Original Research Article

On Enclave Domination Number in Families of Special Graphs and Applications

Abstract

Domination is a classical concept in graph theory that has given rise to numerous refinements and generalizations. In this paper, we investigate a domination variant referred to as the enclave dominating set. A dominating set $E_u \subset V(G)$ is called an enclave dominating set if there exists a unique enclave vertex $u \in E_u$ such that its closed neighborhood is contained in the set E_u, i.e., $N[u] \subseteq E_u$. The minimum cardinality of such a set is the enclave domination number, denoted by $\gamma_\epsilon(G)$, and a vertex serving as the enclave in the minimum enclave dominating set is an enclave dominating vertex. We determine $\gamma_\epsilon(G)$ exactly and identify all enclave dominating vertices with their respective minimum enclave dominating sets for several special graphs and well-defined graph families. These results advance the theoretical framework of domination parameters and offer new perspectives on graph structures, along with an applications of enclave dominating sets.

Keywords: Domination Number; Enclave Dominating vertex; Enclave Dominating set; Enclave Domination Number

2010 Mathematics Subject Classification: 05C69; 05C07; 05C05

1 Introduction

Domination theory constitutes a fundamental branch of graph theory with significant theoretical and practical implications, which were extensively studied in (1), (2), (3). The origins of domination theory can be traced to 1862, when C.F.De Jaenisch examined the classical problem of determining the minimum number of queens required to dominate, or effectively control, all squares of a chessboard. The systematic development of domination as a mathematical concept within graph theory emerged much later, around the 1960s, through the pioneering contributions of Berge and Ore, who laid the foundation for its formal study and subsequent theoretical advancements (4). All graph terminology and notation we follow (5). For a finite, simple, undirected graph $G = (V, E)$, a dominating set $D \subseteq V(G)$ is defined such that every vertex not in D is adjacent to at least one vertex of D. The minimum cardinality of such a dominating set is called the domination number and is denoted by $\gamma(G)$ in (1). This parameter plays a pivotal role in applications such as facility location, network surveillance, and resource optimization, as well as serving as an essential tool in the structural analysis of graphs. Given a vertex $u \in V(G)$, the open neighborhood $N(u)$ denotes the set of vertices adjacent to u, while the closed neighborhood $N[u] = N(u) \cup \{u\}$ includes u itself. The degree $d(u) = |N(u)|$ quantifies the number of vertices adjacent to u. The minimum and maximum degrees among the vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Extending this framework, from (6), a vertex $u \in S \subseteq V(G)$ is called an enclave vertex, if its closed neighborhood $N[u]$ is entirely contained within S, implies that u has no neighbors in the complement $V(G) - S$. A vertex subset containing no enclaves is termed enclaveless. In (7), the Enclave domination number is well defined. Prior research has examined enclave domination numbers for standard graphs, specific graph constructions, including the semi-total line graphs $T_1(G)$ and semi-total point graphs $T_2(G)$ in (8). Various domination parameters have been explored extensively across different families of graphs, with numerous significant examples available in the literature (9)-(14). This study identifies all enclave dominating vertices with their corresponding minimum enclave dominating sets and computes the enclave domination number for specific special graphs and finite graph families.

2 Terminology and Notation

A graph is an ordered pair $G = (V, E)$, where V is a finite, nonempty set whose elements are called vertices (or nodes), and $E \subseteq \{(u, v) : u, v \in V, u \neq v\}$ is a set of unordered pairs of distinct vertices, called edges. The order of a graph G, denoted by $|V(G)|$, is the number of vertices in the graph. The size of a graph G, denoted by $|E(G)|$, is the number of edges in the graph. From (15), the largest integer less than or equal to x is the floor function of a real number x, and it is represented by the symbol $\lfloor x \rfloor$. If n is an integer and $n \leq x < n+1$, then $\lfloor x \rfloor = n$. The ceiling function of a real number x is the lowest integer greater than or equal to x and it is denoted by $\lceil x \rceil$. Suppose that $n - 1 \leq x < n$, where n is an integer, then $\lceil x \rceil = n$. For an integer $n \geq 1$, the path P_n is a graph of order n and size $n - 1$ whose vertices can be labeled by $u_1, u_2, ..., u_n$ whose edges are $u_i u_{i+1}$ for $i = 1, 2, ..., n - 1$ (3). For $n \geq 3$, the cycle C_n is a graph of order n and size n whose vertices can be labeled by $u_1, u_2, ..., u_n$ and whose edges are $u_1u_2, u_2u_3, ..., u_{n-1}u_n, u_nu_1$ (3). For $n \geq 2$, the complete graph K_n is a graph of order n and size $\frac{n(n-1)}{2}$ whose vertices can be labeled by $u_1, u_2, ..., u_n$ and whose edges are represented as u_iu_j for all pairs of vertices where $1 \leq i < j \leq n$ (3).

The Jellyfish graph $J_{m,n}$ is created from a 4-cycle with the vertices x, y, u and v by linking x and y with a prime edge and attaching m pendant edges to u and n pendant edges to v. The edge connecting the vertices x and y is referred to as the prime edge in jellyfish.(9)

The Jewel graph J_n is derived from a four-cycle with the vertices x, y, u, v by linking x and y with a prime edge, and also by adding the edges from u and v that meet at common vertices $v_i, 1 \leq i \leq n$. The edge connecting the vertices x and y in a jewel graph is defined as the prime edge. (9)

The Lollipop graph is represented by the symbol $L_{m,n}$ and consists of a bridge between a complete graph K_m and a path graph P_n. (9)

The Dutch Windmill graph, denoted by D_m^n, is the graph obtained by taking n copies of the cycle C_m with a vertex in common. The friendship graph, denoted by F_n, is a special case of the Dutch windmill graph when $m = 3$. (16)

The Comb graph is the graph obtained from a path P_n by attaching a pendant edge at each vertex of the path, and is denoted by P_n^+. (17)

The Sunlet graph S_n, also called as n-Sunlet graph, is the graph with $2n$ vertices obtained by attaching a pendant edge at each vertex of a cycle C_n. (14)

A Crown graph (also known as a cocktail party graph) $H_{n,n}$ is a graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching. (18)

The Soifer graph is a planar graph on 9 vertices and 20 edges. It is an undirected graph and is known

for tangling the Kempe chains in Kempe's algorithm, exemplifying the failure of Kempe's original proof of the four-colour theorem. (12)
The Franklin graph is a 3-regular graph with 12 vertices and 18 edges. The Franklin graph is named after Philip Franklin. It is a 3-vertex connected and 3-edge connected perfect graph. (12)
The Chavatal graph is an undirected graph with 12 vertices and 24 edges. It is triangle free its girth is four. It is 4-regular; each vertex has exactly four neighbours. (12)
The Moser-spindle is the 7-node unit distance graph with 11 edges. It is sometimes called the Hajos graph. (10)
The Herschel graph is the smallest non-Hamiltonian polyhedral graph. The Herschel graph is a bipartite undirected graph with 11 vertices and 18 edges. (10)
The Goldner Harrary Graph is a simple undirected graph with 11 vertices and 27 edges. It is also a planar graph. (10)
The Fritsch graph is the planar graph on 9 vertices and 21 edges. It is isomorphic to the skeleton of a triaugmented triangular prism, classifying it as a nonchordal graph. (13)
Suppose the graph G has n vertices, namely $u_1, u_2, ..., u_n$. Let E_{u_i} be a minimum enclave dominating set of G for some i, then the vertex u_i is called an enclave dominating vertex. A graph G can have one or more enclave dominating vertices with distinct enclave dominating sets. $\gamma_\epsilon(G)$ denotes the enclave domination number, which is the minimum cardinality of an enclave dominating set in G.

3 Definitions and Results

For the main concepts included in this study, consider the following definitions and results.

Definition 3.1. (1) Let G be a simple connected nontrivial graph. A nonempty set $D \subseteq V(G)$ is a dominating set if every vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D. The minimum cardinality of a dominating set in G is called the domination number of G, denoted by $\gamma(G)$.

Definition 3.2. (7) A dominating set $E_u \subset V(G)$ is called an enclave dominating set if there exists a unique enclave vertex $u \in E_u$ such that its closed neighborhood is contained in the set E_u, i.e., $N[u] \subseteq E_u$. The minimum cardinality among all enclave dominating sets of G is the **enclave domination number**, denoted by $\gamma_\epsilon(G)$. A vertex u is called an **enclave dominating vertex** if it is the enclave of a minimum enclave dominating set.

Example 3.1. *For the graph in Figure 1 the vertex u_4 is the enclave dominating vertex, and the unique minimum enclave dominating set is $E_{u_4} = \{u_4, u_1, u_5\}$. Thus $|E_{u_4}| = 3$ and $\gamma_\epsilon(G) = 3$.*

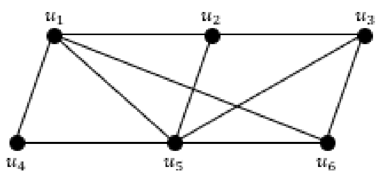

Figure 1: Simple graph G

Theorem 3.2. *(7) If G is a regular graph, then every vertex in G is an enclave dominating vertex. Except $G \cong K_n$.*

Theorem 3.3. *(7) In a graph G, if $d(u) = \delta(G)$ then E_u will be the minimum enclave dominating set, and u is the enclave dominating vertex. But the converse need not be true.*

Theorem 3.4. *(7) If $G \cong K_n$ then $\gamma_\epsilon(G)$ does not exist.*

Theorem 3.5. *(19) If P_n is a path graph with n-vertices, then $\gamma_\epsilon(P_n) = \lceil \frac{n}{3} \rceil$.*

4 Enclave Domination Number of Some Graph Families

The following theorems give the characterization of enclave dominating vertex, and enclave domination number on some well-known special graph families.

Theorem 4.1. *Let $J_{m,n}$ be a Jellyfish graph then $\gamma_\epsilon(J_{m,n}) = 3$.*

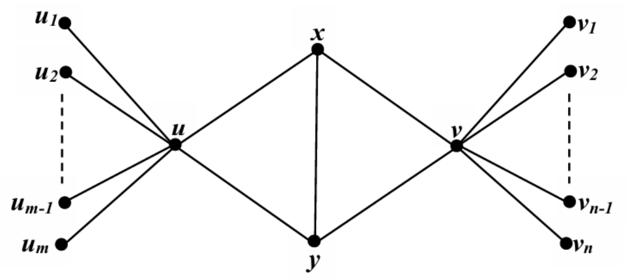

Figure 2: Jellyfish graph $J_{m,n}$

Proof. Let $J_{m,n}$ be a jellyfish graph with vertex set and edge set
$V(J_{m,n}) = \{x, y, u, v, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$
$E(J_{m,n}) = \{ux, xv, vy, yu, xy, uu_i, vv_j : 1 \leq i \leq m, 1 \leq j \leq n\}$.
The order of the graph $J_{m,n}$ is $m + n + 4$, and the size of $J_{m,n}$ is $m + n + 5$. The minimum enclave dominating set of $J_{m,n}$ will obtained in the following cases.
Case (i): If u_i, $(\forall i, 1 \leq i \leq m)$ is the enclave dominating vertex, the m-distinct enclave dominating sets generalized as $E_{u_i} = \{u_i, u, v\}$ and $|E_{u_i}| = 3$.
Case (ii): If v_j, $(\forall j, 1 \leq j \leq n)$ is the enclave dominating vertex, the n-distinct enclave dominating sets are generalized as $E_{v_j} = \{v_j, u, v\}$ and $|E_{v_j}| = 3$.
From the above cases, $\gamma_\epsilon(J_{m,n}) = 3$. □

Theorem 4.2. *Let J_n be a Jewel graph then $\gamma_\epsilon(J_n) = 3$.*

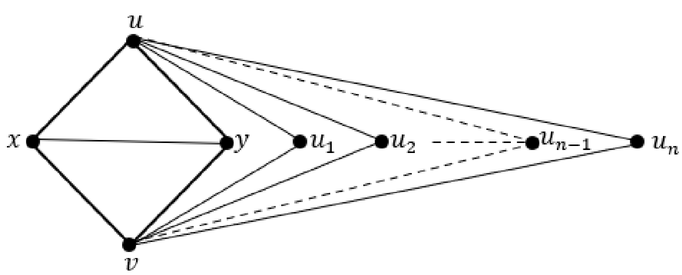

Figure 3: Jewel graph J_n

Proof. Let J_n be a Jewel graph, the vertex set and edge set be
$V(J_n) = \{x, y, u, v, u_i : 1 \leq i \leq n\}$, $E(J_n) = \{ux, uy, vx, vy, xy, uu_i, vu_i : 1 \leq i \leq n\}$.
The order of the graph J_n is $n+4$, the size of the graph J_n is $2n+5$. Here all u_i's act as the enclave dominating vertex, and we get n-distinct minimum enclave dominating set with same cardinality. For $i = 1, 2, \ldots, n$ we have $E_{u_i} = \{u_i, u, v\}$ and $|E_{u_i}| = 3$. Thus $\gamma_\epsilon(J_n) = 3$. □

Theorem 4.3. *If D_m^n be a Dutch windmill graph, then for $m \geq 5, n \geq 2$, we have* $\gamma_\epsilon(D_m^n) = 3 + \lceil \frac{m-5}{3} \rceil + (n-1)\lceil \frac{m-3}{3} \rceil$.

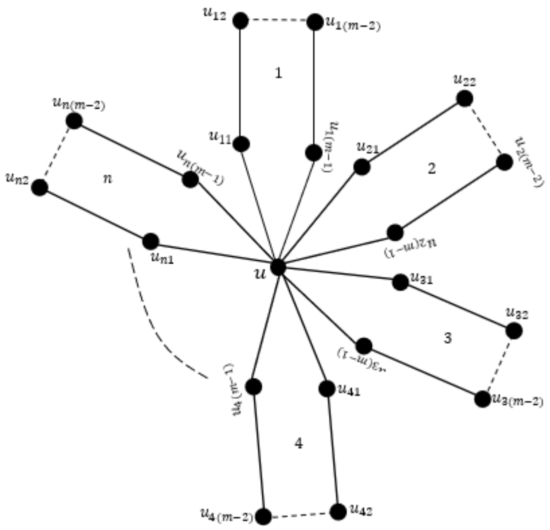

Figure 4: Dutch windmill D_m^n

Proof. Let $V(D_m^n) = \{u\} \cup \{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m-1\}$, $E(D_m^n) = E_1 \cup E_2$, where $E_1 = \{uu_{i1}, uu_{i(m-1)}\}$, and $E_2 = \{u_{ij}u_{i(j+1)} : 1 \leq i \leq n, 1 \leq j \leq m-2\}$. It has mn vertices, $m(n+1)$ edges, and maximum degree $\Delta(D_m^n) = 2n$. Here, the vertex u is the maximum degree vertex, that is, $d(u) = \Delta(G)$. For the graph D_m^n, there exist $2n$ distinct minimum enclave dominating sets for the enclave dominating vertices $u_{i1}, u_{i(m-1)}$. Now choose any vertex u_{i1}, for some fixed i, $1 \leq i \leq n$, adjacent to u as the enclave dominating vertex. Its closed neighborhood is

$$S = N[u_{i1}] = \{u, u_{i1}, u_{i2}\}$$

After removing S we examine the remaining vertices.
In the same $i-th$ cycle the vertices $u_{i3}, u_{i4}, \ldots, u_{i(m-1)}$ remains. Among these vertices u_{i3} is adjacent to $u_{i2} \in S$, and $u_{i(m-1)}$ is adjacent to $u \in S$. Thus they are already dominated. The undominated vertices in the cycle are $u_{i4}, u_{i5}, \ldots, u_{i(m-2)}$, which form a path P_{m-5}.
For each other cycle $j \neq i$, $1 \leq j \leq n$, the vertices $u_{j1}, u_{j2}, \ldots, u_{j(m-1)}$ remains. Here u_{j1} and $u_{j(m-1)}$ are adjacent to $u \in S$, so they are dominated. The remaining vertices $u_{j2}, u_{j3}, \ldots, u_{j(m-2)}$ form a path P_{m-3} in each such cycle, and none of these vertices are adjacent to any vertex in S.
Hence the subgraph induced by the undominated vertices is,

$$P_{(m-5)} \cup \underbrace{(P_{(m-3)} \cup P_{(m-3)} \cup \ldots \cup P_{(m-3)}}_{(n-1)-\text{times}}$$

By theorem 3.5, the domination number of a path P_k is $\lceil \frac{k}{3} \rceil$. Therefore the number of additional vertices needed to dominate the remaining graph is $\lceil \frac{m-5}{3} \rceil + (n-1)\lceil \frac{m-3}{3} \rceil$. Thus, the minimum enclave dominating set corresponding to the enclave dominating vertex u_{i1} has cardinality,

$$|E_{u_{i1}}| = |S| + \lceil \frac{m-5}{3} \rceil + (n-1)\lceil \frac{m-3}{3} \rceil = 3 + \lceil \frac{m-5}{3} \rceil + (n-1)\lceil \frac{m-3}{3} \rceil$$

By symmetry, choosing $u_{i(m-1)}$ as the enclave dominating vertex yields an enclave dominating set of the same cardinality. Hence $\gamma_\epsilon(D_m^n) = 3 + \lceil \frac{m-5}{3} \rceil + (n-1)\lceil \frac{m-3}{3} \rceil$.

□

Theorem 4.4. *For any Lollipop graph $L_{m,n}$, $m \geq 3, n \geq 1$, we have* $\gamma_\epsilon(L_{m,n}) = \begin{cases} \frac{n+6}{3}, & n \equiv 0 \pmod 3 \\ \frac{n+5}{3}, & n \equiv 1 \pmod 3 \\ \frac{n+7}{3}, & n \equiv 2 \pmod 3 \end{cases}$

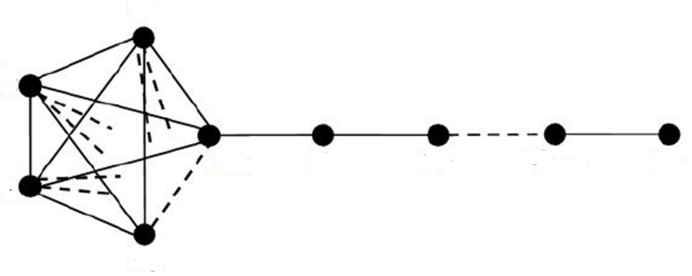

Figure 5: Lollipop graph $L_{m,n}$

Proof. Let $L_{m,n}$ be any lollipop graph with $(m+n)$-vertices, such that the vertex and edge set be $V(L_{m,n}) = \{u_1, u_2, ..., u_m, v_1, v_2, ..., v_n\}$, and $E(L_{m,n}) = E(K_m) \cup E(P_n) \cup \{u_1v_1\}$. The enclave domination number for the graph $L_{m,n}$ arise in the following cases.

Case (i): For $n \equiv 0 \pmod 3$

Let $n = 3k$, in this case the vertices v_n, and $v_i (1 \leq i < n), i \equiv 1 \pmod 3$ are the enclave dominating vertices and their corresponding enclave dominating sets are minimum. They are defined in the following subcases.

Subcase (i)(a): If v_n is the enclave dominating vertex.

Its closed neigborhood is $N[v_n] = \{v_n, v_{n-1}\}$. To obtain the minimum enclave domination we must include u_1 to dominate v_1 and all u_i. The remaining undominated vertices on the path are $v_2, v_3, \ldots, v_{n-2}$. However, v_{n-2} is adjacent to v_{n-1}, so it is dominated. Thus we need only to dominate the subpath induced by $v_2, v_3, \ldots, v_{n-3}$, which has order $n-4 = 3k-4$. A minimum dominating set for a path of order $3k-4$ consists of vertices with indices congruent to $2 \pmod 3$. Hence a minimum enclave dominating set is

$$E_{v_n} = \{v_n, v_{n-1}, u_1\} \cup \{v_p : p \equiv 2 \pmod 3), 2 \leq p \leq 3k-4\}$$

and $|E_{v_n}| = 3 + (k-1) = k+2 = \frac{n+6}{3}$.

Subcase (i)(b): Let the enclave dominating vertex be v_i with $i = 3t+1$ where $0 \leq t \leq k-1$. Then $N[v_i] = \{v_{i-1}, v_i, v_{i+1}\}$ where $i-1 = 3t$, and $i+1 = 3t+2$. The minimum enclave dominating set must contain the vertex u_1, which dominates v_1 and all u_j. To dominate the remaining vertices on the path, we take all vertices with index $\equiv 0 \pmod 3$ from 1 to $i-1$ and all vertices with index $\equiv 2 \pmod 3$ from $i+1$ to n. Thus a minimum enclave dominating set is

$$E_{v_i} = \{v_i, u_1\} \cup \{v_{3s} : 1 \leq s \leq t\} \cup \{v_{3s+2} : t \leq s \leq k-1\}$$

Hence $|E_{v_i}| = 2 + t + (k - t) = k + 2 = \frac{n+6}{3}$.
In both subcases, we obtain the minimum cardinality as $\frac{n+6}{3}$. Thus for $n \equiv 0 \pmod 3$, we have $\gamma_\epsilon(L_{m,n}) = \frac{n+6}{3}$.
Case (ii): For $n \equiv 1 \pmod 3$
Let $n = 3k + 1$. The unique minimum enclave dominating vertex is v_n. Its closed neighborhood is $N[v_n] = \{v_n, v_{n-1}\}$. Include the vertex u_1 to E_{v_n} to dominate K_m and v_1. The remaining path vertices are $v_2, v_3, \dots, v_{n-2}$. Since v_{n-2} is adjacent to v_{n-1}, it is already dominated. Thus the undominated vertices form the path $v_2, v_3, \dots, v_{n-3}$ of order $n - 4 = 3k - 3$. By Theorem 3.5, $\gamma(P_{3k-3}) = k - 1$, achieved by vertices with indices $\equiv 0 \pmod 3$ from 3 to $3k - 3$. Hence the minimum enclave dominating set is,

$$E_{v_n} = \{v_n, v_{n-1}, u_1\} \cup \{v_p : p \equiv 0 \pmod 3, 3 \le p \le 3k - 3\}$$

and

$$|E_{v_n}| = 3 + (k - 1) = k + 2 = \frac{n+5}{3}$$

Thus for $n \equiv 1 \pmod 3$, we have $\gamma_\epsilon(L_{m,n}) = \frac{n+5}{3}$.
Case (iii): For $n \equiv 2 \pmod 3$
Let $n = 3k + 2$. In this case, the vertices v_1, v_n, v_i $(1 < i < n)$,$i \equiv 0 \pmod 3$, and $v_j(1 < j < n - 2)$, $j \equiv 1 \pmod 3$ are the enclave dominating vertices and their corresponding enclave dominating sets are minimum. They are defined in the following subcases.
Subcase (iii)(a): If the vertex v_1 is the enclave dominating vertex.
Then closed neighborhood $N[v_1] = \{v_1, v_2, u_1\}$. The vertex u_1 dominate all the vertices in K_m. To dominate the remaining path vertices $v_4, v_5, \dots, v_n$ we take all the vertices with indices congruent to $2 \pmod 3$ from 2 to n. This set also includes v_2 and provides the minimum enclave dominating set. Thus

$$E_{v_1} = \{v_1, u_1\} \cup \{v_p : p \equiv 2 \pmod 3, 2 \le p \le n\}$$

and $|E_{v_1}| = 2 + (k + 1) = k + 3 = \frac{n+7}{3}$.
Subcase (iii)(b): If the vertex v_n is the enclave dominating vertex.
Then closed neighborhood $N[v_n] = \{v_n, v_{n-1}\}$. Include the vertex u_1 in E_{v_n} to dominate K_m and v_1. The undominated vertices form the path $v_2, v_3, \dots, v_{n-3}$ of order $3k-2$. By Theorem 3.5, $\gamma(P_{3k-2}) = k$, achieved by vertices with indices $\equiv 1 \pmod 3$ from 1 to $3k$. Thus

$$E_{v_n} = \{v_n, v_{n-1}, u_1\} \cup \{v_p : p \equiv 1 \pmod 3, 1 \le p \le 3k\}$$

and $|E_{v_n}| = 3 + k = \frac{n+7}{3}$.
Subcase (iii)(c): If the vertex v_i $(1 < i < n)$, $i \equiv 0 \pmod 3$ is the enclave dominating vertex.
Let $i = 3t$, $1 \le t \le k$. The closed neighborhood $N[v_i] = \{v_{i-1}, v_i, v_{i+1}\}$. Include u_1 in E_{v_i} to obtain the minimum enclave dominating set. Also add the vertices with indices $\equiv 2 \pmod 3$ from 1 to $i - 1$, and those with indices $\equiv 1 \pmod 3$ from $i + 1$ to n. Hence

$$E_{v_i} = \{v_i, u_1\} \cup \{v_{3s-1} : 1 \le s \le t\} \cup \{v_{3s+1} : t \le s \le k\}$$

Hence $|E_{v_i}| = 2 + t + (k - t + 1) = \frac{n+7}{3}$.
Subcase (iii)(d): If the vertex $v_j(1 < j \le n - 2)$, $j \equiv 1 \pmod 3$ is the enclave dominating vertex.
Let $j = 3t + 1$, $0 \le t \le k$. The closed neighborhood is $N[v_j] = \{v_{j-1}, v_j, v_{j+1}\}$. Include u_1 in E_{v_j} to get the minimum enclave dominating set. Also add the vertices with indices $\equiv 0 \pmod 3$ up to $j - 1$, and the vertices with indices $\equiv 2 \pmod 3$ from $j + 1$ to n. Hence

$$E_{v_j} = \{v_j, u_1\} \cup \{v_{3s} : 1 \le s \le t\} \cup \{v_{3s+2} : t \le s \le k\}$$

Hence $|E_{v_j}| = 2 + t + (k - t + 1) = \frac{n+7}{3}$.
Thus, from the above subcases, for $n \equiv 2 \pmod 3$, we have $\gamma_\epsilon(L_{m,n}) = \frac{n+7}{3}$.
From all the cases, the vertices identified above are precisely the enclave dominating vertices that

yield minimum enclave dominating sets; any other vertex either yields a larger enclave dominating set or contains multiple enclave vertices. Therefore, the enclave domination number of $L_{m,n}$ is established. □

Theorem 4.5. *For any Comb graph P_n^+, $n \geq 2$, we have $\gamma_\epsilon(P_n^+) = n + 1$.*

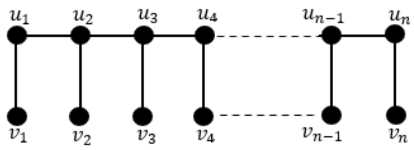

Figure 6: Comb graph P_n^+

Proof. Let the comb graph be denoted by P_n^+. The vertex set and edge set of P_n^+ are $V(P_n^+) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and $E(P_n^+) = E(P_n) \cup \{u_iv_i : 1 \leq i \leq n\}$. The order of the graph P_n^+ be $2n$, and the size of P_n^+ be $2n - 1$. Here, all the pendant vertices $v_i(1 \leq i \leq n)$ are enclave dominating vertex and the coresponding minimum enclave dominating set, for each $i = 1, 2, ..., n$ we have $E_{v_i} = \{u_i, v_1, v_2, ..., v_n\}$, $|E_{v_i}| = n + 1$. Thus $\gamma_\epsilon(P_n^+) = n + 1$. □

Theorem 4.6. *For any n-Sunlet graph S_n, $n \geq 2$ we have $\gamma_\epsilon(S_n) = n + 1$.*

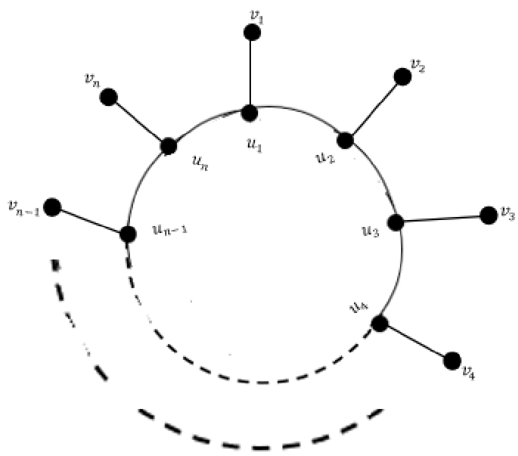

Figure 7: Sunlet graph S_n

Proof. Let S_n be the n-sunlet graph. The vertex set of S_n be $V(S_n) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and the edge set of S_n be $E(S_n) = E(C_n) \cup \{u_iv_i : 1 \leq i \leq n\}$. The order and size of the graph S_n are $2n$. Here, all the pendant vertices $v_i(1 \leq i \leq n)$ are enclave dominating vertices and we have corresponding n-distinct minimum enclave dominating sets. For $i = 1, 2, \ldots, n$ we have $E_{v_i} = \{u_i, v_1, v_2, ..., v_n\}$, $|E_{v_i}| = n + 1$. Thus $\gamma_\epsilon(S_n) = n + 1$. □

Theorem 4.7. *For any Crown graph $H_{n,n}$ and $n \geq 2$ we have $\gamma_\epsilon(H_{n,n}) = n + 1$.*

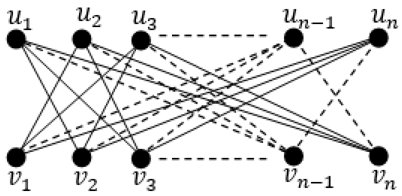

Figure 8: Crown graph $H_{n,n}$

Proof. Let G be the crown graph denoted by $H_{n,n}$. The vertex set and edge set of $H_{n,n}$ be $V(H_{n,n}) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and $E(H_{n,n}) = \{u_i v_j : i \neq j\}$. The order and size of the graph $H_{n,n}$ be $2n$ and $n(n-1)$ respectively. The minimum enclave dominating set of $H_{n,n}$ will arise in the following cases.

Case (i): If any u_i for $1 \leq i \leq n$ is the enclave dominating vertex then $E_{u_i} = \{u_i, v_1, v_2, ..., v_n\}$. And $|E_{u_i}| = n + 1$.

Case (ii): If any v_j for $1 \leq j \leq n$ is the enclave dominating vertex then $E_{v_j} = \{v_j, u_1, u_2, ..., u_n\}$. And $|E_{v_j}| = n + 1$.

From the above cases, $\gamma_\epsilon(H_{n,n}) = n + 1$. □

Theorem 4.8. *For any Friendship graph F_n, $\gamma_\epsilon(F_n)$ does not exist.*

Proof. Let F_n be any friendship graph with $n+1$ vertices. Suppose there exists an enclave dominating set say E_u where u be any vertex in F_n. The closed neighborhood $N[u]$ of u consists of u and the two vertices in its unique triangle. Thus $N[u]$ contains two enclave vertices besides u. This contradicts the definition of enclave dominating set, implies no such enclave dominating set exists. Therefore the enclave domination number of F_n does not exist. □

5 Enclave Domination Number of Some Special Graphs

The following theorems give the characterization of an enclave dominating vertex, and the enclave domination number on some known special graphs.

Theorem 5.1. *If G is a Soifer graph, then $\gamma_\epsilon(G) = 5$.*

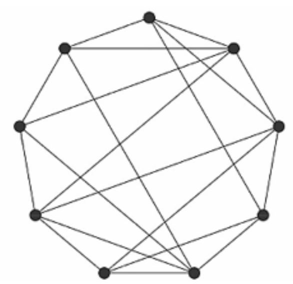

Figure 9: Soifer graph

Proof. Let G be a Soifer graph with vertex set $V(G) = \{u_1, u_2, ..., u_9\}$ and 20 edges. Let partition the vertex set into two as $S = \{u_1, u_2, u_3, u_4\}$, $T = \{u_5, u_6, u_7, u_8, u_9\}$ where the vertices in S and T are of degree 5 and 4 respectively. By Theorem 3.3, the vertices in the set T are the enclave dominating vertices. For $5 \leq i \leq 9$, there exists 5 distinct minimum enclave dominating sets as $E_{u_i} = N[u_i]$, with the enclave dominating vertex u_i. Moreover, every vertex in $V - E_{u_i}$ is dominated by the vertices in E_{u_i}, and the vertex u_i is not adjacent to any other vertices in $V - E_{u_i}$. Since $|E_{u_i}| = 5$, it follows that $\gamma_\epsilon(G) = 5$. □

Theorem 5.2. *If G is a Franklin graph, then $\gamma_\epsilon(G) = 6$.*

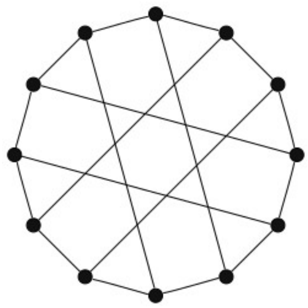

Figure 10: Franklin graph

Proof. Let G be a 3-regular Franklin graph having the vertex set $V(G) = \{u_1, u_2, ..., u_{12}\}$ and 18 edges. By Theorem 3.2, for the Franklin graph, every vertex is anenclave dominating vertex. So, there are 12 distinct minimum enclave dominating sets with equal cardinality. The minimum enclave dominating set of G consists of any 5 consecutive vertices along with an additional vertex that is adjacent to the third vertex of these 5 vertices. Let the third vertex be any $u_i (1 \leq i \leq 12)$ in G. Clearly, $N[u_i]$ is the subset of the enclave dominating set E_{u_i}. And $|E_{u_i}| = 6$, $\gamma_\epsilon(G) = 6$. □

Theorem 5.3. *If G is a Moser spindle graph, then $\gamma_\epsilon(G) = 4$.*

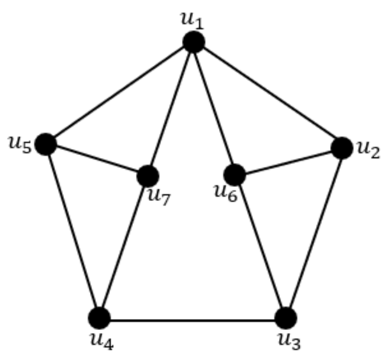

Figure 11: Moser spindle graph

Proof. Let G be a Moser spindle graph with a vertex set $V(G) = \{u_1, u_2, ..., u_7\}$ where the vertices u_1, u_2, u_3, u_4, u_5 form a 5-cycle $u_1u_2u_3u_4u_5u_1$ and the vertex u_6 adjacent to u_1, u_2, u_3 and u_7 adjacent to u_1, u_4, u_5. Here, the vertices u_3 and u_4 are the enclave dominating vertices, the corresponding minimum enclave dominating sets are $E_{u_3} = \{u_3, u_2, u_4, u_6\}$ and $E_{u_4} = \{u_4, u_3, u_5, u_7\}$. The minimum cardinality of enclave dominating sets for Moser spindle graph is 4. Thus $\gamma_\epsilon(G) = 4$ □

Theorem 5.4. *If G is a Chvatal graph, then $\gamma_\epsilon(G) = 5$.*

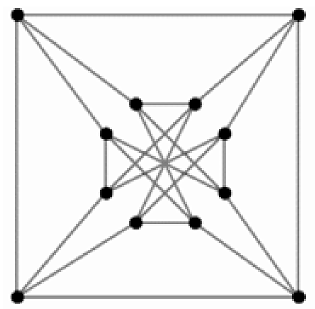

Figure 12: Chvatal graph

Proof. Let G be a Chvatal graph with 12 vertices and 24 edges. Since it is a 4-regular graph, by theorem 3.2, there are 12 distinct minimum enclave dominating sets. Let u be any vertex in G, the enclave dominating set E_u is the closed neighbourhood of the vertex u. Therefore, the set E_u must have 5 vertices, and it satisfies the enclave domination conditions. Thus, $|E_u| = 5$ and $\gamma_\epsilon(G) = 5$. □

Theorem 5.5. *If G is a Fritsch graph, then $\gamma_\epsilon(G) = 5$.*

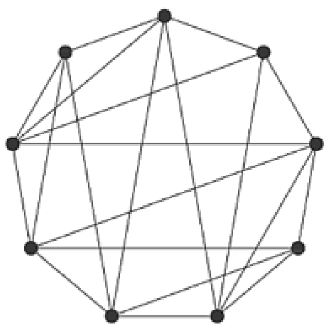

Figure 13: Fritsch graph

Proof. Proof similar to Theorem 5.1. □

Theorem 5.6. *If G is a Herschel graph, then $\gamma_\epsilon(G) = 5$.*

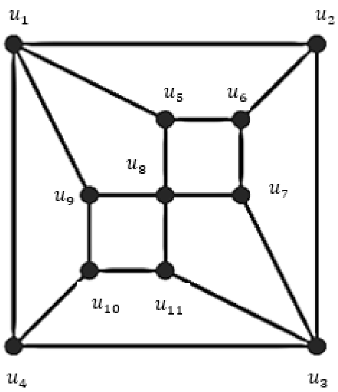

Figure 14: Herschel graph

Proof. Let G be a Herschel graph with the vertex set $\{u_1, u_2, ..., u_{11}\}$.
The vertices $u_2, u_5, u_6, u_7, u_9, u_{10}$ and u_{11} are the enclave dominating vertices. The corresponding minimum enclave dominating sets are as follows,
$E_{u_2} = \{u_2, u_1, u_3, u_6, u_{11}\}$, $E_{u_5} = \{u_5, u_1, u_6, u_8, u_{11}\}$, $E_{u_6} = \{u_6, u_2, u_5, u_7, u_{10}\}$,
$E_{u_7} = \{u_7, u_3, u_6, u_8, u_9\}$, $E_{u_9} = \{u_9, u_1, u_2, u_8, u_{10}\}$, $E_{u_{10}} = \{u_{10}, u_4, u_9, u_{11}, u_6\}$ and
$E_{u_{11}} = \{u_{11}, u_{10}, u_3, u_5, u_8\}$ respectively. The cardinality of all these sets is equal to 5.
Hence $\gamma_\epsilon(G) = 5$. □

Theorem 5.7. *If G is a Goldner-Harary graph, then $\gamma_\epsilon(G) = 4$.*

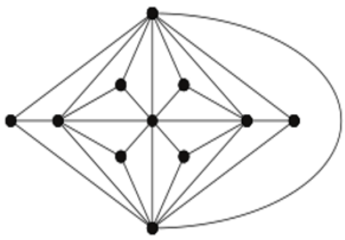

Figure 15: Goldner-Harary graph

Proof. Let G be a Goldner-Harary graph with 11 vertices, and $\delta(G) = 3$. There are 6 vertices in G such that $d(u) = \delta(G)$. By theorem 3.3, there exists 6 distinct minimum enclave dominating sets. Choose a vertex u in G such that $d(u) = \delta(G)$ then $E_u = N[u]$ is the minimum enclave dominating set with enclave dominating set u and $|E_u| = 4$. It follows that $\gamma_\epsilon(G) = 4$. □

6 Applications of Enclave Dominating Sets

6.1 Applications in Security and Surveillance

Consider a scenario where a leader organizes a rally at specific locations within a city and wants to deliver campaign speeches in some places during the rally. The security team or protection

squad must protect the campaign points, their surroundings, and the entire city. This situation can be modeled using enclave dominating sets in graph theory by identifying the optimal positioning for the protection team to ensure protection throughout the rally.
Consider that we represent the city as a graph $G = (V, E)$, where vertices represent key locations and edges denote accessible routes. For the campaign, the chief security officer selects campaign sites that form an enclave dominating set E_u with the enclave dominating vertex u as the primary campaign location in the rally. The enclave dominating condition $N[u] \subseteq E_u$ secures the campaign site u and its immediate surroundings within E_u and uniquely dominates all external areas, ensuring comprehensive coverage of the graph. This approach guarantees city-wide security for the leader's rally, minimizing threats and criminal activities throughout the city. This structure minimizes the number of locations where the security team is placed, equal to the enclave domination number $\gamma_\epsilon(G)$, while ensuring unambiguous monitoring. Peripheral areas are also monitored by E_u, enabling rapid response during rallies.
In a graph G, if there exist multiple enclave dominating vertices, each inducing an enclave dominating set of identical minimum cardinality, then the enclave domination number $\gamma_\epsilon(G)$ is attained by distinct vertices. Consequently, in a rally, a leader can conduct secure campaigning across multiple locations using the same minimal contingent of optimally positioned protection squads. Enclave dominating sets model these ideal squad placements.

6.2 Applications in Other Domains

This versatile framework extends naturally to other domains requiring similar optimization. In each application, the enclave dominating set $E_u \subset V(G)$ contains a unique enclave dominating vertex u whose closed neighborhood $N[u]$ lies entirely within E_u, ensuring that u and all its immediate neighbors are part of the protected set. The minimum cardinality of such a configuration, $\gamma_\epsilon(G)$, represents the optimal resource allocation, while the existence of multiple enclave dominating vertices provides adaptive solutions.
In military operations, forward operating bases form an enclave dominating set E_u, with the command hub serving as the unique enclave dominating vertex u whose surrounding positions $N[u]$ are all secured within E_u. Other bases may have adjacent hostile territory, but the dominating property of E_u ensures surveillance over the entire region.
For wireless sensor networks, data aggregation points constitute E_u, where the central aggregator serves as the unique enclave dominating vertex u with all its neighboring sensors $N[u]$ contained in the trusted set. Other aggregation points may connect to untrusted nodes, but the set E_u as a whole monitors the entire network.
Public health officials can establish minimal quarantine zones as E_u, with the epidemic epicenter u having all directly connected communities $N[u]$ under quarantine. Other quarantined areas may border unquarantined regions, but together they maintain surveillance over all regions.
In cybersecurity, the intrusion detection system (IDS) placements form E_u, where the primary server acts as the unique enclave dominating vertex u with all adjacent devices $N[u]$ directly monitored. Other IDS nodes may leave some neighbors unmonitored, yet together they guarantee comprehensive network coverage.
Thus, the enclave dominating set provides a strategy for achieving comprehensive coverage with minimal resources, where multiple enclave dominating vertices offer adaptive solutions for dynamic environments.

7 Conclusion

In this paper, we determine the enclave domination numbers for several standard special graphs and specific graph families. Through our findings, we have observed how different graph structures influence the behaviour of this parameter. The present study advances the understanding of enclave domination in graph theory and establishes a platform for further investigations. We also explored the practical applications of enclave dominating sets in graphs. Future research may focus on extending these results to unary and binary graph operations.

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