} - \mathscr{J}_{\alpha\gamma} = \begin{bmatrix} \alpha\gamma - 1 & -1 & -1 & \cdots & -1 \\ -1 & \alpha\gamma - 1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & -1 & -1 & \cdots & \alpha\gamma - 1 \end{bmatrix}_{\alpha\gamma \times \alpha\gamma}$$

The determinantal polynomial of each nonzero block $(\alpha\beta)\mathscr{I}_{\alpha\gamma} - \mathscr{J}_{\alpha\gamma}$ is

$$t(t - \alpha\gamma)^{\alpha\gamma - 1}.$$

Since the matrix $L_p(\mathbf{PiG}_{\mathscr{R}})$ have $o(\Upsilon)$ blocks, the determinantal equation of $L_p(\mathbf{PiG}_{\mathscr{R}})$ is

$$t^{\beta}(t - \alpha\gamma)^{\beta(\alpha\gamma - 1)} = 0.$$

Hence L_p-proper values of $L_p(\mathbf{PiG}_{\mathscr{R}})$ are $\alpha\gamma$ of multiplicity $\beta(\alpha\gamma - 1)$ and 0 of multiplicity β; and the L_p-energy $\Omega_L(\mathbf{PiG}_{\mathscr{R}}) = \alpha\beta\gamma(\alpha\gamma - 1)$. ☐

As a consequence of the above two theorems, now we are ready with the characterisation for the largest L_p-proper value of $\mathbf{PiG}_{\mathscr{L}}$ and $\mathbf{PiG}_{\mathscr{R}}$.

Corollary 3.7. *Suppose $\vartheta(\mathbf{PiG}_{\mathscr{L}})$ be the largest L_p-proper value of $\mathbf{PiG}_{\mathscr{L}}$. Then $\vartheta(\mathbf{PiG}_{\mathscr{L}}) \geq 1$ and $\vartheta(\mathbf{PiG}_{\mathscr{L}}) = o(G) \cdot o(\Upsilon)$. Also the multiplicity of the L_p-proper value 0 is $o(\Delta)$.*

Proof. By Theorem 3.5, $\vartheta(\mathbf{PiG}_{\mathscr{L}}) = o(G) \cdot o(\Upsilon)$, and the multiplicity of the L_p-proper value 0 is $o(\Delta)$. Since G and Υ are non-empty sets, $o(G) \geq 1$ and $o(\Upsilon) \geq 1$. Hence $\vartheta(\mathbf{PiG}_{\mathscr{L}}) \geq 1$. ☐

In a similar manner we are able to prove the case of $\mathbf{PiG}_{\mathscr{R}}$. Thus, the proof is let out.

Corollary 3.8. *Let $\vartheta(\mathbf{PiG}_{\mathscr{R}})$ be the largest L_p-proper value of $\mathbf{PiG}_{\mathscr{R}}$, then $\vartheta(\mathbf{PiG}_{\mathscr{R}}) \geq 1$ and $\vartheta(\mathbf{PiG}_{\mathscr{R}}) = o(G) \cdot o(\Delta)$. Also the multiplicity of the L_p-proper value 0 is equal to $o(\Upsilon)$.* ☐

The following Theorems depict the S_l-energy of $\mathbf{PiG}_{\mathscr{L}}$ and $\mathbf{PiG}_{\mathscr{R}}$

Theorem 3.9. *In a completely simple semigroup $\mathscr{M}(G, \Upsilon, \Delta, \Psi)$, the S_l-energy $\Omega_{S_l}(\mathbf{PiG}_{\mathscr{L}})$ of the principal left ideal graph is $2\gamma(\alpha\beta - 1)$.*

Proof. By proposition 2.1, the signless Laplacian matrix $S_l(\mathbf{PiG}_{\mathscr{L}})$ is a block diagonal matrix with $o(\Delta)$ blocks and each block matrix is a $o(G) \cdot o(\Upsilon) \times o(G) \cdot o(\Upsilon)$ square matrix.

$$S_l(\mathbf{PiG}_{\mathscr{L}}) = \begin{bmatrix} (\alpha\beta - 2)\mathscr{I}_{\alpha\beta} + \mathscr{J}_{\alpha\beta} & \mathscr{O}_{\alpha\beta} & \cdots & \mathscr{O}_{\alpha\beta} \\ \mathscr{O}_{\alpha\beta} & (\alpha\beta - 2)\mathscr{I}_{\alpha\beta} + \mathscr{J}_{\alpha\beta} & \cdots & \mathscr{O}_{\alpha\beta} \\ \vdots & \vdots & & \vdots \\ \mathscr{O}_{\alpha\beta} & \mathscr{O}_{\alpha\beta} & \cdots & (\alpha\beta - 2)\mathscr{I}_{\alpha\beta} + \mathscr{J}_{\alpha\beta} \end{bmatrix}_{\alpha\beta\gamma \times \alpha\beta\gamma}$$

where

$$(\alpha\beta-2)\mathscr{I}_{\alpha\beta}+\mathscr{J}_{\alpha\beta}=\begin{bmatrix}\alpha\beta-1 & 1 & 1 & \dots & 1\\ 1 & \alpha\beta-1 & 1 & \dots & 1\\ \vdots & \vdots & \vdots & & \vdots\\ 1 & 1 & 1 & \dots & \alpha\beta-1\end{bmatrix}_{\alpha\beta\times\alpha\beta}$$

The determinantal polynomial of each nonzero diagonal block $(\alpha\beta-2)\mathscr{I}_{\alpha\beta}+\mathscr{J}_{\alpha\beta}$ is

$$t^{\alpha\beta-1}[t-(2\alpha\beta-2)].$$

Since the matrix $S_l(\mathbf{PiG}_{\mathscr{L}})$ consists $o(\varDelta)$ identical blocks, the determinantal equation of $S_l(\mathbf{PiG}_{\mathscr{L}})$ is

$$t^{\alpha\beta\gamma-\gamma}[t-(2\alpha\beta-2)]^{\gamma}=0.$$

Hence S_l-proper value of $S_l(\mathbf{PiG}_{\mathscr{L}})$ are 0 of multiplicity $\gamma(\alpha\beta-1)$ and $2(\alpha\beta-1)$ of multiplicity γ and therefore the S_l-energy $\varOmega_{S_l}(\mathbf{PiG}_{\mathscr{L}})=2\gamma(\alpha\beta-1)$. ☐

Theorem 3.10. *For $T=\mathscr{M}(G,\varUpsilon,\varDelta,\varPsi)$, the S_l-energy $\varOmega_{S_l}(\mathbf{PiG}_{\mathscr{R}})$ of the principal right ideal graph is $2\beta(\alpha\gamma-1)$.*

Proof. By proposition 2.2, the signless Laplacian matrix $S_l(\mathbf{PiG}_{\mathscr{R}})$ is a block diagonal matrix with $o(\varUpsilon)$ identical blocks and each block matrix is a $o(G)\cdot o(\varDelta)\times o(G)\cdot o(\varDelta)$ square matrix.

$$S_l(\mathbf{PiG}_{\mathscr{R}})=\begin{bmatrix}(\alpha\gamma-2)\mathscr{I}_{\alpha\gamma}+\mathscr{J}_{\alpha\gamma} & \mathscr{O}_{\alpha\gamma} & \dots & \mathscr{O}_{\alpha\gamma}\\ \mathscr{O}_{\alpha\gamma} & (\alpha\gamma-2)\mathscr{I}_{\alpha\gamma}+\mathscr{J}_{\alpha\gamma} & \dots & \mathscr{O}_{\alpha\gamma}\\ \vdots & \vdots & & \vdots\\ \mathscr{O}_{\alpha\gamma} & \mathscr{O}_{\alpha\gamma} & \dots & (\alpha\gamma-2)\mathscr{I}_{\alpha\gamma}+\mathscr{J}_{\alpha\gamma}\end{bmatrix}_{\alpha\beta\gamma\times\alpha\beta\gamma}$$

where

$$(\alpha\gamma-2)\mathscr{I}_{\alpha\gamma}+\mathscr{J}_{\alpha\gamma}=\begin{bmatrix}\alpha\gamma-1 & 1 & 1 & \dots & 1\\ 1 & \alpha\gamma-1 & 1 & \dots & 1\\ \vdots & \vdots & \vdots & & \vdots\\ 1 & 1 & 1 & \dots & \alpha\gamma-1\end{bmatrix}_{\alpha\gamma\times\alpha\gamma}$$

The determinantal polynomial of each nonzero diagonal block $(\alpha\gamma-2)\mathscr{I}_{\alpha\gamma}+\mathscr{J}_{\alpha\gamma}$ is

$$t^{\alpha\gamma-1}[t-(2\alpha\gamma-2)].$$

Since the matrix $S_l(\mathbf{PiG}_{\mathscr{R}})$ have $o(\varUpsilon)$ blocks, the determinantal equation of $S_l(\mathbf{PiG}_{\mathscr{R}})$ is

$$t^{\alpha\beta\gamma-\beta}[t-(2\alpha\gamma-2)]^{\beta}=0.$$

Hence S_l-proper value of $S_l(\mathbf{PiG}_{\mathscr{R}})$ are 0 of multiplicity $\beta(\alpha\gamma-1)$ and $2(\alpha\gamma-1)$ of multiplicity β and the S_l-energy $\varOmega_{S_l}(\mathbf{PiG}_{\mathscr{R}})=2\beta(\alpha\gamma-1)$. ☐

As a consequence of the above characterisations for S_l-energies of $\mathbf{PiG}_{\mathscr{L}}$ and $\mathbf{PiG}_{\mathscr{R}}$, we have the folloiwng:

Corollary 3.11. *Let $\omega(\mathbf{PiG}_{\mathscr{L}})$ be the largest S_l-proper value of $\mathbf{PiG}_{\mathscr{L}}$, then $\omega(\mathbf{PiG}_{\mathscr{L}})\geq 0$ and $\omega(\mathbf{PiG}_{\mathscr{L}})=2[o(G)\cdot o(\varUpsilon)-1]$. Also the multiplicity of $\omega(\mathbf{PiG}_{\mathscr{L}})=o(\varDelta)$.*

Proof. By Theorem 3.9, $\omega(\mathbf{PiG}_{\mathscr{L}})=2[o(G)\cdot o(\varUpsilon)-1]$ and the multiplicity of $\omega(\mathbf{PiG}_{\mathscr{L}})=o(\varDelta)$.Since G and $\varUpsilon$ are non-empty sets, $o(G)\geq 1$ and $o(\varUpsilon)\geq 1$. Hence $\omega(\mathbf{PiG}_{\mathscr{L}})\geq 0$. ☐

Likewise we can show the case of $\mathbf{PiG}_{\mathscr{R}}$ and we will state it without providing a proof.

Corollary 3.12. *Let ω $(\mathbf{PiG}_{\mathscr{R}})$ be the largest S_l-proper value of $\mathbf{PiG}_{\mathscr{R}}$, $\omega(\mathbf{PiG}_{\mathscr{R}})\geq 0$ and $\omega(\mathbf{PiG}_{\mathscr{R}})=2[o(G)\cdot o(\varDelta)-1]$. Also the multiplicity of $\omega(\mathbf{PiG}_{\mathscr{R}})=o(\varUpsilon)$.* ☐

3.1 Relationship between different energies

In the previous section, we have characterised the A_d-energy, L_p-energy and S_l-energy of the PiG of the completely simple semigroup. Here we focuses on the connection among A_d-energy, S_l-energy, and L_p-energy of $\mathbf{PiG}_{\mathscr{L}}$ and $\mathbf{PiG}_{\mathscr{R}}$. Consider $\mathscr{M}(G,\Upsilon,\Delta,\Psi)$ with $o(G)=\alpha$, $o(\Upsilon)=\beta$, and $o(\Delta)=\gamma$. Then by Theorem 3.1 the A_d-energy $\Omega_{A_d}(\mathbf{PiG}_{\mathscr{L}})$ is $2\gamma(\alpha\beta-1)$. We obtained the same S_l-energy $\Omega_{S_l}(\mathbf{PiG}_{\mathscr{L}})$ by Theorem 3.9. Similarly, by Theorem 3.2 the A_d-energy $\Omega_{A_d}(\mathbf{PiG}_{\mathscr{R}})$ is $2\beta(\alpha\gamma-1)$. We obtained the same S_l-energy $\Omega_{S_l}(\mathbf{PiG}_{\mathscr{R}})$ by Theorem 3.10. So we have the following theorems.

Theorem 3.13. *For a completely simple semigroup T,*

(i) $\Omega_{A_d}(\mathbf{PiG}_{\mathscr{L}})=\Omega_{S_l}(\mathbf{PiG}_{\mathscr{L}})$.

(ii) $\Omega_{A_d}(\mathbf{PiG}_{\mathscr{R}})=\Omega_{S_l}(\mathbf{PiG}_{\mathscr{R}})$.

Now we state the relationship between L_p-energies and S_l-energies of $\mathbf{PiG}$ of completely simple semigroups.

Theorem 3.14. *For any completely simple semigroup $T=\mathscr{M}(G,\Upsilon,\Delta,\Psi)$,*

$$\Omega_{L_p}(\mathbf{PiG}_{\mathscr{L}})=\frac{o(G)\cdot o(\Upsilon)}{2}\Omega_{S_l}(\mathbf{PiG}_{\mathscr{L}}).$$

Proof. From Theorem 3.5, $\Omega_{L_p}(\mathbf{PiG}_{\mathscr{L}})=\alpha\beta\gamma(\alpha\beta-1)$ and from Theorem 3.9, $\Omega_{S_l}(\mathbf{PiG}_{\mathscr{L}})=2\gamma(\alpha\beta-1)$. Hence $\Omega_{L_p}(\mathbf{PiG}_{\mathscr{L}})=\frac{\alpha\beta}{2}\Omega_{S_l}(\mathbf{PiG}_{\mathscr{L}})=\frac{o(G)\cdot o(\Upsilon)}{2}\Omega_{S_l}(\mathbf{PiG}_{\mathscr{L}})$. ☐

Theorem 3.15. *For any completely simple semigroup $T=\mathscr{M}(G,\Upsilon,\Delta,\Psi)$,*

$$\Omega_{L_p}(\mathbf{PiG}_{\mathscr{R}})=\frac{o(G)\cdot o(\Delta)}{2}\Omega_{S_l}(\mathbf{PiG}_{\mathscr{R}}).$$

Proof. From Theorem 3.6 we have, $\Omega_{L_p}(\mathbf{PiG}_{\mathscr{R}})=\alpha\beta\gamma(\alpha\gamma-1)$ and from Theorem 3.10, $\Omega_{S_l}(\mathbf{PiG}_{\mathscr{R}})=2\beta(\alpha\gamma-1)$. Hence $\Omega_{L_p}(\mathbf{PiG}_{\mathscr{R}})=\frac{\alpha\gamma}{2}\Omega_{S_l}(\mathbf{PiG}_{\mathscr{R}})=\frac{o(G)\cdot o(\Delta)}{2}\Omega_{S_l}(\mathbf{PiG}_{\mathscr{R}})$. ☐

Now we have the specific case when $\Omega_{L_p}(\mathbf{PiG}_{\mathscr{L}}),\Omega_{S_l}(\mathbf{PiG}_{\mathscr{L}})$ are equal.

Theorem 3.16. *If $o(G)=2$ and $o(\Upsilon)=1$ or $o(G)=1$ and $o(\Upsilon)=2$, then*

$$\Omega_{L_p}(\mathbf{PiG}_{\mathscr{L}})=\Omega_{S_l}(\mathbf{PiG}_{\mathscr{L}}).$$

Proof. From Theorem 3.14, we have $\Omega_{L_p}(\mathbf{PiG}_{\mathscr{L}})=\frac{o(G)\cdot o(\Upsilon)}{2}\Omega_{S_l}(\mathbf{PiG}_{\mathscr{L}})$. Hence $o(G)=2$ and $o(\Upsilon)=1$ or $o(G)=1$ and $o(\Upsilon)=2$, then $\Omega_{L_p}(\mathbf{PiG}_{\mathscr{L}})=\Omega_{S_l}(\mathbf{PiG}_{\mathscr{L}})$. ☐

Similar to the case of $\mathbf{PiG}_{\mathscr{L}}$, we can prove the following result.

Theorem 3.17. *If $o(G)=2$ and $o(\Delta)=1$ or $o(G)=1$ and $o(\Delta)=2$, then $\Omega_{L_p}(\mathbf{PiG}_{\mathscr{R}})=\Omega_{S_l}(\mathbf{PiG}_{\mathscr{R}})$.*

Proof. From Corollary 3.15, $\Omega_{L_p}(\mathbf{PiG}_{\mathscr{R}})=\frac{o(G)\cdot o(\Delta)}{2}\Omega_{S_l}(\mathbf{PiG}_{\mathscr{R}})$. Hence $o(G)=2$ and $o(\Delta)=1$ or $o(G)=1$ and $o(\Delta)=2$, then $\Omega_{L_p}(\mathbf{PiG}_{\mathscr{R}})=\Omega_{S_l}(\mathbf{PiG}_{\mathscr{R}})$. ☐

Recall that a semigroup T is said to be a right zero semigroup if and only if $uv=v$ for all $u,v\in T$. Right zero semigroups are completely simple semigroups with $o(G)=o(\Upsilon)=1$. Thus, we can deduce the different energies of right zero semigroups by substituting particular values of α and β in the above theorem.

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Theorem 3.18. *The A_d-energy, L_p-energy, and S_l-energy of PiG vanish for any right zero semigroup.*

Proof. Let $T = \mathscr{M}(G, \Upsilon, \Delta, \Psi)$ be a right zero semigroup with $o(G) = o(\Upsilon) = 1$ and hence we have $\alpha = \beta = 1$. Now by Theorem 3.1, we have $\Omega_{A_d}(\mathbf{PiG}_{\mathscr{L}}) = 2\gamma(\alpha\beta - 1) = 0$. Similarly, by Theorems 3.5 and 3.9, we have $\Omega_{L_p}(\mathbf{PiG}_{\mathscr{L}}) = \alpha\beta\gamma(\alpha\beta - 1) = 0$ and $\Omega_{S_l}(\mathbf{PiG}_{\mathscr{L}}) = 2\gamma(\alpha\beta - 1) = 0,$ $\square$

If $uv = u$ for all u and v in a semigroup T, we say that T is a left zero semigroup. Similar to the case of right zero semigroups, we can prove the following result.

Corollary 3.19. *The A_d-energy, L_p-energy, and S_l-energy of PiG vanish for any left zero semigroup.*

Proof. By Theorem 3.2 $\Omega_{A_d}(\mathbf{PiG}_{\mathscr{R}}) = 2\beta(\alpha\gamma - 1) = 0$. We have, by Theorem 3.6 $\Omega_{L_p}(\mathbf{PiG}_{\mathscr{R}}) = \alpha\beta\gamma(\alpha\gamma - 1) = 0$, and by Theorem 3.10 $\Omega_{S_l}(\mathbf{PiG}_{\mathscr{R}}) = 2m(\alpha\gamma - 1) = 0$. $\square$

4 Conclusions

The present article is devoted to the study of some energies of the PiG of completely simple semigroups. We describe the A_d-energy, L_p-energy and S_l-energy of the PiG, and we establish that A_d-energy and S_l-energy are identical while L_p-energy is $\frac{\alpha\beta}{2}$ times that of S_l-energy. We also see that for left zero and right zero semigroups, all these energies vanish.

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