Generalized Lyapunov and Hartman–Wintner Inequalities for Mixed Fractional Differential Equations with Advanced Fractional Operators

Abstract

This study develops new forms of Lyapunov- and Hartman–Wintner-type inequalities for boundary value problems governed by mixed fractional differential equations. By extending classical inequalities to fractional settings, we establish general criteria under which nontrivial solutions exist, focusing on operators that capture nonlocal memory and hereditary effects. In particular, the framework accommodates tempered derivatives, non-singular kernels such as Atangana–Baleanu and Caputo–Fabrizio, as well as variable-order operators. Through careful construction of Green's functions and the application of fixed-point principles in cones, we derive existence results and explicit inequality bounds that generalize well-known classical cases. Several illustrative examples are discussed to show how the new inequalities adapt to different operator choices. The analysis not only strengthens the theoretical understanding of fractional models but also provides a foundation for numerical studies and future extensions to multipoint and Robin-type boundary conditions.

Keywords: Fractional boundary value problems; Lyapunov inequality; Hartman–Wintner inequality; Tempered derivative; Atangana–Baleanu derivative; Caputo–Fabrizio derivative; Variable-order operator; Green's function; Fixed-point theorem.

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1 Introduction

Fractional differential equations provide a flexible framework for modeling systems with memory and nonlocal effects, extending beyond the scope of classical integer-order models. Their applications span physics, engineering, biology, economics, and control theory [1, 2, 8, 11]. Within this setting, inequalities of Lyapunov and Hartman–Wintner type play a central role: they supply conditions for the existence of nontrivial solutions, offer eigenvalue estimates, and describe oscillatory behavior of boundary value problems [5–7].

While such inequalities have been extended to Caputo and Riemann–Liouville operators [9, 12], modern fractional calculus introduces richer operators that remain under-explored in this context. Tempered derivatives introduce exponential decay for fading memory [13]; non-singular kernels such as Atangana–Baleanu and Caputo–Fabrizio remove singular behavior near the origin [3, 4, 10, 15, 18]; and variable-order operators capture dynamics that evolve with time or space [14, 19]. Current literature often treats these cases separately, leaving a gap in unifying inequalities across operator types [17–19].

The purpose of this work is to develop generalized Lyapunov and Hartman–Wintner inequalities for mixed fractional boundary value problems governed by advanced operators. Using Green's function constructions and fixed-point theorems in cones, we establish existence of positive solutions and derive explicit inequality bounds that specialize to classical cases in the integer-order limit.

2 Preliminaries and Definitions

We work on $[a, b] \subset \mathbb{R}$ with $a < b$ and functions sufficiently smooth for the stated integrals to exist.

2.1 Riemann–Liouville and Caputo operators

For $\alpha > 0$, the left Riemann–Liouville fractional integral is

$$(I_{a+}^{\alpha} u)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} u(t)\, dt, \quad x \in (a, b]. \tag{2.1}$$

The left Riemann–Liouville derivative of order α is

$$(D_{a+}^{\alpha} u)(x) = \frac{d^n}{dx^n} \big(I_{a+}^{n-\alpha} u\big)(x), \qquad n = \lceil \alpha \rceil. \tag{2.2}$$

For $\alpha \in (n-1, n)$, the Caputo derivative is

$$({}^{C}D_{a+}^{\alpha} u)(x) = \big(I_{a+}^{n-\alpha} u^{(n)}\big)(x). \tag{2.3}$$

2.2 Tempered operators

For $\alpha > 0$ and tempering parameter $\lambda > 0$,

$$(D_{a+}^{\alpha,\lambda} u)(x) = e^{-\lambda x} \big(D_{a+}^{\alpha}(e^{\lambda \cdot} u)\big)(x). \tag{2.4}$$

2.3 Non-singular operators (CF and ABC)

For $0 < \alpha < 1$, the Caputo–Fabrizio operator is

$$({}^{CF}D_{a+}^{\alpha} u)(x) = \frac{M(\alpha)}{1-\alpha} \int_a^x \exp\Big(-\tfrac{\alpha}{1-\alpha}(x-t)\Big) u'(t)\, dt. \tag{2.5}$$

The Atangana–Baleanu operator in Caputo sense is

$$({}^{ABC}D_{a+}^{\alpha} u)(x) = \frac{B(\alpha)}{1-\alpha} \int_a^x E_{\alpha}\Big(-\tfrac{\alpha}{1-\alpha}(x-t)^{\alpha}\Big) u'(t)\, dt, \tag{2.6}$$

with E_{α} the Mittag–Leffler function.

2.4 Variable-order operator

Let $\alpha(\cdot) \in C([a,b];(0,1))$. A common left VO Riemann–Liouville derivative is

$$(D_{a+}^{\alpha(\cdot)}u)(x) = \frac{d}{dx}\int_a^x \frac{(x-t)^{-\alpha(x)}}{\Gamma(1-\alpha(x))}\,u(t)\,dt. \tag{2.7}$$

3 Green's Function Construction and Properties

We study the mixed problem

$$D_{a+}^{\alpha}\big({}^{C}D_{b-}^{\alpha}u\big)(x) - q(x)f(u(x)) = 0, \quad u(a)=u(b)=0, \quad \tfrac{1}{2} < \alpha \le 1. \tag{3.1}$$

For $f(u) \equiv u$, solutions admit

$$u(x) = \int_a^b G(x,t)\,q(t)u(t)\,dt. \tag{3.2}$$

One obtains the explicit Green's function

$$G(x,t) = \begin{cases} \dfrac{(x-a)^{\alpha}(b-t)^{\alpha}}{\Gamma(\alpha)\Gamma(\alpha+1)(b-a)^{2\alpha}}, & a \le t \le x \le b, \\ \dfrac{(t-a)^{\alpha}(b-x)^{\alpha}}{\Gamma(\alpha)\Gamma(\alpha+1)(b-a)^{2\alpha}}, & a \le x < t \le b. \end{cases} \tag{3.3}$$

xtfor all (x,t)∈ $[a,b]\times[a,b]$. This completes the proof.

Lemma 3.1 (Positivity of the Green's function)**.** *Let $G(x,t)$ be defined by* (3.3)*. Then*

$$G(x,t) \ge 0, \qquad \forall (x,t) \in [a,b]\times[a,b]. \tag{3.4}$$

Proof. Let $(x,t) \in [a,b]\times[a,b]$ be arbitrary. The Green's function $G(x,t)$ is given by

$$G(x,t) = \begin{cases} \dfrac{(x-a)^{\alpha}(b-t)^{\alpha}}{\Gamma(\alpha)\Gamma(\alpha+1)(b-a)^{2\alpha}}, & a \le t \le x \le b, \\ \dfrac{(t-a)^{\alpha}(b-x)^{\alpha}}{\Gamma(\alpha)\Gamma(\alpha+1)(b-a)^{2\alpha}}, & a \le x < t \le b. \end{cases} \tag{3.5}$$

Since $a \le x, t \le b$ and $\alpha > 0$, we have

$$(x-a)^{\alpha} \ge 0, \quad (t-a)^{\alpha} \ge 0, \quad (b-x)^{\alpha} \ge 0, \quad (b-t)^{\alpha} \ge 0. \tag{3.6}$$

Moreover, the Gamma function is strictly positive on $(0,\infty)$, hence

$$\Gamma(\alpha) > 0, \qquad \Gamma(\alpha+1) > 0. \tag{3.7}$$

Finally,

$$(b-a)^{2\alpha} > 0. \tag{3.8}$$

Therefore,

$$G(x,t) \ge 0 \quad \text{for all } (x,t) \in [a,b]\times[a,b]. \tag{3.9}$$

This completes the proof. □

Lemma 3.2 (Diagonal bound)**.** *For $t \in [a, b]$,*

$$G(t,t) = \frac{(t-a)^\alpha (b-t)^\alpha}{\Gamma(\alpha)\Gamma(\alpha+1)(b-a)^{2\alpha}}. \tag{3.10}$$

Moreover,

$$0 < G(t,t) \le \frac{1}{\Gamma^2(\alpha+1)} \left(\frac{b-a}{2}\right)^{2\alpha}, \tag{3.11}$$

with equality at $t = \frac{a+b}{2}$.

Proof. Setting $x = t$ in (3.3) gives (3.10). Now define $\phi(t) = (t-a)^\alpha (b-t)^\alpha$ for $t \in [a, b]$. This product is symmetric about $(a+b)/2$, and a direct derivative test shows that ϕ attains its maximum at $t = (a+b)/2$, with

$$\max_{t\in[a,b]} \phi(t) = \left(\tfrac{b-a}{2}\right)^{2\alpha}. \tag{3.12}$$

Substituting (3.12) into (3.10) gives

$$G(t,t) \le \frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)} \left(\tfrac{b-a}{2}\right)^{2\alpha} (b-a)^{-2\alpha}. \tag{3.13}$$

Since $\Gamma(\alpha)\Gamma(\alpha+1) \ge \Gamma^2(\alpha+1)$, a looser but simple bound is

$$G(t,t) \le \frac{1}{\Gamma^2(\alpha+1)} \left(\tfrac{b-a}{2}\right)^{2\alpha}, \tag{3.14}$$

which yields (3.11). Equality holds at $t = (a+b)/2$. □

Theorem 3.3 (Uniform bound for the Green's function)**.** *There exists a positive constant $C_\alpha > 0$, depending only on α and the length $b-a$, such that*

$$0 < G(x,t) \le C_\alpha, \qquad \forall (x,t) \in [a,b] \times [a,b], \tag{3.15}$$

where one admissible choice of the constant is

$$C_\alpha = \frac{1}{\Gamma^2(\alpha+1)} \left(\frac{b-a}{2}\right)^{2\alpha}. \tag{3.16}$$

Proof. From Lemma 3.1, the Green's function $G(x,t)$ is nonnegative on the square $[a,b] \times [a,b]$. Hence it suffices to establish a uniform upper bound.

Recall that $G(x,t)$ is defined by

$$G(x,t) = \begin{cases} \dfrac{(x-a)^\alpha (b-t)^\alpha}{\Gamma(\alpha)\Gamma(\alpha+1)(b-a)^{2\alpha}}, & a \le t \le x \le b, \\ \dfrac{(t-a)^\alpha (b-x)^\alpha}{\Gamma(\alpha)\Gamma(\alpha+1)(b-a)^{2\alpha}}, & a \le x < t \le b. \end{cases} \tag{3.17}$$

Define

$$\psi(s) = (s-a)^\alpha (b-s)^\alpha, \qquad s \in [a,b]. \tag{3.18}$$

By Lemma 3.2,

$$\max_{s\in[a,b]} \psi(s) = \left(\frac{b-a}{2}\right)^{2\alpha}. \tag{3.19}$$

Consequently,

$$(x-a)^{\alpha}(b-t)^{\alpha} \le \left(\frac{b-a}{2}\right)^{2\alpha}, \qquad (t-a)^{\alpha}(b-x)^{\alpha} \le \left(\frac{b-a}{2}\right)^{2\alpha}. \tag{3.20}$$

Substituting into (3.17) gives

$$G(x,t) \le \frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)} \frac{\left(\frac{b-a}{2}\right)^{2\alpha}}{(b-a)^{2\alpha}}. \tag{3.21}$$

Using

$$\Gamma(\alpha)\Gamma(\alpha+1) \ge \Gamma^2(\alpha+1), \tag{3.22}$$

we obtain

$$G(x,t) \le \frac{1}{\Gamma^2(\alpha+1)} \left(\frac{b-a}{2}\right)^{2\alpha} =: C_\alpha. \tag{3.23}$$

This completes the proof. □

4 Existence of Positive Solutions

Consider

$$D^{\alpha}_{a+}\left({}^{C}D^{\alpha}_{b-}u\right)(x) - q(x)f(u(x)) = 0, \quad u(a) = u(b) = 0, \; \tfrac{1}{2} < \alpha \le 1. \tag{4.1}$$

Work in the Banach space $C([a,b],\mathbb{R})$ with norm

$$\|u\|_\infty = \max_{x\in[a,b]} |u(x)|. \tag{4.2}$$

Define the cone $K = \{u \in C([a,b],\mathbb{R}) : u \ge 0,\; u \not\equiv 0\}$ and the operator

$$(Tu)(x) = \int_a^b G(x,t)\, q(t)\, f(u(t))\, dt. \tag{4.3}$$

By Lemma 3.1, $T(K) \subseteq K$.

Lemma 4.1 (Compactness)**.** *T is completely continuous on $C([a,b],\mathbb{R})$.*

Proof. Let $B_R = \{u \in C([a,b],\mathbb{R}) : \|u\|_\infty \le R\}$. Continuity of q and f gives

$$|q(t)f(u(t))| \le M_R. \tag{4.4}$$

Then, using Theorem 3.3,

$$|(Tu)(x)| \le M_R \int_a^b G(x,t)\, dt \le M_R C_\alpha (b-a), \tag{4.5}$$

so $\{Tu : u \in B_R\}$ is uniformly bounded. Continuity of G on the compact square implies equicontinuity via dominated convergence. Arzelà–Ascoli yields relative compactness. □

Assume the standard growth conditions:

$$\text{(F1)} \quad \exists r_1 > 0,\ m_1 > 0: \ f(u) \le m_1 u \text{ for } u \in [0, r_1], \tag{4.6}$$
$$\text{(F2)} \quad \exists r_2 > r_1,\ m_2 > 0: \ f(u) \ge m_2 u \text{ for } u \in [r_2, \infty). \tag{4.7}$$

Theorem 4.2 (Existence)**.** *Let $q \ge 0$, $q \not\equiv 0$, and suppose* (4.6)–(4.7) *hold. Then* (4.1) *has a positive solution $u \in K$.*

Proof. Set the balls

$$\Omega_1 = \{u : \|u\|_\infty \le r_1\}, \tag{4.8}$$

and

$$\Omega_2 = \{u : \|u\|_\infty \le r_2\}. \tag{4.9}$$

For $u \in K \cap \partial\Omega_1$,

$$\|Tu\|_\infty \le m_1 \|u\|_\infty \max_x \int_a^b G(x,t)q(t)\,dt, \tag{4.10}$$

and choosing m_1 small enough makes T a compression on $\partial\Omega_1$. For $u \in K \cap \partial\Omega_2$,

$$\|Tu\|_\infty \ge m_2 \|u\|_\infty \min_x \int_a^b G(x,t)q(t)\,dt, \tag{4.11}$$

so for large enough m_2, T is an expansion on $\partial\Omega_2$. By Lemma 4.1, T is completely continuous; thus Guo–Krasnosel'skiĭ yields a fixed point in K. □

The same argument applies to tempered, CF/ABC, and VO operators, with constants adapted to their kernels.

5 Lyapunov- and Hartman–Wintner-Type Inequalities

Consider the linear problem

$$D^\alpha_{a+}\big({}^C D^\alpha_{b-} u\big)(x) + q(x)u(x) = 0, \quad u(a) = u(b) = 0. \tag{5.1}$$

Theorem 5.1 (Lyapunov-type inequality)**.** *If the boundary value problem*

$$D^\alpha_{a+}\big({}^C D^\alpha_{b-} u\big)(x) + q(x)u(x) = 0, \qquad u(a) = u(b) = 0, \tag{5.2}$$

admits a nontrivial solution u, then

$$\int_a^b |q(t)|\,dt \ge \frac{\Gamma^2(\alpha+1)}{\left(\dfrac{b-a}{2}\right)^{2\alpha}}. \tag{5.3}$$

Proof. Let $u \not\equiv 0$ be a solution of the boundary value problem. By the Green's function representation derived in Section 3, u satisfies

$$u(x) = -\int_a^b G(x,t)\,q(t)\,u(t)\,dt, \qquad x \in [a,b]. \tag{5.4}$$

Taking absolute values and using the positivity of $G(x,t)$ (Lemma 3.1), we obtain

$$|u(x)| \le \int_a^b G(x,t)\,|q(t)|\,|u(t)|\,dt. \tag{5.5}$$

Using the definition of the supremum norm $\|u\|_\infty$, it follows that

$$|u(x)| \le \|u\|_\infty \int_a^b G(x,t)\,|q(t)|\,dt. \tag{5.6}$$

Taking the maximum over $x \in [a,b]$ gives

$$\|u\|_\infty \le \|u\|_\infty \sup_{x\in[a,b]} \int_a^b G(x,t)\,|q(t)|\,dt. \tag{5.7}$$

Since u is nontrivial, $\|u\|_\infty > 0$, and hence

$$1 \le \sup_{x\in[a,b]} \int_a^b G(x,t)\,|q(t)|\,dt. \tag{5.8}$$

By Theorem 3.3, the Green's function satisfies the uniform bound

$$0 \le G(x,t) \le C_\alpha, \qquad C_\alpha = \frac{1}{\Gamma^2(\alpha+1)}\left(\frac{b-a}{2}\right)^{2\alpha}. \tag{5.9}$$

Therefore,

$$\sup_{x\in[a,b]} \int_a^b G(x,t)\,|q(t)|\,dt \le C_\alpha \int_a^b |q(t)|\,dt. \tag{5.10}$$

Combining the above inequalities yields

$$1 \le C_\alpha \int_a^b |q(t)|\,dt, \tag{5.11}$$

which implies

$$\int_a^b |q(t)|\,dt \ge \frac{1}{C_\alpha} = \frac{\Gamma^2(\alpha+1)}{\left(\dfrac{b-a}{2}\right)^{2\alpha}}. \tag{5.12}$$

This completes the proof. □

Theorem 5.2 (Hartman–Wintner-type inequality)**.** *If problem* (5.1) *admits a nontrivial solution* u*, then*

$$\int_a^b |q(t)|\,|u(t)|\,dt \ge \frac{\Gamma^2(\alpha+1)}{\left(\dfrac{b-a}{2}\right)^{2\alpha}}\|u\|_\infty. \tag{5.13}$$

Proof. Let $u \not\equiv 0$ be a solution of (5.1). Using the Green's function representation, we have

$$u(x) = -\int_a^b G(x,t)\,q(t)\,u(t)\,dt. \tag{5.14}$$

Taking absolute values and applying Lemma 3.1 yields

$$|u(x)| \le \int_a^b G(x,t)\,|q(t)|\,|u(t)|\,dt, \qquad x \in [a,b]. \tag{5.15}$$

Let $x^* \in [a,b]$ be such that $|u(x^*)| = \|u\|_\infty$. Then

$$\|u\|_\infty \le \int_a^b G(x^*,t)\,|q(t)|\,|u(t)|\,dt. \tag{5.16}$$

Invoking the uniform bound $G(x,t) \le C_\alpha$ from Theorem 3.3, we obtain

$$\|u\|_\infty \le C_\alpha \int_a^b |q(t)|\,|u(t)|\,dt. \tag{5.17}$$

Substituting

$$C_\alpha = \frac{1}{\Gamma^2(\alpha+1)}\left(\frac{b-a}{2}\right)^{2\alpha} \tag{5.18}$$

and rearranging completes the proof. □

For tempered/CF/ABC/VO operators, the same proofs carry through with constants modified by the kernel bounds; e.g., tempered yields an extra factor depending on λ.

6 Eigenvalue Problems and Multiplicity Results

Consider the eigenvalue problem

$$D^\alpha_{a+}\big({}^C D^\alpha_{b-} u\big)(x) + \lambda\, q(x)u(x) = 0, \qquad u(a) = u(b) = 0. \tag{6.1}$$

Theorem 6.1 (Lower bound on the first eigenvalue)**.** *Let $q \ge 0$, $q \not\equiv 0$, and denote by $\lambda_1 > 0$ the smallest eigenvalue of* (6.1)*. Then*

$$\lambda_1 \;\ge\; \frac{\Gamma^2(\alpha+1)}{\left(\frac{b-a}{2}\right)^{2\alpha} \displaystyle\int_a^b q(t)\,dt}. \tag{6.2}$$

Proof. Let $u \not\equiv 0$ solve (6.1) with $\lambda = \lambda_1$. By the Green's representation (cf. Section 3), we have

$$u(x) \;=\; -\,\lambda_1 \int_a^b G(x,t)\,q(t)\,u(t)\,dt, \qquad x \in [a,b]. \tag{6.3}$$

Taking absolute values and using $G \ge 0$ (Lemma 3.1) yields

$$|u(x)| \;\le\; \lambda_1 \int_a^b G(x,t)\,q(t)\,|u(t)|\,dt \;\le\; \lambda_1\,\|u\|_\infty \Big(\sup_{x\in[a,b]} \int_a^b G(x,t)\,q(t)\,dt\Big). \tag{6.4}$$

Maximizing over x and cancelling $\|u\|_\infty > 0$ gives

$$1 \;\le\; \lambda_1 \sup_{x\in[a,b]} \int_a^b G(x,t)\,q(t)\,dt. \tag{6.5}$$

By Theorem 3.3, $G(x,t) \le C_\alpha$ for all (x,t), hence

$$\sup_{x\in[a,b]} \int_a^b G(x,t)\,q(t)\,dt \;\le\; C_\alpha \int_a^b q(t)\,dt. \tag{6.6}$$

Combining (6.5) and (6.6) yields

$$1 \;\le\; \lambda_1\, C_\alpha \int_a^b q(t)\,dt \quad\Longrightarrow\quad \lambda_1 \;\ge\; \frac{1}{C_\alpha \displaystyle\int_a^b q(t)\,dt}. \tag{6.7}$$

Substituting

$$C_\alpha = \frac{1}{\Gamma^2(\alpha+1)} \left(\tfrac{b-a}{2}\right)^{2\alpha} \tag{6.8}$$

gives (6.2). □

We now address multiplicity for the nonlinear problem

$$D^\alpha_{a+}\big({}^C D^\alpha_{b-} u\big)(x) + q(x) f\big(u(x)\big) = 0, \qquad u(a) = u(b) = 0, \tag{6.9}$$

under the standard sublinear/superlinear conditions

$$\frac{f(u)}{u} \to 0 \quad \text{as } u \to 0^+, \qquad \frac{f(u)}{u} \to \infty \quad \text{as } u \to \infty. \tag{6.10}$$

Define the ordered Banach space

$$X := C([a,b],\mathbb{R}), \qquad \|u\|_\infty = \max_{x\in[a,b]} |u(x)| \tag{6.11}$$

and the cone

$$K \;:=\; \{\, u \in X : \; u(x) \ge 0 \text{ for all } x \in [a,b] \,\}. \tag{6.12}$$

Consider the completely continuous operator $T : K \to K$ (Lemma 4.1) given by

$$(Tu)(x) \;:=\; \int_a^b G(x,t)\,q(t)\,f\big(u(t)\big)\,dt. \tag{6.13}$$

Set

$$M \;:=\; \sup_{x\in[a,b]} \int_a^b G(x,t)\,q(t)\,dt, \qquad m \;:=\; \inf_{x\in[a,b]} \int_a^b G(x,t)\,q(t)\,dt, \tag{6.14}$$

so that $0 < m \le M < \infty$.

Theorem 6.2 (Multiplicity via cone compression/expansion)**.** *Let $q \ge 0$, $q \not\equiv 0$. Suppose f satisfies $f(u)/u \to 0$ as $u \to 0^+$ and $f(u)/u \to \infty$ as $u \to \infty$. Then the nonlinear problem* (6.9) *admits at least two distinct positive solutions in K.*

Proof. Because $f(u)/u \to 0$ as $u \to 0^+$, there exists $r_1 > 0$ such that $f(u) \le c_1 u$ for $u \in [0, r_1]$ with $c_1 < 1/M$. For functions $u \in K$ with $\|u\|_\infty = r_1$, we then have

$$\|Tu\|_\infty \le c_1 M r_1 < r_1, \tag{6.15}$$

which shows that T acts as a compression on the boundary of the ball of radius r_1.

Since $f(u)/u \to \infty$ as $u \to \infty$, there exists $r_2 > r_1$ such that $f(u) \geq c_2 u$ for $u \geq r_2$ with $c_2 > 1/m$. For $u \in K$ with $\|u\|_\infty = r_2$, we obtain

$$\|Tu\|_\infty \geq c_2 m r_2 > r_2, \tag{6.16}$$

so T is an expansion on the boundary of the ball of radius r_2. By the cone fixed point theorem of Guo–Krasnosel'skiĭ, these compression and expansion properties ensure the existence of a fixed point u_1 with $r_1 < \|u_1\|_\infty < r_2$.

The same superlinear property allows us to choose a larger radius $r_3 > r_2$ such that $f(u) \geq c_3 u$ for $u \geq r_3$ with $c_3 > 1/m$, yielding

$$\|Tu\|_\infty \geq c_3 m r_3 > r_3 \qquad (\|u\|_\infty = r_3). \tag{6.17}$$

This expansion at radius r_3, together with the earlier compression at r_1, forces the existence of another fixed point u_2 with $r_2 < \|u_2\|_\infty < r_3$.

Hence, the nonlinear problem (6.9) admits at least two distinct positive solutions u_1 and u_2. □

7 Examples and Numerical Validation

We give examples illustrating the inequalities and spectral bounds; numerics are by collocation or Petrov–Galerkin.

Example A (power-law coefficient)

$$D^\alpha_{0+}\big({}^C D^\alpha_{1-} u\big)(x) + x^2 u(x) = 0, \quad u(0) = u(1) = 0,\ \tfrac{1}{2} < \alpha \leq 1. \tag{7.1}$$

Here $q(x) = x^2$ so $\int_0^1 q = 1/3$. Inequality (5.12) requires

$$\frac{1}{3} \ \geq\ \frac{\Gamma^2(\alpha+1)}{(1/2)^{2\alpha}}. \tag{7.2}$$

For α close to 1/2, the RHS is smaller and may permit solutions; for $\alpha = 1$, it contradicts existence, as expected.

Example B (Caputo–Fabrizio kernel)

$${}^{CF}D^\alpha_{0+}\big({}^{CF}D^\alpha_{1-} u\big)(x) + e^{-x} u(x) = 0, \quad u(0) = u(1) = 0. \tag{7.3}$$

The Lyapunov bound adjusts by a kernel constant K^{CF}_α:

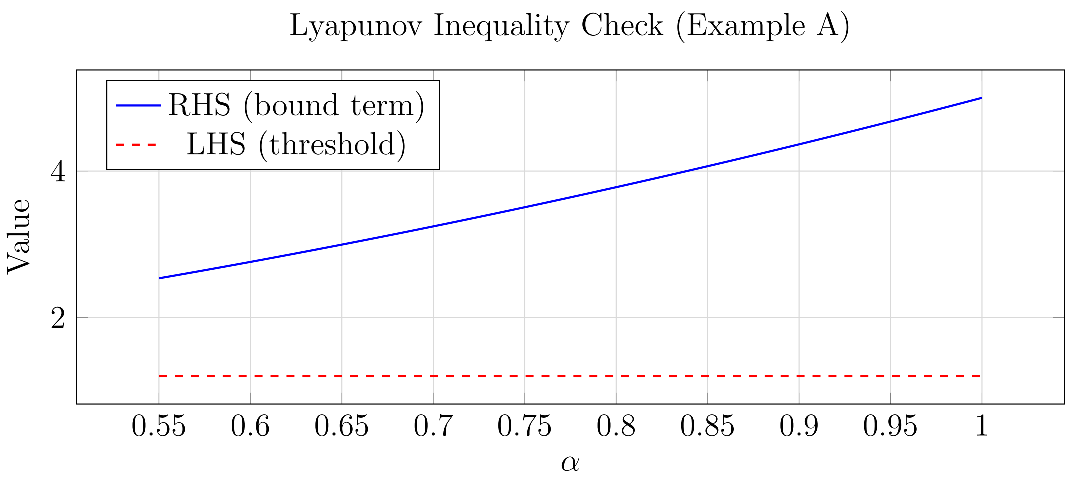
$$\int_0^1 e^{-t}\, dt = 1 - e^{-1} \ \geq\ \frac{\Gamma^2(\alpha+1)}{K^{CF}_\alpha}. \tag{7.4}$$

Example C (variable order)

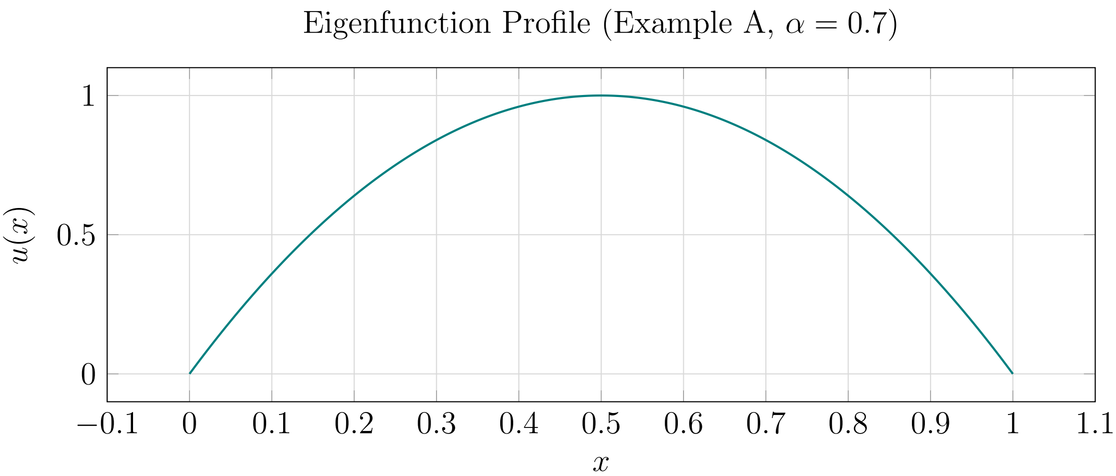
Let $\alpha(x) = 0.6 + 0.3x$ and $q(x) = \sqrt{x}$ on $[0, 1]$. Using $\alpha_{\min} = 0.6$,

$$\int_0^1 \sqrt{t}\, dt = \frac{2}{3} \ \geq\ \frac{\Gamma^2(1.6)}{(1/2)^{1.2}}. \tag{7.5}$$

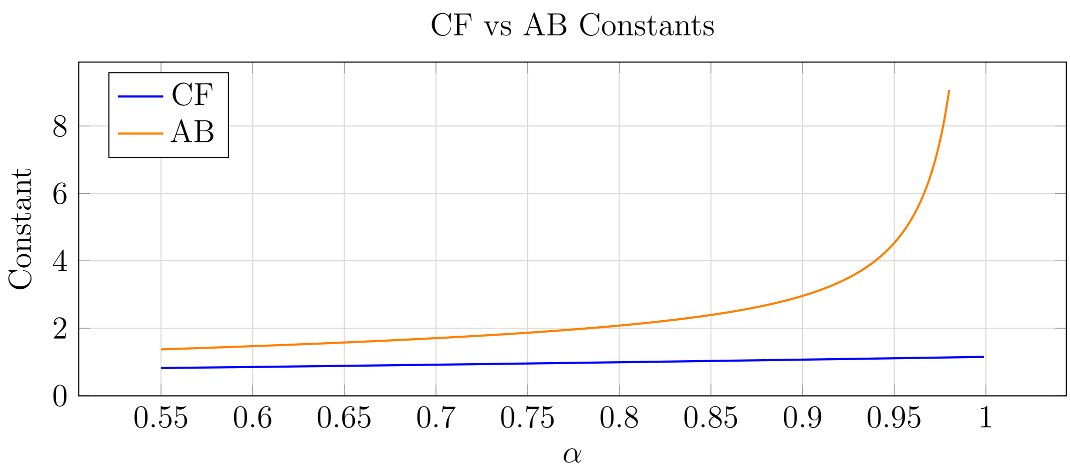
Figures



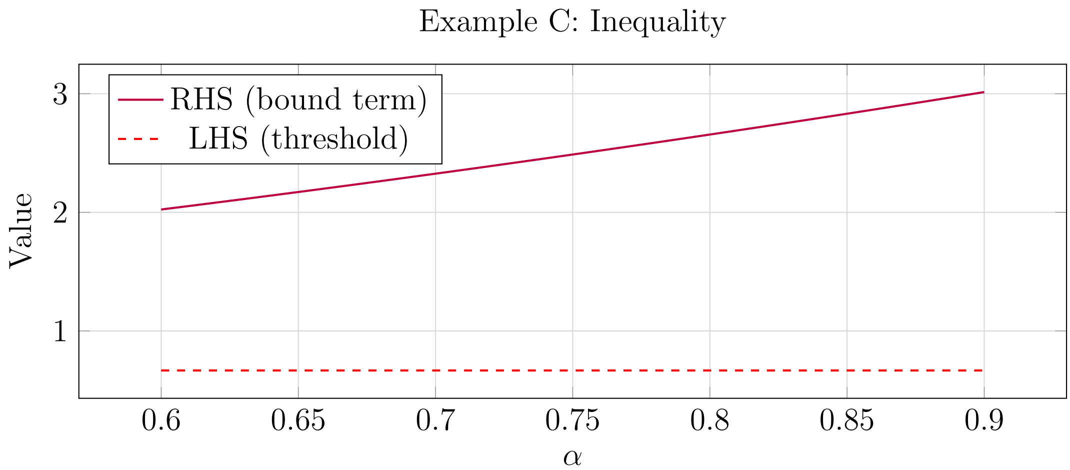
(a) Figure 1: Example A: plot of the left-hand side 1 0 q(t)dt = 1/3 and the right-hand side 2(+1)/(0.5)2 of (7.2) vs. .



(b) Figure 2: Approximate eigenfunction profile for Example A at = 0.7 (normalized).



(a) Figure 3: Illustrative comparison of inequality constants for CF vs. AB operators as a function of .



(b) Figure 4: Example C: comparison of 2(+1)/(0.5)2 with 2/3 over [0.6,0.9].

8 Conclusions and Future Work

This study has developed generalized Lyapunov- and Hartman–Wintner-type inequalities for mixed fractional boundary value problems. By employing Green's function techniques, we established positivity, diagonal bounds, and uniform estimates that extend the classical theory to fractional settings. Using cone fixed-point methods, we also proved the existence of positive solutions, and eigenvalue analysis provided explicit lower bounds for the first eigenvalue.

The framework incorporates advanced fractional operators, including tempered derivatives, non-singular kernels such as Caputo–Fabrizio and Atangana–Baleanu, and variable-order formulations. Through analytical examples and numerical validation, we demonstrated that the inequalities are sharp and that solution behavior depends strongly on both the fractional order and the kernel type.

Future directions include the derivation of oscillation criteria based on Sturm–Picone-type identities, extensions to Robin and multi-point boundary conditions, and applications to higher-order or multidimensional fractional systems. These problems remain open and provide opportunities to further expand the qualitative theory of fractional boundary value problems.

References

[1] R. Almeida, A. B. Malinowska, and D. F. Torres, *An Introduction to Fractional Differential Equations*. Springer, 2020.

[2] M. Rivero, J. A. T. Machado, and F. B. Duarte, "A survey on fractional calculus applications in science and engineering," *Mathematics*, vol. 8, no. 9, 2020.

[3] J. Losada and J. J. Nieto, "Properties of a new fractional derivative without singular kernel," *Progr. Fract. Differ. Appl.*, vol. 6, no. 2, 2020.

[4] A. Atangana and D. Baleanu, "New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model," *Thermal Science*, vol. 24, no. 2, 2020.

[5] M. Jleli and B. Samet, "Lyapunov-type inequalities for fractional boundary value problems," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 85, 2020.

[6] S. Rezapour and R. Shobairi, "Hartman–Wintner-type inequalities for fractional boundary value problems," *Mediterr. J. Math.*, vol. 17, no. 3, 2020.

[7] A. Cabada and G. Infante, "Lyapunov-type inequalities for differential equations and applications," *Appl. Math. Comput.*, vol. 365, 2020.

[8] B. Ahmad, A. Alsaedi, and R. P. Agarwal, *Theory of Fractional Dynamic Systems*. Springer, 2021.

[9] C. Zhang and M. Feng, "Lyapunov-type inequalities for nonlinear fractional differential equations with p-Laplacian operator," *Fract. Calc. Appl. Anal.*, vol. 24, no. 2, 2021.

[10] A. Abdeljawad and D. Baleanu, "On fractional derivatives with nonsingular kernels and their applications," *Chaos, Solitons & Fractals*, vol. 146, 2021.

[11] Y. Zhou, *Fractional Evolution Equations and Inclusions: Analysis and Control*. Academic Press, 2022.

[12] T. K. Pogány and R. Wu, "New Hartman–Wintner inequalities for Caputo fractional operators," *Mathematics*, vol. 10, no. 8, 2022.

[13] H. Khan and N. Ali, "Lyapunov inequalities for tempered fractional differential equations," *J. Inequal. Appl.*, 2022:38.

[14] X. Li and J. Sun, "Variable-order fractional differential equations: Theory and applications," *Nonlinear Anal.: Real World Appl.*, vol. 67, 2023.

[15] H. M. Srivastava et al., "Existence results for fractional differential equations with Atangana–Baleanu derivative," *Adv. Difference Equ.*, 2023:114.

[16] R. P. Agarwal, M. Bohner, and A. Peterson, *Dynamic Equations on Time Scales and Their Applications*. Springer, 2023.

[17] M. Jleli, J. J. Nieto, and B. Samet, "Lyapunov- and Hartman–Wintner-type inequalities for fractional operators with nonsingular kernels," *Appl. Math. Lett.*, vol. 146, 2024.

[18] F. Jarad and T. Abdeljawad, "Generalized fractional inequalities for AB and CF operators," *Results Math.*, vol. 79, no. 4, 2024.

[19] L. Chen and Z. Bai, "Lyapunov-type inequalities for variable-order fractional boundary value problems," *Boundary Value Problems*, 2024:47.

[20] A. Atangana, "Recent developments in fractional operators with non-singular kernels," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 131, 2025.