Analysis of a two-server queue with consultation in Markovian environment

Abstract

In this paper, a two-server queueing model is analysed where consultations in Markovian environment are given by the main server to the regular server. The main server serves customers and also provides consultation to the regular server with a preemptive priority over customer service. The service of the customer at the main server will be interrupted due to higher-priority consultations. The upper bounds for the number of interruptions at the main server and the number of consultations to the regular server are finite. The consultations are arised from M Markovian environmental factors which are related through the transition probability matrix F. The arrival process and consultation requirements are according to mutually independent Poisson processes. The service times at the main server and the regular server are governed by mutually independent phase type distributions. The duration of the threshold clock is exponentially distributed whereas the super clock has a phase-type distribution. The stability condition is established and some performance measures are investigated numerically.

Key words : main server, regular server, consultation, interruption, Markovian environmental factors

1 Introduction

Chakravarthy [3] introduced the concept of a multi-server queueing system in which timely consultations are offered by one of the main servers to the other servers. This well-experienced server is capable of resolving the doubts of the other servers, and thereby ensuring smoother and more efficient service. Such queueing systems can be observed in hospitals, petrol pumps, banks, and similar service environments. Such timely consultations are essential for maintaining a high service quality.

In Krishnamoorthy et.al [8], we can see a single server queueing model with interruptions to the server. The number of interruptions are controlled by a super clock and a finite upper bound. If the super clock realises or the number of interruptions reaches this limit then no further interruptions are allowed to that customer. The service of the customer will be resumed or restarted after each interruption according to the realisation of of a threshold clock.

Queues with service interruptions are first studied by White and Christie [11]. Here the duration of interruption is an exponentially distributed random variable. At the end of the interruption, the service of the customer will be resumed.

Gaver [5] discusses a queueing system with interruption. When the interruption is completed, the service is repeated or resumed, but there is no particular rule to determine the repetition or resumption.

Keilson [7], Ibe and Trivedi [6], Avi-Izhak and Naor [1] and Fiems et. al [4] are some papers which discuss queueing systems with generally distributed service and duration of interruptions.

Bhaskar Senguptha [2] discusses a queueing system in an alternating random environment. The server suffers random breakdown and is unable to serve customers during it's repair. In the period of break down, some of the arriving customers are directed to another service facility.

Krishnamoorthy et.al [9] considers a two-server queueing model with consultations in random environment. This is a service system with a main server and a regular server to which customers arrive according to a Poisson process. There are upper bounds for interruptions to a customer at the main server and that for the consultations to the regular server. The consultations are occurred due to a few random environmental factors. The main server immediately offers consultation to the regular server if either the number of interruptions to the customer at the main server has not reached its upper bound and the super clock has not expired. Otherwise the regular server is forced to wait until the completion of the service of the customer at the main server. The threshold clock determines the restart or resumption of services at both the servers.

In this paper, we consider the consultations are because of some Markovian environmental factors where the factors are related by a transition matrix.

2 Description of model

In this model we consider a queueing system with one main server and one regular server. The customers arrive to the system according to a Poisson process with rate λ. The service times of customers at the main and regular servers follow independent phase type distributions with representations ($\boldsymbol{\alpha}$,V) and ($\boldsymbol{\beta}$,Y) respectively. The number of phases are a and b, respectively. Denote $V^0 = -V\mathbf{e}$ and $Y^0 = -Y\mathbf{e}$ where $\mathbf{e}$ is a column vector of $1's$ of appropriate order. The main server offers consultation to the regular server whenever it is needed. Let $f_1, f_2, ..., f_M$ be M random environmental factors due to which consultations occur. The environmental factors are related to each other by the transition probability matrix F. Requirement of consultation in random environment is a Poisson process with rate θ where the i^{th} factor occurring with probability δ_i, i=1,2,...,M. The duration of consultation for the i^{th} factor is exponentially distributed with parameter ξ_i. The threshold clock determines the restart or resumption of services at both the servers. The duration of threshold clock has exponential distribution with parameter ω.

The upper bound for interruptions to a customer at the main server is K and that for the consultations to the regular server is L. The duration of super clock follows a phase type distribution with representation $(\boldsymbol{\gamma}, G)$ with number of phases c where $\mathbf{G}^0 = -\mathbf{Ge}$. The duration of threshold clock has exponential distribution with parameter η.

Notations :- We use the following notations in this model.

- $K_0 = K(c+1)$ and $K_1 = K_0 + 1$
- $L_1 = LM + 1$, $L_2 = 2LM$
- $D_1 = L_1K_1ab + L_2b + L_2K_0ab + L_2ab$
- $D_0 = K_1a + L_1b + L_2b$
- $\tilde{\gamma} = (\boldsymbol{\gamma}, 0)$, $\tilde{\boldsymbol{\eta}} = (1, 0)$
- $\dot{I}_1 = \begin{bmatrix} \mathbf{0} & I_{K_0} \end{bmatrix}_{K_0 \times K_1}$, $\dot{I}_2 = \begin{bmatrix} \mathbf{0} & I_{LM} \end{bmatrix}_{LM \times L_1}$
- $\tilde{G} = \begin{bmatrix} G & G^0 \\ \underline{\mathbf{0}} & 0 \end{bmatrix}$ and $F^* = \begin{bmatrix} \boldsymbol{\delta} & \underline{\mathbf{0}} \\ O & I_{L-1} \otimes F \\ O & O \end{bmatrix}$
- $\boldsymbol{\xi} = (\xi_1, \xi_2, ..., \xi_K)^{'}$

Consider the queueing model $X = \{X(t), t \geq 0\}$,
where $X(t) = \{N(t), S(t), C_1(t), E(t), C_2(t), V_1(t), V_2(t), Q_1(t), Q_2(t)\}$.

The variables are defined as follows:

$N(t)$	$-$	the number of customers in the system
$C_1(t)$	$-$	number of consultations already received by the regular server during the service of a particular customer
$C_2(t)$	$-$	number of interruptions already befell to a customer at the main server

$V_1(t), Q_1(t)$ and $Q_2(t)$ represent the phases of the super clock the main server and the regular server, respectively.

Here $S(t)$ denotes the status of the servers at time t such that

$$S(t) = \begin{cases} \tilde{0}, & \text{if only the regular server is busy} \\ 0, & \text{if the main together with or without} \\ & \text{the regular server is busy} \\ 1, & \text{if the main server is giving consultation only} \\ 2, & \text{if the main server is giving consultation} \\ & \text{with one interrupted customer at the main server} \\ 3, & \text{if the regular server is waiting for getting consultation} \\ & \text{after the present service at the main server} \end{cases}$$

$V_2(t)$ represents the status of the threshold clock. It take values 0 and 1 corresponding to the expiry and running , respectively of this clock.

The variable $E(t)$ is described as follows:
If the regular server is busy serving a customer after a consultation, then $E(t)$ represents the environmental factor due to which that consultation has occurred; if the regular server is getting a consultation or waiting to get a consultation, then $E(t)$ represents the environmental factor for which that consultation is going on.
Thus $E(t) = i$, where $1 \leq i \leq M$ for the following cases:
(1)$N(t) = 1$ and $S(t) = \tilde{0}$ or $N(t) \geq 2$ and $S(t) = 0$ with $1 \leq C_1(t) \leq L$
(2)$N(t) = 1$ and $S(t) = 1$
(3)$N(t) \geq 2$ and $S(t) = \{1, 2, 3\}$.

$C_1(t)$ is '0' means the regular server has not obtained a consultation yet and so the phases of the environmental factors $E(t)$ do not present when $C_1(t) = 0$.

$\{X(t), t \geq 0\}$ is a Continuous Time Markov Chain with state space

$$\zeta = \{0\} \cup \bigcup_{i=1}^{\infty} \zeta(i).$$

The terms $\zeta(i)$'s are defined as
$\zeta(1) = \zeta(1, 0) \cup \zeta(1, \tilde{0}) \cup \zeta(1, 1)$ and
$\zeta(i) = \zeta(i, 0) \cup \zeta(i, 1) \cup \zeta(i, 2) \cup \zeta(i, 3)$, for $i \geq 2$,
where
$\zeta(1, 0) = \{(1, 0, 0, t_1)\} \cup \{(1, 0, k, l_1, t_1) : 1 \leq k \leq K\}$,
$\zeta(1, \tilde{0}) = \{(1, \tilde{0}, 0, t_2) \cup (1, \tilde{0}, j, l_3, t_2) : 1 \leq j \leq L\}$,
$\zeta(1, 1) = \{(1, 1, j, l_2, l_3, t_2) : 0 \leq j \leq L - 1\}$,
$\zeta(i, 0) = \{(i, 0, 0, 0, t_1, t_2) \cup (i, 0, j, l_3, 0, t_1, t_2) \cup (i, 0, 0, k, l_1, t_1, t_2)$
$\cup (i, 0, j, l_3, k, l_1, t_1, t_2) : 1 \leq j \leq L, 1 \leq k \leq K\}$,
$\zeta(i, 1) = \{(i, 1, j, l_2, l_3, t_2) : 0 \leq j \leq L - 1\}$,
$\zeta(i, 2) = \{(i, 2, j, l_2, l_3, k, l_1, t_1, t_2) : 0 \leq j \leq L - 1, 0 \leq k \leq K - 1\}$,
$\zeta(i, 3) = \{(i, 3, j, l_2, l_3, t_1, t_2) : 0 \leq j \leq L - 1\}$,
for $i \geq 2$ with $0 \leq l_1 \leq c$, $l_2 = \{1, 0\}$, $1 \leq l_3 \leq M$,

$1 \leq t_1 \leq a$ and $1 \leq t_2 \leq b$.
The infinitesimal generator P is given by

$$P = \begin{bmatrix} -\lambda & A_1 & & & & \\ A_2 & A_3 & A_4 & & & \\ & A_5 & B_1 & B_0 & & \\ & & B_2 & B_1 & B_0 & \\ & & & \ddots & \ddots & \ddots \\ & & & & & \end{bmatrix} \tag{1}$$

where

$$A_1 = \lambda \left[\begin{array}{cc} \mathbf{e}^{'}_{K_1}(1) \otimes \boldsymbol{\alpha} & \underline{\mathbf{0}} \end{array} \right], \; A_2 = \begin{bmatrix} \mathbf{e}_{K_1} \otimes V^0 \\ \mathbf{e}_{LM+1} \otimes Y^0 \\ \mathbf{0} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} I_{K_1} \otimes V & O & O \\ O & A_{31} & \theta F^* \otimes \tilde{\boldsymbol{\eta}} \otimes I_b \\ O & \dot{I}_2 \otimes \xi \otimes \Delta & I_L \otimes \nabla \otimes I_b \end{bmatrix} - \lambda I,$$

$$A_4 = \lambda \left[\begin{array}{ccc} A_{41} & A_{42} & O \end{array} \right], \; A_5 = \left[\begin{array}{ccc} A_{51} & A_{52} & A_{53} \end{array} \right],$$

$$B_0 = \lambda I, \; B_1 = \begin{bmatrix} B_{11} & O & B_{12} & B_{13} \\ B_{14} & B_{15} & O & O \\ B_{16} & O & B_{17} & O \\ O & O & O & B_{18} \end{bmatrix} - \lambda I,$$

$$B_2 = \left[\begin{array}{ccc} B_{21} & B_{53} & O \end{array} \right].$$

Order of the square matrices B_0, B_1 and B_2 is D_1;
that of A_3 is D_0. The orders of A_1, A_2, A_4, A_5 are
are $1 \times D_0$, $D_0 \times 1$, $D_0 \times D_1$ and $D_1 \times D_0$, respectively.
The block matrices are described as follows:
$A_{31} = I_{L_1} \otimes Y - \theta I^* \otimes I_b$,

$$A_{41} = \begin{bmatrix} \mathbf{e}^{'}_{L_1}(1) \otimes I_{K_1} \otimes I_a \otimes \boldsymbol{\beta} \\ I_{L_1} \otimes \mathbf{e}^{'}_{K_1}(1) \otimes \boldsymbol{\alpha} \otimes I_b \\ O \end{bmatrix}_{C_0 \otimes K_1 M_1 ab}, \; A_{42} = \begin{bmatrix} O \\ I_{L_2 b} \end{bmatrix}_{D_0 \otimes L_2 b},$$

$$A_{51} = \begin{bmatrix} \mathbf{e}_{L_1} \otimes I_{K_1} \otimes I_a \otimes Y^0 \\ O \end{bmatrix}_{D_1 \otimes K_1 a},$$

$$A_{52} = \begin{bmatrix} I_{L_1} \otimes \mathbf{e}_{K_1} \otimes V^0 \otimes I_b \\ O \end{bmatrix}_{D_1 \otimes L_1 b},$$

$$A_{53} = \begin{bmatrix} O \\ I_{L_2} \otimes V^0 \otimes I_b \end{bmatrix}_{D_1 \otimes L_2 b},$$

$B_{11} = I_{L_1} \otimes I_{K_1} \otimes (V \oplus Y) - \theta I^* \otimes I_{K_1} \otimes I_{ab}$,

$B_{12} = \theta F^* \otimes J \otimes \tilde{\boldsymbol{\eta}} \otimes I_{ab}$, $B_{13} = \theta F^* \otimes J^* \otimes \tilde{\boldsymbol{\eta}} \otimes I_{ab}$,

$B_{14} = \dot{I}_2 \otimes \xi \otimes \tilde{\Delta}$, $B_{15} = I_L \otimes \nabla \otimes I_b$, $B_{16} = \dot{I}_2 \otimes \xi \otimes \dot{I}_1 \otimes \Delta^*$,

$B_{17} = I_L \otimes \tilde{\nabla} \otimes I_{ab}$, $B_{18} = I_{LM} \otimes (H \oplus V) \otimes I_b$,

$$B_{21} = \begin{bmatrix} \tilde{V}^0 + \tilde{Y}^0 \\ O \end{bmatrix}_{D_1 \otimes L_1 K_1 ab}.$$

Here

$$I^* = \begin{bmatrix} I_{(L-1)M+1} & O \\ O & O \end{bmatrix}_{L_1 \otimes L_1},$$

$$J = \begin{bmatrix} diag(\tilde{\boldsymbol{\gamma}}, I_{K-1} \otimes \hat{I}_c) \\ O \end{bmatrix}, \; J^* = \begin{bmatrix} 0 \\ \mathbf{e}_{K-1} \otimes \hat{\mathbf{e}_c} \\ \mathbf{e}_{c+1} \end{bmatrix},$$

$$\tilde{V}^0 = I_{L_1} \otimes \mathbf{e}_{K_1} \otimes V^0 \otimes \boldsymbol{\alpha} \otimes I_b, \; \tilde{Y}^0 = \mathbf{e}_{LM+1} \otimes I_{K_1} \otimes I_a \otimes Y^0 \otimes \boldsymbol{\beta},$$

$$\Delta = \begin{bmatrix} I_b \\ \mathbf{e}_b \otimes \boldsymbol{\beta} \end{bmatrix}, \; H = \omega \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \; \nabla = I_L \otimes (\boldsymbol{\xi} \oplus H) \otimes I_b,$$

$$\tilde{\nabla} = I_L \otimes [\boldsymbol{\xi} \otimes I_{2K_0} + I_{MK} \otimes (\tilde{G} \oplus H)] \otimes I_{ab},$$

$$\tilde{\Delta} = \begin{bmatrix} \mathbf{e}'_{K_1}(1) \otimes \boldsymbol{\alpha} \otimes I_b \\ \mathbf{e}_b \otimes \mathbf{e}'_{K_1}(1) \otimes \boldsymbol{\alpha} \otimes \boldsymbol{\beta} \end{bmatrix}, \; \Delta^* = \begin{bmatrix} I_{ab} \\ \mathbf{e}_{ab} \otimes \boldsymbol{\alpha} \otimes \boldsymbol{\beta} \end{bmatrix}.$$

3 Steady state analysis

In this section we perform the steady-state analysis of the queueing model under study. We first establish the stability condition of the queueing system.

3.1 Stability condition

Let $\boldsymbol{\pi}$ denote the steady-state probability vector of the generator $B_0 + B_1 + B_2$. The LIQBD description of the model indicates that the queueing system is stable (see, Neuts [10]) if and only if

$$\boldsymbol{\pi} B_0 \mathbf{e} < \boldsymbol{\pi} B_2 \mathbf{e}. \tag{2}$$

The vector $\boldsymbol{\pi}$ cannot be obtained explicitly in terms of the parameters of the model.

Define the traffic intensity ρ as

$$\rho = \frac{\boldsymbol{\pi} B_0 \mathbf{e}}{\boldsymbol{\pi} B_2 \mathbf{e}}. \tag{3}$$

Note that the stability condition in equation (2) is equivalent to $\rho < 1$. We will discuss the impact of the input parameters of the model on the traffic intensity in Section 3.6.

3.2 Steady state probability vector

Since the model studied as a QBD process, its steady-state distribution has a matrix-geometric solution under the stability condition. Assume that the stability condition (2) holds. Let $\boldsymbol{p}$ denote the steady-state probability vector

of the generator Q given in equation (1).
Partitioning $\boldsymbol{p}$ as

$$\boldsymbol{p} = (\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, ...) \tag{4}$$

we see that the sub-vectors of $\boldsymbol{p}$, under the assumption that the stability condition (2) holds, are obtained as (see, Neuts [10])

$$\boldsymbol{p}_n = \boldsymbol{p}_2 R^{n-2}, n \geq 3 \tag{5}$$

where R is the minimal non-negative solution to the matrix quadratic equation:

$$R^2 B_2 + RB_1 + B_0 = 0. \tag{6}$$

$\boldsymbol{p}_0$, $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are obtained using the boundary equations

$$\begin{aligned} -\lambda \boldsymbol{p}_0 + \boldsymbol{p}_1 A_2 &= 0 \\ \boldsymbol{p}_0 A_1 + \boldsymbol{p}_1 A_3 + \boldsymbol{p}_2 A_5 &= 0 \\ \boldsymbol{p}_1 A_4 + \boldsymbol{p}_2 (B_1 + RB_2) &= 0 \end{aligned} \tag{7}$$

The normalizing condition results in

$$\boldsymbol{p}_0 + \boldsymbol{p}_1 \mathbf{e} + \boldsymbol{p}_2 (I - R)^{-1} \mathbf{e} = 1. \tag{8}$$

Once the rate matrix R is obtained, the vector $\boldsymbol{p}$ can be computed by exploiting the special structure of the coefficient matrices.

3.3 Performance measures

In this section we list a number of key system performance measures to bring out the qualitative aspects of the model under study. These are listed below along with their formulae for computation.

Towards this end, we further partition the vectors $\boldsymbol{p}_n$ as $\boldsymbol{p}_1 = (\boldsymbol{p}_{10}, \boldsymbol{p}_{1\tilde{0}}, \boldsymbol{p}_{11})$ and $\boldsymbol{p}_n = (\boldsymbol{p}_{n0}, \boldsymbol{p}_{n1}, \boldsymbol{p}_{n2}, \boldsymbol{p}_{n3})$, where $n \geq 2$.

Note that $\boldsymbol{p}_0$ is a scalar, $\boldsymbol{p}_1 = (\boldsymbol{p}_{10}, \boldsymbol{p}_{1\tilde{0}}, \boldsymbol{p}_{11})$ and $\boldsymbol{p}_n = (\boldsymbol{p}_{n0}, \boldsymbol{p}_{n1}, \boldsymbol{p}_{n2}, \boldsymbol{p}_{n3})$, where $n \geq 2$.
Here $\boldsymbol{p}_{10}, \boldsymbol{p}_{1\tilde{0}}, \boldsymbol{p}_{11}, \boldsymbol{p}_{n0}, \boldsymbol{p}_{n1}, \boldsymbol{p}_{n2}, \boldsymbol{p}_{n3}$, for $n \geq 2$ are vectors of dimensions M_1a, K_1b, K_2b, K_1M_1ab, K_2b, M_0K_2ab and K_2ab, respectively.

Now we compute some more performance measures.

(1) Expected number of customers in the system

$$E_1 = \sum_{n=1}^{\infty} n \boldsymbol{p}_n \mathbf{e}. \tag{9}$$

(2) Expected number of customers in the queue

$$E_2 = \sum_{n=2}^{\infty} (n-1) \boldsymbol{p}_{n1} \mathbf{e} + \sum_{n=3}^{\infty} (n-2)(\boldsymbol{p}_{n0} \mathbf{e} + \boldsymbol{p}_{n2} \mathbf{e} + \boldsymbol{p}_{n3} \mathbf{e}). \tag{10}$$

(3) Effective rate of consultation

$$E_3 = \theta \sum_{m=0}^{K-1} \boldsymbol{p}_{1\tilde{0}m}\mathbf{e} + \theta \sum_{n=2}^{\infty}\sum_{m=0}^{K-1} \boldsymbol{p}_{n0m}\mathbf{e}. \quad (11)$$

(4) Effective rate of interruption

$$E_4 = \theta \sum_{n=2}^{\infty}\sum_{m=0}^{K-1} \boldsymbol{p}_{n0m0}\mathbf{e} + \theta \sum_{n=2}^{\infty}\sum_{m=0}^{K-1}\sum_{k=1}^{M-1}\sum_{l_1=1}^{c} \boldsymbol{p}_{n0mkl_1}\mathbf{e} \quad (12)$$

(5) Fraction of time the main server is idle

$$\Gamma_{mi} = \boldsymbol{p}_0\mathbf{e} + \boldsymbol{p}_{1\tilde{0}}\mathbf{e}. \quad (13)$$

(6) Fraction of time the regular server is idle

$$\Gamma_{ri} = \boldsymbol{p}_0\mathbf{e} + \boldsymbol{p}_{10}\mathbf{e}. \quad (14)$$

(7) Fraction of time the main server is busy serving a customer

$$\Gamma_{mb} = \boldsymbol{p}_{10}\mathbf{e} + \sum_{n=2}^{\infty} \boldsymbol{p}_{n0}\mathbf{e} + \sum_{n=2}^{\infty} \boldsymbol{p}_{n3}\mathbf{e}. \quad (15)$$

(8) Fraction of time the regular server is busy serving a customer

$$\Gamma_{rb} = \boldsymbol{p}_{1\tilde{0}}\mathbf{e} + \sum_{n=2}^{\infty} \boldsymbol{p}_{n0}\mathbf{e}. \quad (16)$$

(9) Fraction of time regular server is getting consultation

$$\Gamma_{rc} = \sum_{n=1}^{\infty} \boldsymbol{p}_{n1}\mathbf{e} + \sum_{n=2}^{\infty} \boldsymbol{p}_{n2}\mathbf{e}. \quad (17)$$

(10) Fraction of time regular server is waiting to get consultation

$$\Gamma_{wc} = \sum_{n=2}^{\infty} \boldsymbol{p}_{n3}\mathbf{e}. \quad (18)$$

(11) Fraction of time main server remains interrupted

$$\Gamma_{min} = \sum_{n=2}^{\infty} \boldsymbol{p}_{n2}\mathbf{e}. \quad (19)$$

4 Numerical examples

Let $Y = \begin{bmatrix} -12 & 6 \\ 5 & -10 \end{bmatrix}$; $V = \begin{bmatrix} -9 & 3 \\ 2 & -8 \end{bmatrix}$; $G = \begin{bmatrix} -12 & 8 \\ 8 & -12 \end{bmatrix}$;

$\boldsymbol{\alpha} = \begin{bmatrix} 0.3 & 0.7 \end{bmatrix}$; $\boldsymbol{\beta} = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}$; $\boldsymbol{\gamma} = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix}$; $\boldsymbol{\delta} = \begin{bmatrix} 0.3 & 0.5 & 0.2 \end{bmatrix}$;

$F = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.4 & 0.5 & 0.1 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$; $\boldsymbol{\xi} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$; $\omega = 2$; $K = 3$; $L = 3$.

Note that, for the above matrices, vectors and values, the stability condition $\rho < 1$ holds.

Table 1: Effect of θ on the various performance measures

$\lambda = 3$

θ	3	3.5	4	4.5	5
ρ	0.7211	0.7849	0.8445	0.9001	0.9519
E_1	4.4827	5.9070	7.6951	9.9176	12.6401
E_2	3.3316	4.5717	6.0350	7.6359	9.2266
E_3	0.3886	0.4544	0.5156	0.5706	0.6186
E_4	0.5579	0.6477	0.7288	0.7971	0.8482
Γ_{mi}	0.3388	0.2896	0.2446	0.2045	0.1696
Γ_{ri}	0.4435	0.3799	0.3214	0.2692	0.2236
Γ_{mb}	0.3135	0.3053	0.2962	0.2859	0.2737
Γ_{rb}	0.1947	0.1969	0.1972	0.1981	0.1993
Γ_{min}	0.2461	0.2882	0.3263	0.3580	0.3811
Γ_{rc}	0.3471	0.4030	0.4532	0.4950	0.5256
Γ_{rw}	0.0140	0.0181	0.0222	0.0261	0.0294

From Table 1 it is observed that as the consultation rate θ increases, then traffic intensity ρ will increase and consequently E_3 and E_4 will increase. Thus, there is an increase in Γ_{min} and Γ_{rc}. As θ increases, there are more frequent consultations, so the attainment of the upper bounds of number of interruptions to the customer at the main server or the realisation of super clock happens sooner. Thus the main server is compelled to complete the service of his customer before further consultations. This results in greater waiting time of the regular server to receive consultation. Thus Γ_{rw} increases. Also, the possibility of the restart of the service at the regular server increases and hence Γ_{rb} increases. Since there are increse in Γ_{min}, Γ_{rc} and Γ_{rw} , the customers are forced to be in the system and in the queue for a longer time and there will be an increase in E_1 and E_2. So the idle times Γ_{mi} and Γ_{ri} of the main server and the regular server, respectively decrease. Also note that, the main server needs to spend more time in consultation, it gets a lesser time to serve customers. Therefore Γ_{mb} decreases.

Table 2 shows that as the arrival rate λ increases, then the traffic intensity ρ increases. There are more customers in the system and so accumulation of customers increases. So E_1 and E_2 increase. As a result, E_3 and E_4 will also increase. Thus there is a hike in Γ_{min} and Γ_{rc}. Hence Γ_{rw} also increases. Since the arrival rate is increased and the queue is equipped with more customers, the service time of the servers increases. Thus Γ_{mb} and Γ_{rb} show an increase. This

Table 2: Effect of λ on the various performance measures

$\theta = 2$

λ	3	3.5	4	4.5	5
ρ	0.5808	0.6775	0.7743	0.8711	0.9679
E_1	2.4768	3.9794	6.2527	9.6005	14.3406
E_2	1.5690	2.8554	4.7760	7.2039	9.6201
E_3	0.2504	0.3313	0.4116	0.4848	0.5465
E_4	0.3657	0.4607	0.5479	0.6166	0.6544
Γ_{mi}	0.4446	0.3535	0.2716	0.2024	0.1471
Γ_{ri}	0.5796	0.4695	0.3662	0.2763	0.2030
Γ_{mb}	0.3282	0.3599	0.3853	0.4015	0.4036
Γ_{rb}	0.1865	0.2349	0.2794	0.3143	0.3334
Γ_{min}	0.1573	0.2079	0.2569	0.2969	0.3199
Γ_{rc}	0.2272	0.2861	0.3397	0.3805	0.4004
Γ_{rw}	0.0067	0.0090	0.0113	0.0131	0.0142

results in a decrease in Γ_{mi} and Γ_{ri}.

Conclusion

In this paper we analyse a queueing model having two servers with consultations in Markovian environment. The consultation is provided by the main server to the regular server. The main server serves customers and also offers consultation to the regular server with a preemptive priority over customers. The number of interruptions are controlled by an upper bound and a super clock. The consultations are due to Markovian environmental factors where the environmental factors are related to each other by the transition probability matrix. We establish the stability condition and study some performance measures numerically.

References

[1] *Avi-Itzhak B, Naor P* : Some queueing problems with the service station subject to breakdowns, Oper. Res. 11(3), 303-320, 1963.

[2] *Bhaskar Sengupta*: A queue with service interruptions in an alternating random environment, Oper. Res., 38(2), 308-318, 1990.

[3] *Chakravarthy S. R.* : A multi-server queueing model with server consultations, Eur.J. of Oper. Res., 233(3), 625-639, 2014.

[4] *Fiems D., Maertens T., Bruneel H.* : Queueing systems with different types of interruptions, Eur J Oper Res 188(3), 838-845, 2008.

[5] *Gaver D. P., Jacobs P. A., Latouche G.* : Finite birth and death models in randomly changing environments, Advances in Applied Probability, 16(4), 715-731, 1984.

[6] *Ibe O. C., Trivedi K. S.*: Two queues with alternating service and server breakdown, Queueing Systems 7(3), 253-268, 1960.

[7] *Keilson J.* : Queues subject to service interruptions, The Annals of Mathematical Statistics 33(4), 1314-1322, 1962.

[8] *Krishnamoorthy A., Pramod P. K., Chakravarthy S. R.* : A Note on characterizing service interruptions with phase type distribution, Stochastic Analysis and Applications , 31(4), 671-683, 2013.

[9] *Krishnamoorthy A., Resmi T., Lakshmy B.* : On a Two-Server Queue with Consultation in Random Environment, Information Technologies and Mathematical Modelling., Queueing Theory and Applications. Communications in Computer and Information Science, 1109, 21-229, 2019.

[10] *Neuts M.F.* : Matrix-geometric solutions in stochastic models, An Algorithmic Approach, The Johns Hopkins University Press, Baltimore, 1981.

[11] *White H., Christie L. S.* : Queuing with Preemptive Priorities or with Breakdown, Operations Research, 6, 79–95, 1958