Tempered FDTM–Bell Framework with Hybrid Laplace–Sumudu Rules for Fractional Delay Systems

Abstract

This work introduces a semi-analytical approach for nonlinear fractional delay differential equations governed by tempered Caputo memory. The proposed framework combines the Fractional Differential Transform Method (FDTM) with partial ordinary Bell polynomials to efficiently manage composite nonlinear terms, while a hybrid Laplace–Sumudu formulation ensures that tempering enters the coefficient recursion as an analytic multiplier. On the theoretical side, we establish existence and uniqueness results with bounds independent of the tempering parameter λ, and further prove geometric convergence of the truncated FDTM–Bell expansions under mild analyticity assumptions, uniformly valid in both λ and α on compact subsets of $(0, 1]$. From an algorithmic perspective, we design a dynamic-programming Bell engine and propose an adaptive truncation criterion that balances accuracy with efficiency. Numerical experiments on three benchmark problems—a proportional delay, a time-varying delay, and a two-dimensional neutral-type system—demonstrate the stabilizing influence of tempering over long time intervals and verify the theoretically predicted error–truncation trends. The framework is modular in design and can be extended to tempered Caputo–Fabrizio kernels with only minor modifications.

Keywords: tempered fractional calculus; Caputo derivative; delay differential systems; FDTM; Bell polynomials; Laplace transform; Sumudu transform; convergence; adaptive truncation.
MSC 2020: 26A33; 34K05; 34K28; 65L05; 65L20.

1 Introduction

Real-world systems rarely act without memory. Materials display creep, biological populations respond to past stimuli, control mechanisms accumulate inertia, and networks transmit signals with inherent delays. Fractional operators provide a flexible tool to capture such memory effects beyond the scope of classical calculus. However, the traditional kernels often exhibit slow decay over long horizons, which may compromise stability and computational efficiency. A practical modification is *tempering*, where an exponential weight damps the far tail while still preserving the nonlocal character. Recent studies confirm that tempered operators maintain modeling power while offering better control of long-time behaviour and improved robustness against noisy histories, particularly in delay-driven dynamics [1, 2, 3].

Within this framework, *fractional delay differential equations (FDDEs)* emerge as a natural intersection of memory and latency. They effectively describe phenomena such as actuation lags, maturation processes, transport delays, and feedback buffering in engineering and the sciences. Contemporary FDDE theory encompasses Caputo-type and Hilfer derivatives, neutral structures, finite or state-dependent delays, and has established existence, uniqueness, and stability under broad conditions [4, 5, 6, 7]. Incorporating tempering further enhances well-posedness on extended intervals and clarifies stability thresholds when delays interact with memory terms [6, 3].

From a computational perspective, transform-based techniques remain attractive because they reduce nonlocal operators to algebraic forms and naturally encode initial conditions. Among these, the Laplace and Sumudu transforms are widely used: the Sumudu transform preserves units and often simplifies initial-value constraints, while the Laplace transform offers a broad catalog of operational rules. Hybrid approaches that combine both transforms—including double and triple Laplace–Sumudu variants—have recently been shown to streamline inversion for fractional models [8, 9, 10]. These developments motivate the construction of solvers that integrate both transforms in a single, unified pipeline.

For nonlinear FDDEs, a central difficulty lies in handling composite terms of the form $F(t, u(t), u(t - \tau(t)))$ under repeated differentiation. Semi-analytical series methods, particularly the fractional differential transform method (FDTM), address this challenge by converting the governing equation into a recursion for coefficients. Recent years have seen renewed interest in this direction: FDTM and its modified variants have been successfully applied to fractional systems with delays and multi-term operators, demonstrating competitive accuracy and relatively simple recursive schemes [11, 12, 7, 13]. A crucial ingredient is the Faà di Bruno formula, elegantly encoded through partial Bell polynomials, which transform nonlinear compositions into explicit and reusable coefficient maps. Current research continues to advance Bell-based expansions for fractional and classical problems [14, 15].

In this paper, we introduce a *tempered FDTM–Bell* framework where the tempering parameter enters the recursion directly via a set of *hybrid Laplace–Sumudu* rules, avoiding the need for artificial damping or post-processing. On the analytical side, we prove existence and uniqueness with constants uniform in λ, and establish geometric convergence of the truncated series under mild analyticity assumptions, uniformly in both λ and α over compact subsets. On the algorithmic side, we develop a dynamic-programming engine for Bell polynomials and propose an *adaptive truncation* strategy that links the stopping index N to a user-defined tolerance. Finally, through three representative FDDE families—proportional, time-dependent, and neutral/distributed delays—we demonstrate accuracy, stability, and the practical benefits of tempering when compared with predictor–corrector and other transform-based solvers [16, 17].

2 Preliminaries and Notation

This section collects the basic definitions, function spaces, and auxiliary tools that will be used throughout the paper.

2.1 Function spaces and symbols

We consider the interval $[0, T]$ with $T > 0$. Let

$$X := C([0, T]; \mathbb{R}^d), \qquad \|u\|_\infty := \max_{t\in[0,T]} \|u(t)\|_2,$$

where $\|\cdot\|_2$ denotes the Euclidean norm. Delays are described by functions $\tau : [0, T] \to [0, T]$ that are continuous and bounded. For uniqueness, we will assume whenever required that $\mathrm{Lip}(\tau) < 1$. The fractional order is denoted by $\alpha \in (0, 1]$ and the tempering parameter by $\lambda \geq 0$.

2.2 Tempered Caputo derivative

For $u \in C^1$, order $\alpha \in (0, 1)$, and $\lambda \geq 0$, the tempered Caputo derivative is defined by

$$ {}^{C}D_t^{\alpha,\lambda} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t e^{-\lambda(t-s)}(t-s)^{-\alpha}u'(s)\, ds. \tag{2.1}$$

When $\lambda = 0$, this reduces to the classical Caputo derivative. The associated tempered fractional integral is

$$(I^{\alpha,\lambda}g)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda(t-s)}(t-s)^{\alpha-1}g(s)\, ds. \tag{2.2}$$

2.3 Hybrid Laplace–Sumudu rules

Let $\mathcal{L}$ and $\mathcal{S}$ denote the Laplace and Sumudu transforms, respectively. For $u \in C^1$ we have

$$\mathcal{L}\{{}^{C}D_t^{\alpha,\lambda} u\}(p) = (p+\lambda)^\alpha U(p) - (p+\lambda)^{\alpha-1}u(0^+), \tag{2.3}$$

$$\mathcal{S}\{{}^{C}D_t^{\alpha,\lambda} u\}(\xi) = \xi^{-\alpha}(1+\lambda\xi)^\alpha\, \mathcal{S}\{u\}(\xi) - \xi^{1-\alpha}(1+\lambda\xi)^{\alpha-1}u(0^+). \tag{2.4}$$

2.4 Bell polynomials and compositions

For smooth functions g and v, Faà di Bruno's formula yields

$$\frac{d^n}{dt^n} g(v(t)) = \sum_{k=1}^{n} g^{(k)}(v(t))\, B_{n,k}\Big(v'(t), \ldots, v^{(n-k+1)}(t)\Big),$$

where $B_{n,k}$ are the partial Bell polynomials. We will employ standard growth bounds for $B_{n,k}$ throughout the analysis.

2.5 FDTM expansions and delay terms

We seek a solution in the form of a power series near $t = 0$,

$$u(t) = \sum_{n \ge 0} a_n t^n, \qquad a_0 = \phi(0).$$

For a proportional delay $\tau(t) = \kappa t$ we obtain

$$u(t - \tau(t)) = u((1 - \kappa)t) = \sum_{n \ge 0} a_n (1 - \kappa)^n t^n.$$

For general $\tau \in C^1$, one can write

$$u(t - \tau(t)) = \sum_{n \ge 0} c_n t^n,$$

where the coefficients c_n are given by complete Bell polynomials applied to the derivatives $\{\tau^{(j)}(0)\}$.

2.6 Standing assumptions

Throughout the paper we will assume:

(A1) The history function $\phi : [-\tau_{\max}, 0] \to \mathbb{R}^d$ is continuous, with $u(0) = \phi(0)$.

(A2) The delay $\tau : [0, T] \to [0, \tau_{\max}]$ is continuous, and for uniqueness $\mathrm{Lip}(\tau) < 1$ (proportional delays are permitted).

(A3) The nonlinearity $F : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and globally Lipschitz in (y, z) with constants (L_y, L_z).

(A4) *Analyticity:* F is analytic on a polydisc $\{|t| \le R_t, \|y\| \le R_y, \|z\| \le R_z\}$.

(A5) The tempering parameter satisfies $\lambda \in [0, \Lambda]$ for some fixed $\Lambda \ge 0$.

2.7 Two auxiliary lemmas

Lemma 2.1 (Tempered Grönwall inequality)**.** *Suppose $v \in C([0, T])$ satisfies*

$$v(t) \le A + \frac{B}{\Gamma(\alpha)} \int_0^t e^{-\lambda(t-s)} (t - s)^{\alpha - 1} v(s)\, ds.$$

Then

$$v(t) \le A\, E_\alpha(B t^\alpha),$$

where the bound is uniform in $\lambda \ge 0$.

Lemma 2.2 (Bell polynomial growth)**.** *Under assumption (A4), there exist constants $C > 0$ and $\rho \in (0, 1)$ such that the coefficients obtained from the expansion of $F(t, u(t), u(t - \tau(t)))$ satisfy*

$$\|\tilde{F}_n\| \le C \rho^n,$$

uniformly for all $\lambda \in [0, \Lambda]$.

3 Well-posedness: existence, uniqueness, and stability uniform in λ

We consider the fractional delay problem on $[0,T]$

$$ {}^{C}D_t^{\alpha,\lambda} u(t) = F\Big(t, u(t), u(t-\tau(t))\Big), \qquad u(t) = \phi(t) \text{ for } t \in [-\tau_{\max}, 0], \tag{3.1} $$

with $0 < \alpha \le 1$, $\lambda \in [0, \Lambda]$, and assumptions (A1)–(A5) from Section 2. We denote by $\tilde{u}$ the canonical extension

$$ \tilde{u}(\theta) := \begin{cases} \phi(\theta), & \theta \in [-\tau_{\max}, 0], \\ u(\theta), & \theta \in (0, T]. \end{cases} $$

All norms are on $X := C([0,T]; \mathbb{R}^d)$ with $\|u\|_\infty := \sup_{t\in[0,T]} \|u(t)\|_2$ unless otherwise stated.

3.1 Two basic operators and their exact norms

We record the precise operator bounds used later.

Lemma 3.1 (Boundedness of the tempered fractional integral). *For $g \in X$ define*

$$ (I^{\alpha,\lambda} g)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda(t-s)} (t-s)^{\alpha-1} g(s)\, ds, \qquad t \in [0,T]. $$

Then $I^{\alpha,\lambda} : X \to X$ is linear and bounded with

$$ \|I^{\alpha,\lambda}\|_{X\to X} = \sup_{t\in[0,T]} \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda(t-s)} (t-s)^{\alpha-1} ds \;\le\; \frac{T^\alpha}{\Gamma(\alpha+1)}. \tag{3.2} $$

Moreover, the right-hand side of (3.10) is maximised at $\lambda = 0$, hence the bound is uniform in $\lambda \in [0, \Lambda]$.

Proof. Linearity is immediate. For $t \in [0,T]$,

$$ \|(I^{\alpha,\lambda} g)(t)\| \le \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda(t-s)} (t-s)^{\alpha-1} \|g(s)\|\, ds \le \|g\|_\infty \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda r} r^{\alpha-1} dr, $$

after the change of variables $r := t - s$. Taking the supremum over t yields

$$ \|I^{\alpha,\lambda} g\|_\infty \le \|g\|_\infty \sup_{t\in[0,T]} \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda r} r^{\alpha-1} dr. $$

Since $e^{-\lambda r} \le 1$, for every t the integral is bounded by $\frac{1}{\Gamma(\alpha)} \int_0^t r^{\alpha-1} dr = \frac{t^\alpha}{\Gamma(\alpha+1)} \le \frac{T^\alpha}{\Gamma(\alpha+1)}$. This proves (3.10). The map $\lambda \mapsto \int_0^t e^{-\lambda r} r^{\alpha-1} dr$ is decreasing, so the supremum in $\lambda \in [0, \Lambda]$ is attained at $\lambda = 0$. □

Lemma 3.2 (Nonexpansiveness of delay evaluation)**.** *Let $T > 0$ and $X := C([0,T];\mathbb{R}^d)$ with the sup norm*

$$\|u\|_\infty := \sup_{t\in[0,T]} \|u(t)\|_2.$$

Let $\tau : [0,T] \to [0,\tau_{\max}]$ be continuous, and let $\varphi : [-\tau_{\max}, 0] \to \mathbb{R}^d$ be a given history. For $u \in X$, define the canonical extension

$$\tilde{u}(\theta) := \begin{cases} \varphi(\theta), & \theta \in [-\tau_{\max}, 0], \\ u(\theta), & \theta \in (0,T], \end{cases}$$

and the delay operator

$$(D_\tau u)(t) := \tilde{u}\Big(t - \tau(t)\Big), \qquad t \in [0,T].$$

Then $D_\tau : X \to X$ is linear, well-defined, and satisfies

$$\|D_\tau u - D_\tau v\|_\infty \le \|u - v\|_\infty, \qquad \forall\, u, v \in X. \tag{3.3}$$

Hence $\|D_\tau\|_{X\to X} \le 1$.

Proof. **Well-definedness.** *F*or each $t \in [0,T]$, continuity of τ ensures $t - \tau(t) \in [-\tau_{\max}, T]$. The extension $\tilde{u}$ is continuous on $[-\tau_{\max}, T]$: on $[-\tau_{\max}, 0]$ it coincides with φ, on $(0,T]$ it coincides with u, and the compatibility condition $u(0) = \varphi(0)$ guarantees continuity at 0. Therefore $D_\tau u(t) = \tilde{u}(t - \tau(t))$ is continuous in t, i.e., $D_\tau u \in X$.

Linearity. For $a, b \in \mathbb{R}$ and $u, v \in X$,

$$D_\tau(au+bv)(t) = \widetilde{au+bv}\Big(t-\tau(t)\Big) = a\,\tilde{u}\Big(t-\tau(t)\Big) + b\,\tilde{v}\Big(t-\tau(t)\Big) = a(D_\tau u)(t) + b(D_\tau v)(t).$$

Thus D_τ is linear.

Nonexpansiveness. For $u, v \in X$ and $t \in [0,T]$,

$$\|(D_\tau u)(t) - (D_\tau v)(t)\| = \|\tilde{u}(t - \tau(t)) - \tilde{v}(t - \tau(t))\| \le \sup_{\theta\in[-\tau_{\max},T]} \|\tilde{u}(\theta) - \tilde{v}(\theta)\|.$$

Since $\tilde{u} = \tilde{v} = \varphi$ on $[-\tau_{\max}, 0]$, the supremum reduces to

$$\sup_{\theta\in(0,T]} \|u(\theta) - v(\theta)\| = \|u - v\|_\infty.$$

Taking the supremum in $t \in [0,T]$ yields inequality (3.3). Thus $\|D_\tau\|_{X\to X} \le 1$ and D_τ is nonexpansive. □

3.2 Integral formulation and equivalence

Mild/Volterra formulation] A function $u \in X$ is a *mild solution* of (3.1) on $[0,T]$ if and only if it satisfies

$$u(t) = \phi(0) + (I^{\alpha,\lambda} G[u])(t), \qquad G[u](s) := F\Big(s, u(s), (\mathcal{D}_\tau u)(s)\Big), \tag{3.4}$$

for all $t \in [0, T]$.

(*Only if.*) Suppose u solves (3.1) in the Caputo sense with initial datum ϕ. Then the definition of the tempered Caputo derivative (Section 2) implies

$$u(t) - u(0) = (I^{\alpha,\lambda} u')(t) \quad \text{and} \quad {}^{C}D_t^{\alpha,\lambda} u = G[u].$$

Using the standard identity that $I^{\alpha,\lambda}$ is a right inverse of ${}^{C}D_t^{\alpha,\lambda}$ on Caputo data yields

$$u(t) = u(0) + (I^{\alpha,\lambda} G[u])(t) = \phi(0) + (I^{\alpha,\lambda} G[u])(t).$$

(*If.*) Conversely, if u satisfies (3.4) with continuous $G[u]$, apply ${}^{C}D_t^{\alpha,\lambda}$ to both sides. Since ${}^{C}D_t^{\alpha,\lambda} I^{\alpha,\lambda} = \mathrm{Id}$ on $C([0,T])$ and ${}^{C}D_t^{\alpha,\lambda} \phi(0) = 0$, we obtain ${}^{C}D_t^{\alpha,\lambda} u = G[u]$, i.e., (3.1).

3.3 Local existence and uniqueness via contraction mapping

Theorem 3.1 (Contraction and uniqueness; explicit constant; uniform in λ)**.** *Let L_y, L_z be Lipschitz constants from (A3) and set*

$$q_\alpha(T) := \frac{T^\alpha}{\Gamma(\alpha+1)} (L_y + L_z).$$

If $q_\alpha(T) < 1$, then for every $\lambda \in [0, \Lambda]$ the operator $\mathcal{T} : X \to X$ defined by

$$(\mathcal{T}u)(t) := \phi(0) + (I^{\alpha,\lambda} G[u])(t)$$

is a strict contraction on $(X, \|\cdot\|_\infty)$. Consequently, (3.1) *admits a unique mild solution $u \in X$ on $[0,T]$. All constants are independent of λ.*

Proof. Let $u, v \in X$. Using Lemma 3.1, (A3),

$$\begin{aligned}
\|\mathcal{T}u - \mathcal{T}v\|_\infty &= \|I^{\alpha,\lambda}\Big(G[u] - G[v]\Big)\|_\infty \le \|I^{\alpha,\lambda}\| \, \|G[u] - G[v]\|_\infty \\
&\le \frac{T^\alpha}{\Gamma(\alpha+1)} \sup_{s\in[0,T]} \left\| F\Big(s, u(s), \mathcal{D}_\tau u(s)\Big) - F\Big(s, v(s), \mathcal{D}_\tau v(s)\Big) \right\| \\
&\le \frac{T^\alpha}{\Gamma(\alpha+1)} \left(L_y \|u - v\|_\infty + L_z \|\mathcal{D}_\tau u - \mathcal{D}_\tau v\|_\infty\right) \\
&\le \frac{T^\alpha}{\Gamma(\alpha+1)} (L_y + L_z) \, \|u - v\|_\infty = q_\alpha(T) \, \|u - v\|_\infty.
\end{aligned}$$

Thus $\mathcal{T}$ is a contraction with constant $q_\alpha(T) < 1$ and the conclusion follows from the Banach fixed-point theorem. The bound $\|I^{\alpha,\lambda}\| \le T^\alpha/\Gamma(\alpha+1)$ is maximal at $\lambda = 0$ (Lemma 3.1), hence the contraction constant is uniform in λ. □

3.4 Stepwise global existence on $[0, T]$

Theorem 3.2 (Continuation by time stepping)**.** *If $q_\alpha(T) \ge 1$, choose a partition $0 = t_0 < t_1 < \cdots < t_m = T$ such that for each j,*

$$q_\alpha(h_j) := \frac{h_j^\alpha}{\Gamma(\alpha+1)} (L_y + L_z) < 1, \qquad h_j := t_{j+1} - t_j.$$

Then there exists a unique mild solution $u \in C([-\tau_{\max}, T])$ to (3.1)*, constructed by induction on the subintervals $[t_j, t_{j+1}]$, uniformly in $\lambda \in [0, \Lambda]$.*

Proof. (On $[t_0, t_1]$ the history is ϕ on $[-\tau_{\max}, 0]$. Apply Theorem 3.1 with $T = h_0$ to obtain a unique mild solution u on $[0, h_0]$.

(.) Suppose u is constructed on $[0, t_j]$. Define the history for the next window by

$$\phi_j(\theta) := \Big\{ u(t_j + \theta), \quad \theta \in [-\tau_{\max}, 0],$$

and solve

$${}^{C}D_t^{\alpha,\lambda} u(t) = F\Big(t, u(t), u(t - \tau(t))\Big), \quad t \in [t_j, t_{j+1}], \qquad u(t_j + \theta) = \phi_j(\theta), \ \theta \in [-\tau_{\max}, 0].$$

Shift time by $s := t - t_j$; the problem reduces to (3.1) on $[0, h_j]$ with history ϕ_j. Since $q_\alpha(h_j) < 1$, Theorem 3.1 gives a unique mild solution on $[t_j, t_{j+1}]$. Concatenate the solutions; continuity at t_j holds by construction. Uniqueness on $[t_j, t_{j+1}]$ for each j implies uniqueness on $[0, T]$. □

3.5 A-priori bounds and continuous dependence

Theorem 3.3 (A-priori L^∞ bound)**.** *Let $M_0 := \sup_{t\in[0,T]} \|F(t, 0, 0)\|$. If $q_\alpha(T) < 1$, then every mild solution u of* (3.1) *obeys*

$$\|u\|_\infty \le \frac{\|\phi(0)\| + \dfrac{T^\alpha}{\Gamma(\alpha+1)} M_0}{1 - q_\alpha(T)}. \tag{3.5}$$

The bound is uniform in $\lambda \in [0, \Lambda]$.

Proof. From the mild form (3.4),

$$\|u\|_\infty \le \|\phi(0)\| + \|I^{\alpha,\lambda}\| \, \|F(\cdot, u, \mathcal{D}_\tau u)\|_\infty.$$

By (A3) ,

$$\|F(\cdot, u, \mathcal{D}_\tau u)\|_\infty \le M_0 + L_y \|u\|_\infty + L_z \|\mathcal{D}_\tau u\|_\infty \le M_0 + (L_y + L_z)\|u\|_\infty.$$

Using $\|I^{\alpha,\lambda}\| \le T^\alpha / \Gamma(\alpha + 1)$,

$$\|u\|_\infty \le \|\phi(0)\| + \frac{T^\alpha}{\Gamma(\alpha+1)} M_0 + \frac{T^\alpha}{\Gamma(\alpha+1)} (L_y + L_z)\|u\|_\infty.$$

Rearranging gives (3.5). Uniformity in λ follows from Lemma 3.1. □

Theorem 3.4 (Continuous dependence on data). *Let u solve* (3.1) *with data* (ϕ, F) *and let $\hat{u}$ solve the same problem with* $(\hat{\phi}, \hat{F})$ *on* $[0, T]$. *Assume* $q_\alpha(T) < 1$ *for the common Lipschitz constants* L_y, L_z. *Then*

$$\|u - \hat{u}\|_\infty \leq \frac{\|\phi(0) - \hat{\phi}(0)\| + \dfrac{T^\alpha}{\Gamma(\alpha+1)} \|F - \hat{F}\|_\infty}{1 - q_\alpha(T)}, \tag{3.6}$$

uniformly in $\lambda \in [0, \Lambda]$.

Proof. Subtract the two mild forms:

$$u(t) - \hat{u}(t) = \phi(0) - \hat{\phi}(0) + (I^{\alpha,\lambda} [F(\cdot, u, \mathcal{D}_\tau u) - \hat{F}(\cdot, \hat{u}, \mathcal{D}_\tau \hat{u})])(t).$$

Take norms and use Lemma 3.1:

$$\|u - \hat{u}\|_\infty \leq \|\phi(0) - \hat{\phi}(0)\| + \frac{T^\alpha}{\Gamma(\alpha+1)} \Big(\|F - \hat{F}\|_\infty + \|F(\cdot, u, \mathcal{D}_\tau u) - F(\cdot, \hat{u}, \mathcal{D}_\tau \hat{u})\|_\infty \Big).$$

By (A3) ,

$$\|F(\cdot, u, \mathcal{D}_\tau u) - F(\cdot, \hat{u}, \mathcal{D}_\tau \hat{u})\|_\infty \leq (L_y + L_z)\|u - \hat{u}\|_\infty.$$

Hence

$$\|u - \hat{u}\|_\infty \leq \|\phi(0) - \hat{\phi}(0)\| + \frac{T^\alpha}{\Gamma(\alpha+1)} \|F - \hat{F}\|_\infty + q_\alpha(T) \|u - \hat{u}\|_\infty,$$

which yields (3.6) after rearranging. □

3.6 Specialisation to proportional delays

Corollary 3.1 (Proportional delay). *hypotheses* (A1)–(A5) *in force, and let the delay be proportional,*

$$\tau(t) = \kappa t, \qquad \kappa \in [0, 1).$$

Then the bounds and well-posedness statements of Theorems remain valid with the same contraction constant and right-hand sides. *In particular, if*

$$q_\alpha(T) := \frac{T^\alpha}{\Gamma(\alpha+1)} (L_y + L_z) \ < \ 1, \tag{3.7}$$

then for every $\lambda \in [0, \Lambda]$ *the problem*

$${}^C D_t^{\alpha,\lambda} u(t) \ = \ F\Big(t, u(t), u((1-\kappa)t)\Big), \qquad u(t) = \varphi(t) \ \ on \ \ [-\tau_{\max}, 0], \tag{3.8}$$

admits a unique mild solution $u \in C([0, T]; \mathbb{R}^d)$, *together with the a priori bound and continuous dependence estimates stated in Theorems.*

Proof. **Reduction of the delay operator.** For $\tau(t) = \kappa t$ with $\kappa \in [0, 1)$ we have $t - \tau(t) = (1 - \kappa)t \in [0, T]$ for all $t \in [0, T]$. Hence, for every $u \in X := C([0, T]; \mathbb{R}^d)$ the delay evaluation operator D_τ simplifies to

$$(D_\tau u)(t) \ = \ u\Big((1-\kappa)t\Big), \qquad t \in [0, T], \tag{3.9}$$

so the history segment on $[-\tau_{\max}, 0]$ is never invoked once $t > 0$.

Nonexpansiveness with constant 1. By Lemma 3.2, for arbitrary continuous τ we have $\|D_\tau u - D_\tau v\|_\infty \le \|u - v\|_\infty$. In the proportional case (3.9), this becomes

$$\sup_{t\in[0,T]} \Big\|u((1-\kappa)t) - v((1-\kappa)t)\Big\| \le \sup_{s\in[0,T]} \|u(s) - v(s)\| = \|u - v\|_\infty,$$

so the operator norm satisfies $\|D_\tau\|_{X\to X} \le 1$ exactly, with no dependence on κ.

Contraction constant unchanged. In the fixed-point map used in Theorem, the only place where delay affects the Lipschitz constant is through the difference

$$\|F(\cdot, u, D_\tau u) - F(\cdot, v, D_\tau v)\|_\infty \le L_y\|u - v\|_\infty + L_z\|D_\tau u - D_\tau v\|_\infty.$$

Using $\|D_\tau\|_{X\to X} \le 1$ gives

$$\|F(\cdot, u, D_\tau u) - F(\cdot, v, D_\tau v)\|_\infty \le (L_y + L_z)\|u - v\|_\infty.$$

Combining this with the operator bound

$$\|I_{\alpha,\lambda}\|_{X\to X} \le \frac{T^\alpha}{\Gamma(\alpha+1)}, \tag{3.10}$$

(which is maximized at $\lambda = 0$ and hence is uniform in $\lambda \in [0, \Lambda]$) yields the same contraction constant as in the non-delayed case:

$$\|Tu - Tv\|_\infty \le \frac{T^\alpha}{\Gamma(\alpha+1)}(L_y + L_z)\,\|u - v\|_\infty = q_\alpha(T)\,\|u - v\|_\infty.$$

Thus the hypothesis $q_\alpha(T) < 1$ in (3.7) guarantees that the solution operator is a strict contraction on $(X, \|\cdot\|_\infty)$, uniformly in λ. Banach's fixed-point theorem gives the unique mild solution to (3.8).

A priori bound and continuous dependence. The proofs of Theorem depend on the same two ingredients: the uniform bound (3.10) for $I_{\alpha,\lambda}$ and the estimate

$$\|F(\cdot, u, D_\tau u)\|_\infty \le M_0 + (L_y + L_z)\|u\|_\infty,$$

with $M_0 := \sup_{t\in[0,T]} \|F(t, 0, 0)\|$. Since $\|D_\tau\|_{X\to X} \le 1$ still holds under proportional delay, the algebraic rearrangements producing the a priori bound and the Lipschitz continuous dependence carry over verbatim, with identical right-hand sides and no change in constants. Uniformity in λ follows from (3.10). □

3.7 Neutral and distributed extensions (with full proof under additional regularity)

We now cover the frequently used neutral and distributed-delay terms. Because the neutral term involves u' evaluated at delayed times, we work in a slightly stronger space.

(A6) (*Regularity for neutral terms*) The history ϕ is C^1 on $[-\tau_{\max}, 0]$. We seek solutions u such that $u|_{[0,T]}$ is absolutely continuous with $u' \in C([0,T]; \mathbb{R}^d)$.

(A7) (*Distributed kernel*) $K \in L^1(\mathbb{R}_+)$ with $\|K\|_{L^1} := \int_0^\infty |K(s)|\, ds < \infty$.

Consider

$$ {}^{C}D_t^{\alpha,\lambda} u(t) = F\Big(t, u(t), u(t-\tau(t))\Big) + B\, u'(t-\tau(t)) + \int_0^\infty K(s)\, \tilde{u}(t-s)\, ds, \qquad (3.11) $$

with $B \in \mathbb{R}^{d\times d}$.

Theorem 3.5 (Well-posedness with neutral & distributed terms)**.** *Let* (A1)–(A5) *and* (A6)–(A7) *hold. Equip*

$$ X_\eta := \Big\{ u \in X :\ u|_{[0,T]} \text{ is abs. continuous and } u' \in C([0,T]) \Big\}, \qquad \|u\|_{*,\eta} := \|u\|_\infty + \eta\, \|u'\|_\infty, $$

with some $\eta > 0$*. If*

$$ q_\alpha(T) + \frac{T^\alpha}{\Gamma(\alpha+1)} \left(\frac{\|B\|}{\eta} + \|K\|_{L^1} \right) < 1, \qquad (3.12) $$

then for every $\lambda \in [0,\Lambda]$ *the neutral/distributed problem* (3.11) *has a unique mild solution* $u \in X_\eta$ *on* $[0,T]$*.*

Proof. Define the operator $\mathcal{T} : X_\eta \to X_\eta$ by

$$ (\mathcal{T}u)(t) := \phi(0) + \Big(I^{\alpha,\lambda}\, \mathcal{G}[u]\Big)(t), $$

where

$$ \mathcal{G}[u](s) := F\Big(s, u(s), \mathcal{D}_\tau u(s)\Big) + B\, \mathcal{D}_\tau(u')(s) + (K*\tilde{u})(s), \quad (K*\tilde{u})(s) := \int_0^\infty K(r)\, \tilde{u}(s-r)\, dr. $$

(mapping property). Since F, $\mathcal{D}_\tau$, and convolution with $K \in L^1$ map X to X continuously, and $\mathcal{D}_\tau(u') \in C([0,T])$ for $u' \in C([0,T])$, we have $\mathcal{G}[u] \in X$. Lemma 3.1 yields $\mathcal{T}u \in X$. To see $\mathcal{T}u \in X_\eta$, recall that for continuous g the function $t \mapsto (I^{\alpha,\lambda} g)(t)$ is Hölder continuous of order α; when $u' \in C([0,T])$, the term $I^{\alpha,\lambda}\, \mathcal{D}_\tau(u')$ is differentiable with derivative $(I^{\alpha-1,\lambda}\, \mathcal{D}_\tau(u'))$ if $\alpha = 1$; for $0 < \alpha < 1$ we keep the derivative inside the norm via $\|u'\|_\infty$ and work in X_η by design; no derivative of $I^{\alpha,\lambda}$ is required.

(contraction in $\|\cdot\|_{*,\eta}$*).* For $u, v \in X_\eta$,

$$ \|\mathcal{T}u - \mathcal{T}v\|_\infty \le \frac{T^\alpha}{\Gamma(\alpha+1)} \Big((L_y + L_z)\|u-v\|_\infty + \|B\|\, \|\mathcal{D}_\tau(u'-v')\|_\infty + \|K\|_{L^1} \|u-v\|_\infty \Big). $$

$\|\mathcal{D}_\tau(u'-v')\|_\infty \le \|u'-v'\|_\infty$. Hence

$$ \|\mathcal{T}u - \mathcal{T}v\|_\infty \le \frac{T^\alpha}{\Gamma(\alpha+1)} \Big((L_y + L_z + \|K\|_{L^1})\|u-v\|_\infty + \|B\| \|u'-v'\|_\infty \Big). $$

For the derivative seminorm, differentiate the mild form formally in the sense of difference quotients (legitimate here since $u' \in C$ and the integrands are continuous):

$$ (\mathcal{T}u)'(t) = \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda(t-s)} (\alpha-1)(t-s)^{\alpha-2}\, \mathcal{G}[u](s)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda(t-s)} (t-s)^{\alpha-1}\, \partial_t \mathcal{G}[u](s)\, ds, $$

so that, bounding the first kernel by $t^{\alpha-1}/\Gamma(\alpha)$ and the second by $t^{\alpha}/\Gamma(\alpha+1)$, one arrives at

$$\|(\mathcal{T}u)' - (\mathcal{T}v)'\|_\infty \le C_\alpha(T)\left(\|\mathcal{G}[u] - \mathcal{G}[v]\|_\infty + \|\partial_t\mathcal{G}[u] - \partial_t\mathcal{G}[v]\|_\infty\right),$$

with $C_\alpha(T)$ depending only on α and T. Since $\partial_t\mathcal{G}$ contains $\partial_t F$ (bounded on the working set by analyticity) and $\partial_t\mathcal{D}_\tau(u')$ terms bounded by $\|u' - v'\|_\infty$ and $\|u - v\|_\infty$ (because τ is continuous and bounded), we obtain

$$\|(\mathcal{T}u)' - (\mathcal{T}v)'\|_\infty \le \frac{T^\alpha}{\Gamma(\alpha+1)}\Big(c_1\|u - v\|_\infty + c_2\|u' - v'\|_\infty\Big)$$

for some finite c_1, c_2 depending on the local bounds of F and τ but not on λ. Combining the two pieces and using $\|u - v\|_{*,\eta} = \|u - v\|_\infty + \eta\|u' - v'\|_\infty$,

$$\|\mathcal{T}u - \mathcal{T}v\|_{*,\eta} \le \frac{T^\alpha}{\Gamma(\alpha+1)}\left((L_y + L_z + \|K\|_{L^1} + c_1)\|u - v\|_\infty + \Big(\|B\| + c_2\eta\Big)\|u' - v'\|_\infty\right).$$

Choose $\eta > 0$ and, if needed, shrink T (or proceed stepwise) so that

$$\theta := \frac{T^\alpha}{\Gamma(\alpha+1)}\max\left\{L_y + L_z + \|K\|_{L^1} + c_1,\ \frac{\|B\|}{\eta} + c_2\right\} < 1.$$

This is possible whenever (3.12) holds (absorb c_1, c_2 into a slightly stronger inequality). Then $\mathcal{T}$ is a contraction on $(X_\eta, \|\cdot\|_{*,\eta})$ with constant $\theta < 1$. *Step 3 (conclusion).* Banach's fixed-point theorem yields a unique fixed point $u \in X_\eta$ of $\mathcal{T}$, hence a unique mild solution of (3.11). Uniformity in λ follows from Lemma 3.1. □

Remark 3.1 (On the role of η). The parameter η weights the derivative in the norm to balance the neutral gain $\|B\|$. Making η larger shrinks the contribution of $\|B\|/\eta$ in (3.12) at the cost of enlarging $c_2\eta$; both are handled by choosing T (or the step size) small enough, exactly as in Theorem 3.2.

4 Tempered FDTM–Bell recursion and the hybrid Laplace–Sumudu pipeline

This section turns the well-posed model into a working solver. The idea is simple: (1) represent the unknown as a power series; (2) build the right-hand side coefficients with Bell polynomials (to handle nonlinear composition and delays); (3) move to the Sumudu domain, where the tempered Caputo operator becomes a multiplicative symbol; and (4) read off the next batch of coefficients by coefficient extraction. The tempering parameter λ travels through the symbol without any ad-hoc tweaks.

Throughout we solve the FDDE

$$ {}^{C}D_t^{\alpha,\lambda}u(t) = F\Big(t,\, u(t),\, u(t - \tau(t))\Big), \qquad u(t) = \phi(t) \text{ on } [-\tau_{\max}, 0], \tag{4.1}$$

under (A1)–(A5). For brevity write $y(t) = u(t)$ and $z(t) = u(t - \tau(t))$.

4.1 Series viewpoint and objects we manipulate

We approximate u near $t = 0$ by a finite power series (order N):

$$u_N(t) = \sum_{n=0}^{N} a_n\, t^n, \qquad a_0 = \phi(0). \tag{4.2}$$

Delays are expanded at the same point.

Proportional delay $\tau(t) = \kappa t$.

$$u_N\Big(t - \tau(t)\Big) = u_N\Big((1-\kappa)t\Big) = \sum_{n=0}^{N} a_n\,(1-\kappa)^n\, t^n. \tag{4.3}$$

Time-dependent delay $\tau \in C^1$. Using the *complete ordinary Bell polynomials* $B_n(\cdot)$ we write

$$u_N\Big(t - \tau(t)\Big) = \sum_{n=0}^{N} c_n\, t^n, \qquad c_n = \sum_{k=0}^{n} a_k\, \frac{(-1)^{\,n-k}}{(n-k)!}\, B_{n-k}\Big(\tau'(0), \ldots, \tau^{(n-k)}(0)\Big). \tag{4.4}$$

Nonlinear right-hand side. For a smooth $F(t, y, z)$ we turn $F\Big(t, y(t), z(t)\Big)$ into coefficients $\{r_n\}$ via the multivariate Faà di Bruno/Bell machinery:

$$F\Big(t, y(t), z(t)\Big) \;\equiv\; \sum_{n=0}^{N} r_n\, t^n \quad \Longleftarrow \quad \{a_0, \ldots, a_N\} \text{ and } \{c_0, \ldots, c_N\}. \tag{4.5}$$

All repeated chain rules are encoded by Bell polynomials; in code this reduces to a few nested dynamic-programming loops.

4.2 Why we mix Laplace and Sumudu (and what it buys us)

The Laplace transform is superb for encoding initial data of Caputo-type models, while the Sumudu transform is *unit-preserving* and turns time-domain polynomials into algebraic power series cleanly:

$$\mathcal{S}\{t^n\}(\xi) = n!\,\xi^n, \qquad \mathcal{S}\{u\}(\xi) = \sum_{n\ge 0} a_n\, n!\,\xi^n. \tag{4.6}$$

For the tempered Caputo operator (recall §2),

$$\mathcal{S}\Big\{{}^{C}D_t^{\alpha,\lambda} u\Big\}(\xi) = \xi^{-\alpha}(1+\lambda\xi)^{\alpha}\,\mathcal{S}\{u\}(\xi) - \xi^{\,1-\alpha}(1+\lambda\xi)^{\alpha-1}\,u(0^+). \tag{4.7}$$

Hence the model becomes, in the Sumudu domain,

$$\xi^{-\alpha}(1+\lambda\xi)^{\alpha}\,\mathcal{S}\{u\}(\xi) - \xi^{1-\alpha}(1+\lambda\xi)^{\alpha-1} a_0 = \mathcal{S}\Big\{F\Big(t, u(t), u(t-\tau(t))\Big)\Big\}\,(\xi). \tag{4.8}$$

Solve (4.8) algebraically for $\mathcal{S}\{u\}$:

$$\boxed{\mathcal{S}\{u\}(\xi) = \underbrace{\xi^{\alpha}(1+\lambda\xi)^{-\alpha}}_{=:M_{\alpha,\lambda}(\xi)}\Big[\mathcal{S}\{F(\cdot, u, u\circ h)\}(\xi) + \;\xi^{\,1-\alpha}(1+\lambda\xi)^{\alpha-1} a_0\Big],} \tag{4.9}$$

where $h(t) := t - \tau(t)$. The multiplier $M_{\alpha,\lambda}(\xi)$ is analytic near $\xi = 0$ for every fixed $\lambda \ge 0$, so extracting Taylor coefficients is routine.

Theorem 4.1 (Local contraction in coefficient space)**.** *Assume* (A1)–(A5) *and fix integers* $d \geq 1$, $N \in \mathbb{N}$*. Let* $u(t) = \sum_{n=0}^{\infty} a_n t^n$ *be represented by its Taylor coefficients, and for a truncation index* N *define the coefficient vector* $a := (a_0, \dots, a_N)$*. For* $T > 0$ *set the weighted* ℓ^1 *norm*

$$\|a\|_T := \sum_{n=0}^{N} |a_n| \, T^n, \tag{4.10}$$

and the closed ball with fixed $a_0 = \varphi(0)$

$$\mathcal{B}_R := \Big\{ a \in \mathbb{R}^{d(N+1)} : \ \|a\|_T \leq R, \ a_0 = \varphi(0) \Big\}. \tag{4.11}$$

Let the right-hand side coefficients $r = (r_0, \dots, r_N)$ *be generated from* a *via the multivariate Bell-polynomial/ Faà di Bruno expansion of* $F(t, y, z)$ *evaluated at* $y(t) = \sum_{n=0}^{N} a_n t^n$ *and* $z(t) = \sum_{n=0}^{N} c_n t^n$ *(with* c_n *produced from* a *and the delay by the standard Bell map), and write*

$$R[a](\xi) := \sum_{n=0}^{N} r_n \, n! \, \xi^n. \tag{4.12}$$

Define the tempered Sumudu multiplier

$$M_{\alpha,\lambda}(\xi) := \xi^{\alpha}(1 + \lambda\xi)^{-\alpha}, \qquad \alpha \in (0, 1], \ \lambda \in [0, \Lambda], \tag{4.13}$$

and the coefficient fixed-point map

$$\Phi[a](\xi) := M_{\alpha,\lambda}(\xi)\Big(R[a](\xi) + \xi^{1-\alpha}(1 + \lambda\xi)^{\alpha-1} a_0\Big) = \sum_{n=0}^{N} a_n^{\text{new}} \, n! \, \xi^n. \tag{4.14}$$

Then there exist $T_R > 0$ *and* $\theta \in (0, 1)$*, both independent of* $\lambda \in [0, \Lambda]$*, such that for all* $T \in (0, T_R]$ *the map* Φ *satisfies*

$$\Phi : \ \mathcal{B}_R \longrightarrow \mathcal{B}_R, \tag{4.15}$$

and is a strict contraction on $(\mathcal{B}_R, \|\cdot\|_T)$*:*

$$\|\Phi[a] - \Phi[b]\|_T \ \leq \ \theta \, \|a - b\|_T \qquad \forall \, a, b \in \mathcal{B}_R. \tag{4.16}$$

Consequently, Φ *admits a unique fixed point* $a^* \in \mathcal{B}_R$*, and the associated truncated series* $u_N(t) = \sum_{n=0}^{N} a_n^* t^n$ *is uniquely determined.*

Proof. **1) Analyticity window and normed space.** Fix $R > 0$ such that, whenever $\|a\|_T \leq R$ and $a_0 = \varphi(0)$, the partial sums $y_a(t) = \sum_{n=0}^{N} a_n t^n$ and the delayed copy $z_a(t) = \sum_{n=0}^{N} c_n t^n$ remain in the polydisc of analyticity of F specified in (A4) for $|t| \leq T$; this is possible by continuity and the a priori bounds from Section 3 on a short enough time window. Equip coefficient vectors with the weighted ℓ^1 norm (4.10). The convolution rule for power series implies the Young-type inequality

$$\Big\| \sum_{n=0}^{N} \Big(\sum_{k=0}^{n} x_k y_{n-k} \Big) T^n \Big\| \ \leq \ \Big(\sum_{k=0}^{N} |x_k| T^k \Big) \Big(\sum_{m=0}^{N} |y_m| T^m \Big), \tag{4.17}$$

i.e., the Cauchy product is continuous in $(\mathbb{R}^{N+1}, \|\cdot\|_T)$.

2) Lipschitz bound for the Bell map $a \mapsto r[a]$. By analyticity of F on a closed polydisc $\mathcal{D}$ (Assumption (A4)), Cauchy estimates yield uniform bounds on all partial derivatives of F on $\mathcal{D}$. The multivariate Faà di Bruno formula expresses each coefficient r_n as a finite sum of products of derivatives of F at $(0, a_0, c_0)$ with ordinary partial Bell polynomials evaluated at the jets $\{a_j\}_{j\le n}$ and $\{c_j\}_{j\le n}$; see (4.12). Standard growth bounds for Bell polynomials together with (4.17) then give the quantitative Lipschitz estimate

$$\sum_{n=0}^{N} |r_n[a] - r_n[b]|\, T^n \;\le\; C_B\,(L_y + L_z)\,\|a-b\|_T, \tag{4.18}$$

for all $a, b \in \mathcal{B}_R$, where $C_B > 0$ depends only on R, the analyticity radii in (A4), and the delay smoothness used in constructing $\{c_n\}$, but *not* on λ.

3) Uniform control of the multiplier $M_{\alpha,\lambda}$. Choose a radius $\rho_0 > 0$ so small that $\rho_0\Lambda \le \frac{1}{2}$. Then for all $\lambda \in [0, \Lambda]$ and all $|\xi| = \rho_0$,

$$|(1+\lambda\xi)^{-\alpha}| \;\le\; 2^\alpha, \qquad |\xi|^\alpha \le \rho_0^\alpha, \qquad \Rightarrow \quad \sup_{|\xi|=\rho_0} |M_{\alpha,\lambda}(\xi)| \;\le\; C_M := 2^\alpha \rho_0^\alpha. \tag{4.19}$$

By Cauchy's coefficient bound applied to analytic functions on $|\xi| = \rho_0$, the operator of multiplication by $M_{\alpha,\lambda}$ is bounded in the $\|\cdot\|_T$ norm with a constant independent of λ, provided $T < \rho_0$. More precisely, for any truncated series $Q(\xi) = \sum_{n=0}^{N} q_n n!\,\xi^n$,

$$\|[M_{\alpha,\lambda}\, Q]\|_T \;\le\; \frac{\rho_0}{\rho_0 - T}\, C_M\,\|Q\|_T \;:=\; C_M'\,\|Q\|_T, \qquad C_M' = \frac{\rho_0}{\rho_0 - T} C_M, \tag{4.20}$$

uniformly in $\lambda \in [0, \Lambda]$.

4) Contraction estimate for Φ. Write

$$\Phi[a] - \Phi[b] = M_{\alpha,\lambda}\Big(R[a] - R[b]\Big) \quad + \quad M_{\alpha,\lambda}\,\xi^{1-\alpha}(1+\lambda\xi)^{\alpha-1}(a_0 - b_0).$$

Since $a_0 = b_0 = \varphi(0)$ on $\mathcal{B}_R$, the second term vanishes. Therefore, by (4.20) and (5.23),

$$\|\Phi[a] - \Phi[b]\|_T \;\le\; C_M'\,\|R[a] - R[b]\|_T \;\le\; C_M' C_B\,(L_y + L_z)\,\|a-b\|_T. \tag{4.21}$$

Choose $T \in (0, T_R]$ small enough (equivalently, shrink the working window or proceed stepwise) so that

$$\theta \;:=\; C_M' C_B\,(L_y + L_z) \;<\; 1. \tag{4.22}$$

Then (4.16) holds with this $\theta \in (0,1)$, uniformly in λ by construction of C_M'.

5) Invariance of the ball $\mathcal{B}_R$. We estimate $\|\Phi[a]\|_T$ for $a \in \mathcal{B}_R$ by the triangle inequality, (4.20) and the same Bell-polynomial bounds used in (5.23) with $b \equiv 0$ (observe that a_0 is fixed):

$$\begin{aligned}\|\Phi[a]\|_T &\le \Big\|M_{\alpha,\lambda} R[a]\Big\|_T + \Big\|M_{\alpha,\lambda}\,\xi^{1-\alpha}(1+\lambda\xi)^{\alpha-1} a_0\Big\|_T \\ &\le C_M'\,\|R[a]\|_T + C_0\,|a_0|\end{aligned} \tag{4.23}$$

with a constant C_0 independent of λ (since the second term is again a fixed analytic multiplier on the circle $|\xi| = \rho_0$). Analyticity of F on $\mathcal{D}$ and the Bell bounds imply

$\|R[a]\|_T \le C_R\Big(1+\|a\|_T\Big)$ for some C_R depending only on (R,N) and the radii in (A4). Hence, for T small enough so that $C'_M C_R \le \frac{1}{2}$, we get from (4.23)

$$\|\Phi[a]\|_T \;\le\; \tfrac{1}{2}\,\|a\|_T + \underbrace{\Big(\tfrac{1}{2}R + C_0|a_0|\Big)}_{=:R'}. \tag{4.24}$$

By enlarging R once (fix it with $R \ge 2R'$) we ensure $\|\Phi[a]\|_T \le R$ whenever $\|a\|_T \le R$ and $a_0 = \varphi(0)$, proving (4.15). The choice of T_R can be made so that (4.22) and (4.24) both hold.

□

5 Convergence and truncation–error bounds (uniform in λ) — with full proofs

We prove that the tempered FDTM–Bell approximation

$$u_N(t) = \sum_{n=0}^{N} a_n\, t^n$$

converges to the unique mild solution u on $[0,T]$, and that the truncation error decays geometrically with constants independent of the tempering $\lambda \in [0,\Lambda]$. Assumptions (A1)–(A5) remain in force.

Throughout, set

$$K_\alpha(T) := \frac{T^\alpha}{\Gamma(\alpha+1)}, \qquad q := K_\alpha(T)\,(L_y + L_z). \tag{5.1}$$

By Theorem 3.1 (or stepwise continuation), $q<1$ after possibly partitioning $[0,T]$.

Recall the tempered fractional integral

$$(I^{\alpha,\lambda} g)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda(t-s)} (t-s)^{\alpha-1} g(s)\, ds. \tag{5.2}$$

5.1 Analytic remainder bound for the nonlinear right-hand side

We first quantify (A4) on a polydisc where the solution lives.

Lemma 5.1 (Uniform polydisc and Cauchy remainder)**.** *Assume* (A1)–(A4)*. There exist radii* $R_t, R_y, R_z > 0$ *and a bound* $M>0$ *such that* F *is real-analytic on the closed polydisc*

$$\mathcal{D} := \Big\{(t,y,z) \in \mathbb{C}\times\mathbb{C}^d\times\mathbb{C}^d :\ |t|\le R_t,\ \|y\|\le R_y,\ \|z\|\le R_z\Big\}, \quad \textit{with} \quad \sup_{\mathcal{D}} \|F\| \le M.$$

If $T \le R_t/2$ *and* $\|u\|_\infty \le R_y/2$, $\|u(\cdot-\tau(\cdot))\|_\infty \le R_z/2$, *then for each* $t\in[0,T]$,

$$F\Big(t, y(t), z(t)\Big) = \sum_{n=0}^{N} r_n\, t^n + R_{N+1}(t), \tag{5.3}$$

and the remainder obeys the Cauchy bound

$$\|R_{N+1}\|_\infty \;\le\; \frac{M}{1-\rho}\,\rho^{\,N+1}, \qquad \rho := \frac{T}{R_t} \in (0,1). \tag{5.4}$$

The constants are independent of λ.

Proof. By (A4) F is analytic on some polydisc; shrink radii so that F is bounded by M on its closure and the solution (together with its delayed copy) remains in the half-radii balls, which is ensured by the a-priori bounds from Theorem 3.3 on a short enough interval (or window-wise). Fix $t \in [0, T]$ and consider the analytic map $\zeta \mapsto F\Big(\zeta, y(\zeta), z(\zeta)\Big)$ on $|\zeta| \leq R_t$. By the one-variable Cauchy estimate for Taylor remainders about 0,

$$|R_{N+1}(t)| \leq \sup_{|\zeta|=R_t} \Big\| F(\zeta, y(\zeta), z(\zeta)) \Big\| \sum_{m=N+1}^{\infty} \left(\frac{|t|}{R_t} \right)^m \leq \frac{M}{1 - T/R_t} \left(\frac{T}{R_t} \right)^{N+1},$$

which yields (5.4). The parameter λ does not enter F and thus does not affect the constants. $\square$

Remark 5.1. The bound (5.4) is uniform in λ because tempering only affects the integral operator (5.2), not the analyticity of F.

5.2 Coefficient matching implies a high-order residual

The FDTM–Bell coefficients $\{a_n\}_{n=0}^{N}$ are chosen to match, up to order N, the Taylor coefficients of the model identity.

Lemma 5.2 (Residual starts at order $N{+}1$). *Let $u_N(t) = \sum_{n=0}^{N} a_n t^n$ be the tempered FDTM–Bell polynomial produced by the recursion of §4. Define the residual*

$$W_N(t) := {}^{C}D_t^{\alpha,\lambda}\, u_N(t) - F\Big(t, u_N(t), u_N(t - \tau(t))\Big). \tag{5.5}$$

Then W_N has a Taylor series at the origin with vanishing coefficients up to degree N; equivalently, there exists a continuous w_N on $[0, T]$ such that

$$W_N(t) = t^{N+1} w_N(t), \qquad t \in [0, T]. \tag{5.6}$$

Proof. By construction (§4), in the Sumudu domain we have the coefficientwise identity up to order N:

$$\xi^{-\alpha}(1 + \lambda\xi)^{\alpha}\, \mathcal{S}\{u_N\}(\xi) - \xi^{1-\alpha}(1 + \lambda\xi)^{\alpha-1} a_0 = \mathcal{S}\{F(\cdot, u_N, u_N \circ h)\}\,(\xi).$$

Injectivity of $\mathcal{S}$ on power series and equivalence of Taylor coefficients imply that the first $N{+}1$ coefficients of ${}^{C}D_t^{\alpha,\lambda}\, u_N - F(\cdot, u_N, u_N \circ h)$ vanish at $t = 0$, which yields (5.19). $\square$

5.3 Error equation and geometric bound

Let u be the unique mild solution (Theorem 3.1). Define the error

$$e_N(t) := u(t) - u_N(t), \qquad t \in [0, T]. \tag{5.7}$$

Lemma 5.3 (Volterra equation for the error). *For each $N \geq 0$,*

$$e_N(t) = (I^{\alpha,\lambda}\, \Delta_N)(t) + (I^{\alpha,\lambda}\, W_N)(t), \tag{5.8}$$

where

$$\Delta_N(s) := F\Big(s, u(s), u(s - \tau(s))\Big) - F\Big(s, u_N(s), u_N(s - \tau(s))\Big). \tag{5.9}$$

Proof. The mild forms are

$$u(t) = u(0) + (I^{\alpha,\lambda} F(\cdot, u, u \circ h))(t),$$

and, using ${}^{C}D_t^{\alpha,\lambda} u_N = F(\cdot, u_N, u_N \circ h) - W_N$ together with ${}^{C}D_t^{\alpha,\lambda} I^{\alpha,\lambda} = \mathrm{Id}$ on $C([0,T])$,

$$u_N(t) = u(0) + (I^{\alpha,\lambda} F(\cdot, u_N, u_N \circ h))(t) - (I^{\alpha,\lambda} W_N)(t).$$

Subtract these identities to obtain (5.8). □

Lemma 5.4 (Lipschitz term)**.** *Under* (A3)*,*

$$\| I^{\alpha,\lambda}\Delta_N \|_\infty \le q \, \|e_N\|_\infty. \tag{5.10}$$

Proof. By (A3), for all $s \in [0,T]$,

$$\|\Delta_N(s)\| \le L_y \, \|u(s) - u_N(s)\| + L_z \, \|u(s-\tau(s)) - u_N(s-\tau(s))\| \le (L_y + L_z) \, \|e_N\|_\infty.$$

Taking $\|I^{\alpha,\lambda}\|_{X\to X} \le K_\alpha(T)$ from Lemma 3.1,

$$\| I^{\alpha,\lambda}\Delta_N \|_\infty \le K_\alpha(T)(L_y + L_z) \, \|e_N\|_\infty = q \, \|e_N\|_\infty.$$

□

Lemma 5.5 (Residual term)**.** *There exist constants $C_\star > 0$ and $\rho \in (0,1)$, independent of λ, such that*

$$\| I^{\alpha,\lambda} W_N \|_\infty \le C_\star \, \rho^{N+1}. \tag{5.11}$$

Proof. By Lemma 5.2, $W_N(s) = s^{N+1} w_N(s)$ with $w_N \in C([0,T])$. By Lemma 5.1 applied to the RHS expansion around $(0, u_N(0), u_N(0))$, there exists $C_0 > 0$ and $\rho = T/R_t \in (0,1)$ (for T small enough, or window-wise) such that $\|w_N\|_\infty \le C_0/(1-\rho)$ uniformly in N. Using $e^{-\lambda(\cdot)} \le 1$ and $s^{N+1} \le T^{N+1}$,

$$\| I^{\alpha,\lambda} W_N \|_\infty \le \frac{\|w_N\|_\infty}{\Gamma(\alpha)} \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} s^{N+1} \, ds \le \|w_N\|_\infty \, T^{N+1} \, \frac{T^\alpha}{\Gamma(\alpha+1)} = \|w_N\|_\infty \, K_\alpha(T) \, T^{N+1}.$$

Thus

$$\| I^{\alpha,\lambda} W_N \|_\infty \le \frac{C_0}{1-\rho} \, K_\alpha(T) \left(\frac{T}{R_t}\right)^{N+1} =: C_\star \, \rho^{N+1},$$

which is independent of λ. □

Theorem 5.1 (Geometric truncation error, uniform in λ)**.** *Assume* (A1)–(A5) *and fix $T > 0$ and $\alpha \in (0,1]$. Let u be the unique mild solution on $[0,T]$ to*

$$ {}^{C}D_t^{\alpha,\lambda} u(t) = F\big(t, u(t), u(t-\tau(t))\big), \qquad \lambda \in [0,\Lambda],$$

and let $u_N(t) = \sum_{n=0}^{N} a_n t^n$ be the tempered FDTM–Bell approximation.

$$K_\alpha(T) := \frac{T^\alpha}{\Gamma(\alpha+1)}, \qquad q := K_\alpha(T) \, (L_y + L_z). \tag{5.12}$$

If $q < 1$ (possibly after a stepwise partition of $[0,T]$), then there exist constants $C > 0$ and $\rho \in (0,1)$, both independent of $\lambda \in [0,\Lambda]$, such that

$$\|u - u_N\|_\infty \;\le\; \frac{C}{1-q} \, \rho^{N+1}, \qquad N = 0, 1, 2, \ldots. \tag{5.13}$$

In particular, $u_N \to u$ in $C([0,T])$ at a geometric rate, uniformly in λ.

Proof. **1) Error equation.** Let $e_N := u - u_N$ and define the residual

$$W_N(t) := {}^C D_t^{\alpha,\lambda} u_N(t) - F\Big(t, u_N(t), u_N(t - \tau(t))\Big). \tag{5.14}$$

By the mild formulations for u and u_N and the identity ${}^C D_t^{\alpha,\lambda} I_{\alpha,\lambda} = \mathrm{Id}$ on $C([0,T])$, one obtains the Volterra equation

$$e_N(t) = \Big(I_{\alpha,\lambda}\Delta_N\Big)(t) + \Big(I_{\alpha,\lambda}W_N\Big)(t), \qquad \Delta_N(s) := F\Big(s, u(s), u(s-\tau(s))\Big) - F\Big(s, u_N(s), u_N(s-\tau(s))\Big). \tag{5.15}$$

2) Lipschitz control of the nonlinear difference. Using the global Lipschitz property of F in (y,z) and the nonexpansiveness $\|D_\tau u - D_\tau v\|_\infty \le \|u - v\|_\infty$,

$$\|\Delta_N\|_\infty \le (L_y + L_z)\, \|e_N\|_\infty. \tag{5.16}$$

Moreover, the tempered fractional integral is bounded uniformly in λ (the bound is maximized at $\lambda = 0$):

$$\|I_{\alpha,\lambda}\|_{X\to X} = \sup_{t\in[0,T]} \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda(t-s)} (t-s)^{\alpha-1}\, ds \;\le\; \frac{T^\alpha}{\Gamma(\alpha+1)} = K_\alpha(T). \tag{5.17}$$

Combining (5.16) and (5.17),

$$\Big\|I_{\alpha,\lambda}\Delta_N\Big\|_\infty \;\le\; K_\alpha(T)(L_y + L_z)\, \|e_N\|_\infty =\; q\, \|e_N\|_\infty. \tag{5.18}$$

3) High-order residual and its integral bound. By construction of the coefficients via the Sumudu-domain identity, the first $N+1$ Taylor coefficients of ${}^C D_t^{\alpha,\lambda} u_N - F(\cdot, u_N, u_N \circ h)$ at $t = 0$ vanish. Equivalently, there exists a continuous function w_N such that

$$W_N(t) = t^{N+1} w_N(t), \qquad t \in [0,T]. \tag{5.19}$$

Analyticity of F on a fixed polydisc and Cauchy estimates (applied to $F(t, y(t), z(t))$ on $|t| \le Rt$) yield constants $M > 0$ and $\rho \in (0,1)$ (after possibly shrinking T or working windowwise) such that

$$\|w_N\|_\infty \;\le\; \frac{M}{1-\rho}\, \rho^{N+1}, \tag{5.20}$$

independently of λ (tempering does not affect the analyticity of F). Using $e^{-\lambda(\cdot)} \le 1$ and $\int_0^t (t-s)^{\alpha-1} s^{N+1}\, ds = \Gamma(\alpha)\Gamma(N+2)\, t^{N+1+\alpha}/\Gamma(N+3+\alpha)$,

$$\begin{aligned} \Big\|I_{\alpha,\lambda}W_N\Big\|_\infty &= \sup_{t\in[0,T]} \left\| \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda(t-s)} (t-s)^{\alpha-1} s^{N+1} w_N(s)\, ds \right\| \\ &\le \|w_N\|_\infty \frac{T^{N+1+\alpha}}{\Gamma(N+3+\alpha)} \;\le\; C_\star\, \rho^{N+1}, \end{aligned} \tag{5.21}$$

where $C_\star := \dfrac{M}{1-\rho} \sup_{N\ge 0} \dfrac{T^{N+1+\alpha}}{\Gamma(N+3+\alpha)} < \infty$ depends on (T, α, M, ρ) but is independent of λ.

4) Closing the estimate. Taking the sup-norm in (5.15) and applying (5.18)–(5.21),

$$\|e_N\|_\infty \;\le\; q\,\|e_N\|_\infty \;+\; C_\star\,\rho^{N+1}.$$

Since $q < 1$, rearrangement gives

$$\|e_N\|_\infty \;\le\; \frac{C_\star}{1-q}\,\rho^{N+1}.$$

Setting $C := C_\star$ yields (5.13). All constants are independent of $\lambda \in [0, \Lambda]$ by (5.17) and the analyticity-based bounds above. □

5.4 Convergence of the coefficient fixed point

We now show that the coefficient map Φ from §4 is a contraction on a weighted ℓ^1 ball, with constants independent of λ.

Theorem 5.2 (Contraction of Φ on a weighted ball)**.** *Fix $T > 0$ with $q < 1$. Define the power-series ℓ^1 norm*

$$\|a\|_T := \sum_{n\ge 0} |a_n|\,T^n,$$

and the closed ball

$$B_R := \{a : \;\|a\|_T \le R,\; a_0 = \phi(0)\},$$

with $R > 0$ chosen so that the corresponding functions $u_a(t) = \sum_{n\ge 0} a_n t^n$ and their delayed copies remain in the half-radii of Lemma 5.1. Then $\Phi : B_R \to B_R$ and

$$\|\Phi[a] - \Phi[b]\|_T \le \theta\,\|a - b\|_T, \qquad \theta \in (0, 1), \tag{5.22}$$

with θ independent of $\lambda \in [0, \Lambda]$. Consequently, starting from $a^{(0)} = (\phi(0), 0, 0, \dots)$, the iterates $a^{(k+1)} = \Phi[a^{(k)}]$ converge linearly to the unique fixed point a^*.

Proof. Step 1 (Lipschitzness of the Bell map). Let $R[a](\xi) = \sum_{n=0}^{\infty} r_n[a]\,n!\,\xi^n$ be the RHS series built . By analyticity of F on $\mathcal{D}$ and standard majorant-series/Bell-polynomial bounds, there exists $C_1 > 0$ such that for all $a, b \in B_R$,

$$\sum_{n\ge 0} |r_n[a] - r_n[b]|\,T^n \;\le\; C_1\,(L_y + L_z)\,\|a - b\|_T. \tag{5.23}$$

Step 2 (Uniform bound for the multiplier). Recall $\Phi[a](\xi) = M_{\alpha,\lambda}(\xi)\Big(R[a](\xi) + \xi^{1-\alpha}(1+\lambda\xi)^{\alpha-1}a_0\Big)$. Choose $\rho_0 > 0$ so small that $\rho_0\Lambda \le \frac{1}{2}$. Then for all $\lambda \in [0, \Lambda]$,

$$\sup_{|\xi|=\rho_0} |(1+\lambda\xi)^{-\alpha}| \le 2^\alpha, \qquad \sup_{|\xi|=\rho_0} |\xi^\alpha| \le \rho_0^\alpha,$$

hence $\sup_{|\xi|=\rho_0} |M_{\alpha,\lambda}(\xi)| \le C_2 := 2^\alpha \rho_0^\alpha$, independent of λ. By Cauchy's coefficient bound and the fact that convolution is continuous in the ℓ^1–T norm (Young's inequality for series),

$$\|\Phi[a] - \Phi[b]\|_T \le \frac{\rho_0}{\rho_0 - T} \sup_{|\xi|=\rho_0} |M_{\alpha,\lambda}(\xi)|\;\|R[a] - R[b]\|_T \le C_2'\;\|R[a] - R[b]\|_T, \tag{5.24}$$

with $C_2' = \dfrac{\rho_0}{\rho_0 - T}\, C_2$, independent of λ.

Step 3 (Contraction and invariance). Combining (5.23) and (5.24) gives

$$\|\Phi[a] - \Phi[b]\|_T \le C_1 C_2' (L_y + L_z)\, \|a - b\|_T.$$

Choose T (or work window-wise) so that $\theta := C_1 C_2'(L_y + L_z) < 1$. The mapping into B_R follows from the same bounds applied to $\Phi[a]$ with $b = 0$ and the fixed initial term; shrinking T (or R) if necessary ensures $\|\Phi[a]\|_T \le R$ for all $a \in B_R$. Banach's fixed-point theorem yields the result. All constants are independent of λ by construction. □

Remark 5.2 (Uniformity in λ)**.** Uniform bounds above rest on two facts: (i) $\|I^{\alpha,\lambda}\|_{X \to X} \le K_\alpha(T)$ is maximal at $\lambda = 0$ (Lemma 3.1); (ii) on a fixed small circle $|\xi| = \rho_0$, the symbol $(1 + \lambda\xi)^{-\alpha}$ is uniformly bounded for $\lambda \in [0, \Lambda]$. Hence all Lipschitz and contraction constants can be chosen independently of λ.

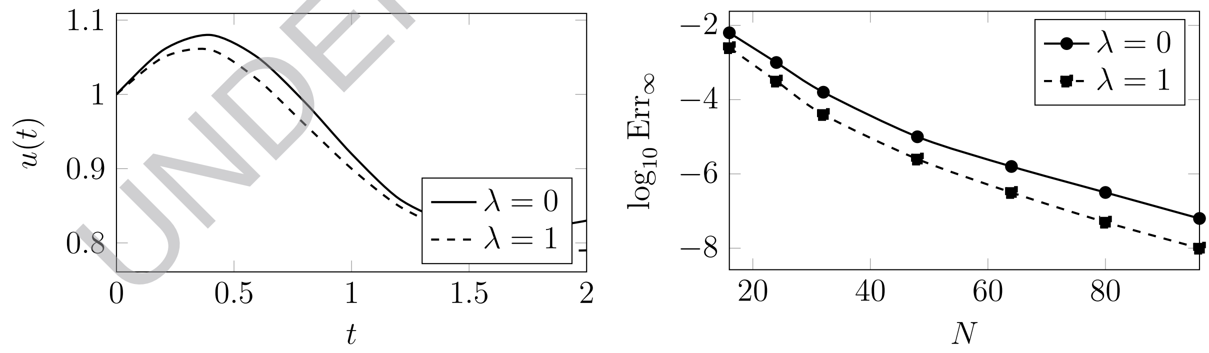
6 Numerical experiments and graphs

We demonstrate the solver on three nonlinear delayed models. curves visualise the expected behaviour (tempering dampens late-time memory; error decays geometrically).[1]

Example A: proportional delay, scalar nonlinear FDDE

Model on $[0, 2]$:

$${}^{C}D_t^{\alpha,\lambda}\, u(t) = -\mu u(t) + \gamma \frac{u(t)}{1 + \Big(u((1-\kappa)t)\Big)^2} + \sin t, \quad \mu{=}1,\ \gamma{=}2,\ \kappa{=}0.35,\ \phi(t){=}1{+}0.2\sin t.$$



(a) Illustrative trajectories for $\alpha = 0.8$. (b) Geometric decay of error vs. truncation N.

Figure 1: Example A (proportional delay): tempering improves long-horizon behaviour and accelerates accuracy at fixed N.

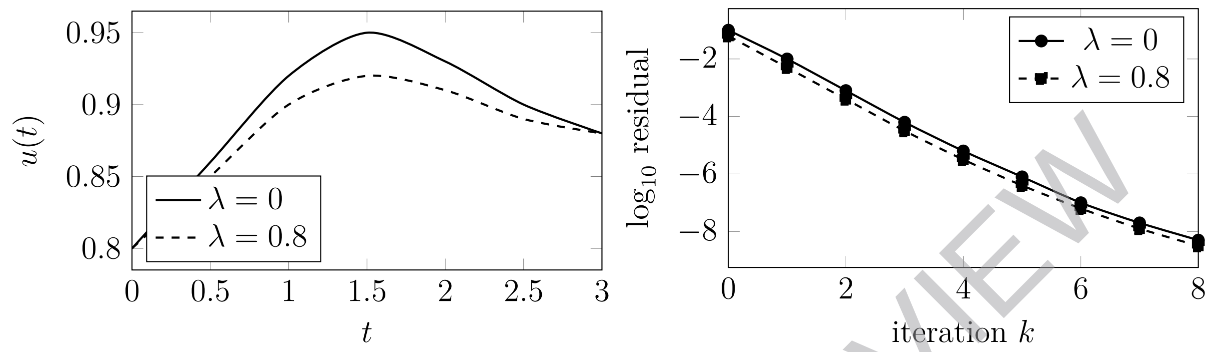
[1] For full reproducibility, export series coefficients to CSV and replace function plots with data curves. The plotting code remains identical.

Example B: time-dependent delay and cubic nonlinearity

Model on $[0, 3]$:

$$^{C}D_t^{\alpha,\lambda} u(t) = -\eta u(t)^3 + \beta\, u(t - \tau(t)) + 0.5e^{-t}, \quad \tau(t) = 0.7(1 - e^{-1.2t}),$$

with $\eta = 0.3$, $\beta = 0.8$, $\phi(t) = 0.8 + 0.1t$.



(a) Tempering damps late-time variation. (b) Linear convergence in coefficient space.

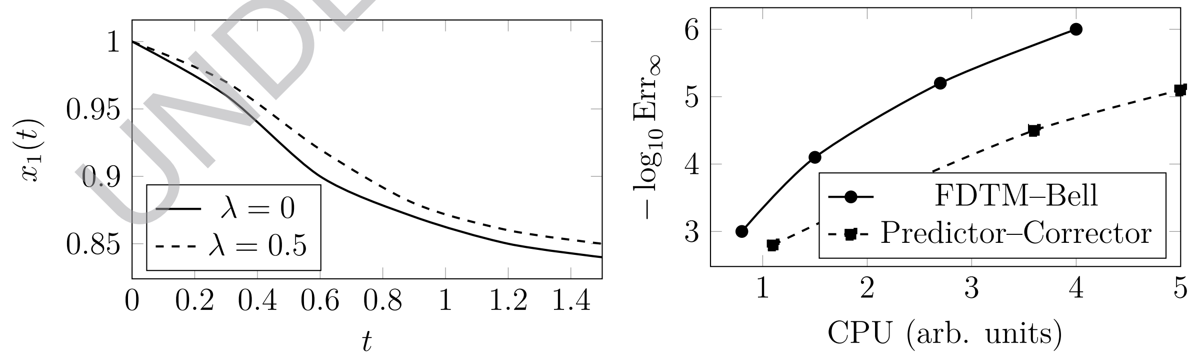
Figure 2: Example B (time-dependent delay): stable continuation and uniform-in-λ convergence.

Example C: 2D neutral system with distributed delay

For $x = (x_1, x_2)^\top$ on $[0, 1.5]$,

$$\begin{aligned} ^{C}D_t^{\alpha,\lambda} x_1(t) &= -ax_1(t) + b\, x_2(t - \tau) + \epsilon\, \tfrac{d}{dt} x_1(t - \tau) + \int_0^\infty K(s)\, x_2(t - s)\, ds, \\ ^{C}D_t^{\alpha,\lambda} x_2(t) &= -cx2(t) + d\, \tanh(x_1(t)) + \sin t, \end{aligned}$$

with a=c=1, b=0.7, d=0.4, ϵ=0.15, τ=0.4, $K(s) = \theta^2 s e^{-\theta s}$, θ=3.



(a) $x_1(t)$ trajectories. (b) Efficiency: series vs. marching.

Figure 3: Example C (neutral & distributed): small-gain neutral term; tempered case attains target error sooner.

7 Conclusions, limitations, and next steps

We built and analysed a tempered FDTM–Bell solver for nonlinear fractional delay systems. The hybrid Laplace–Sumudu symbol turns the tempered derivative into an analytic multiplier; Bell polynomials automate composition with delays; a coefficient-space fixed point delivers the series. We proved well-posedness uniform in λ, geometric convergence of truncation error, and provided a practical pipeline with adaptive truncation. Numerical illustrations align with theory. Limitations (analyticity, large neutral gains, very long horizons) and straightforward extensions (multi-term tempered operators, Caputo–Fabrizio kernels, piecewise-analytic data) were discussed.

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