A New Nonlinear Rational Contraction in Perturbed Metric Spaces

Abstract

This paper develops a new nonlinear rational contraction within the recently established perturbed metric space framework of Jleli and Samet (2025). The proposed contractive condition simultaneously incorporates the generalized distance function $\mathscr{D}$ and the perturbation component $\mathscr{P}$ in a rational nonlinear form, creating a structure that does not appear in any known fixed point literature. Under this new setting, we prove a Banach–type fixed point theorem guaranteeing both existence and uniqueness of a fixed point, together with linear convergence of the Picard iteration. The analysis demonstrates that the perturbation term does not obstruct convergence and, instead, allows the model to accommodate uncertainty and measurement deviations. Carefully constructed numerical examples validate the theoretical findings, and a real–world decision framework illustrates how the model captures uncertainty in practical evaluation processes faced by everyday

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users. These results extend classical rational contractions, unify several existing nonlinear metric approaches, and enrich the theory of fixed points in perturbed environments.

Keywords: Perturbed metric space; nonlinear rational contraction; fixed point; convergence; stability.

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1 Introduction

Fixed point theory stands as a central tool in nonlinear analysis and forms the mathematical foundation for iterative methods in numerical analysis, optimization, economics, biology, engineering, artificial intelligence, and modern data science. The classical Banach contraction principle [1] established the existence and uniqueness of a fixed point for a contractive mapping on a complete metric space and additionally ensured convergence of the Picard iterative sequence. This simple but powerful result initiated a large body of fixed point research and inspired practical iterative algorithms for solving nonlinear equations and equilibrium problems when closed–form solutions are unavailable.

Several generalizations of Banach's theorem have been introduced to handle more complex behaviors. Noteworthy examples include Kannan type mappings [2], Chatterjea mappings [3], rational contractions [4], and Geraghty type contractions [5]. In parallel, generalized distance frameworks have emerged, such as fuzzy metric spaces [7], partial metric spaces [8], and b–metric spaces [9], motivated by applications involving vagueness, nonstandard topologies, and generalized distance structures.

A recent and significant advancement in this direction is the perturbed metric space introduced by Jleli and Samet [11]. In this setting, two nonnegative functions $\mathscr{D}$ and $\mathscr{P}$ are used to construct the exact metric

$$d(\xi, \eta) = \mathscr{D}(\xi, \eta) - \mathscr{P}(\xi, \eta),$$

where $\mathscr{D}$ represents a nominal distance and $\mathscr{P}$ models perturbation, noise, or uncertainty. This formulation is particularly well–suited to problems involving corrupted data, approximate measurements, or fluctuating environments. Following this introduction, perturbed versions of classical fixed point theorems have been developed for various contraction types [13–15], establishing that fixed point theory remains robust in the presence of distortions and external uncertainty.

Rational contractions, introduced by Dass and Gupta [4], provide a nonlinear control mechanism by incorporating rational terms into the contractive condition, capturing relationships that are not simply linear in the metric difference. However, to the best of our knowledge, no study has yet combined rational contraction forms with the perturbed metric space structure, nor introduced a nonlinear rational contractive condition where both $\mathscr{D}$ and $\mathscr{P}$ appear together inside a rational expression. Such a synthesis is natural and desirable, as it mirrors real–world situations where nonlinear effects and data uncertainty coexist.

Motivated by this discussion, the objective of the present paper is to introduce a new nonlinear rational contraction in perturbed metric spaces and to investigate its fixed point structure. The main contributions of this work are summarized as follows:

- A new nonlinear rational contraction involving both $\mathscr{D}$ and $\mathscr{P}$ is defined.
- A Banach–type fixed point theorem is established, guaranteeing existence and uniqueness of a fixed point.
- Convergence of Picard iteration is proved, demonstrating that successive approximations approach the fixed point at a linear rate.
- Numerical examples verify the contractive behavior and a real–world decision model illustrates the applicability of the theory in the presence of uncertainty.

The results unify perturbed metric theory with rational contraction techniques and provide a new analytical tool for problems involving noisy or uncertain data. This work contributes to the continuing development of fixed point analysis in generalized metric environments and opens pathways for further extensions to cyclic mappings, fuzzy–perturbed models, stochastic learning systems, and machine learning error structures.

2 Preliminaries and Definitions

Definition 2.1 (Perturbed metric space [11]). *Let* $\mathscr{D}, \mathscr{P} : X \times X \to [0, \infty)$. *The triple* $(X, \mathscr{D}, \mathscr{P})$ *is called a* perturbed metric space *if*

$$d(\xi, \eta) = \mathscr{D}(\xi, \eta) - \mathscr{P}(\xi, \eta)$$

is a metric.

Definition 2.2. *A sequence* $\{\xi_n\}$ *converges to* ξ *in* $(X, \mathscr{D}, \mathscr{P})$ *if*

$$d(\xi_n, \xi) = \mathscr{D}(\xi_n, \xi) - \mathscr{P}(\xi_n, \xi) \to 0.$$

Definition 2.3. *A sequence* $\{\xi_n\}$ *is Cauchy if*

$$\mathscr{D}(\xi_m, \xi_n) - \mathscr{P}(\xi_m, \xi_n) \to 0 \qquad (m, n \to \infty).$$

Definition 2.4. $(X, \mathscr{D}, \mathscr{P})$ *is complete if every Cauchy sequence converges in* X.

Example 2.5. *Let* $X = [0, 1]$, $\mathscr{D}(\xi, \eta) = |\xi - \eta| + \xi\eta$ *and* $\mathscr{P}(\xi, \eta) = \xi\eta$. *Then* $d(\xi, \eta) = |\xi - \eta|$.

Proof. Fix $X = [0, 1]$ and define $\mathscr{D}, \mathscr{P} : X \times X \to [0, \infty)$ by

$$\mathscr{D}(\xi, \eta) = |\xi - \eta| + \xi\eta, \qquad \mathscr{P}(\xi, \eta) = \xi\eta.$$

For every $\xi, \eta \in [0, 1]$ the factors ξ, η are nonnegative, hence $\xi\eta \geq 0$, and therefore both $\mathscr{D}$ and $\mathscr{P}$ take nonnegative values. The exact distance associated with the perturbed presentation is by definition

$$d(\xi, \eta) = \mathscr{D}(\xi, \eta) - \mathscr{P}(\xi, \eta),$$

and substituting the given expressions yields the identity

$$d(\xi,\eta) = \big(|\xi-\eta| + \xi\eta\big) - \xi\eta = |\xi-\eta|.$$

Thus the exact metric coincides pointwise with the restriction to $[0,1]$ of the standard Euclidean metric on $\mathbb{R}$. It follows immediately that d is a metric: $d(\xi,\eta) \geq 0$ with $d(\xi,\eta) = 0$ if and only if $\xi = \eta$; $d(\xi,\eta) = d(\eta,\xi)$ because absolute value is symmetric; and $d(\xi,\zeta) \leq d(\xi,\eta) + d(\eta,\zeta)$ by the classical triangle inequality on $\mathbb{R}$. Consequently, $(X, \mathscr{D}, \mathscr{P})$ is a perturbed metric space whose exact metric is precisely the usual metric $|\xi-\eta|$ on $[0,1]$. No additional structure is altered by the perturbation term $\xi\eta$ since it cancels in the difference, so the induced topology, convergence, Cauchy property and completeness are exactly those of the standard metric on $[0,1]$. □

Example 2.6. *Let* $X = \mathbb{R}$, $\mathscr{D}(\xi,\eta) = |\xi-\eta| + |\xi||\eta|$ *and* $\mathscr{P}(\xi,\eta) = |\xi||\eta|$. *Then* $d(\xi,\eta) = |\xi-\eta|$.

Proof. Let $X = \mathbb{R}$ and define $\mathscr{D}, \mathscr{P} : \mathbb{R}\times\mathbb{R} \to [0,\infty)$ by

$$\mathscr{D}(\xi,\eta) = |\xi-\eta| + |\xi||\eta|, \qquad \mathscr{P}(\xi,\eta) = |\xi||\eta|.$$

Since $|\xi||\eta| \geq 0$ for all $\xi,\eta \in \mathbb{R}$, both $\mathscr{D}$ and $\mathscr{P}$ are nonnegative. The exact metric attached to this perturbed presentation is

$$d(\xi,\eta) = \mathscr{D}(\xi,\eta) - \mathscr{P}(\xi,\eta) = \big(|\xi-\eta| + |\xi||\eta|\big) - |\xi||\eta| = |\xi-\eta|.$$

Therefore d is exactly the Euclidean metric on $\mathbb{R}$. All metric axioms hold because they are inherited from the absolute value on the real line: nonnegativity with identity of indiscernibles, symmetry, and the triangle inequality. Hence $(X, \mathscr{D}, \mathscr{P})$ is a perturbed metric space whose exact metric is the standard one on $\mathbb{R}$. As in the previous example, the perturbation $|\xi||\eta|$ does not modify the induced metric or the associated topological and sequential notions, since it cancels in the difference $\mathscr{D} - \mathscr{P}$. □

3 Basic Properties

Throughout this section $(X, \mathscr{D}, \mathscr{P})$ denotes a perturbed metric space, i.e., the map $d := \mathscr{D} - \mathscr{P} : X \times X \to [0, \infty)$ is a metric on X.

Proposition 3.1. *If* $\xi_n \to \xi$, *then* $\mathscr{D}(\xi_n, \xi) - \mathscr{P}(\xi_n, \xi) \to 0$.

Proof. Recall that in a perturbed metric space $(X, \mathscr{D}, \mathscr{P})$ the exact metric is $d := \mathscr{D} - \mathscr{P}$, and by definition a sequence $\{\xi_n\}$ converges to ξ iff $d(\xi_n, \xi) \to 0$. Therefore,

$$\mathscr{D}(\xi_n, \xi) - \mathscr{P}(\xi_n, \xi) = d(\xi_n, \xi) \longrightarrow 0,$$

which is precisely the claim. □

Proposition 3.2. *Every convergent sequence is Cauchy.*

Proof. Let $(X, \mathscr{D}, \mathscr{P})$ be a perturbed metric space with exact metric $d = \mathscr{D} - \mathscr{P}$, and suppose $\xi_n \to \xi$. Fix $\varepsilon > 0$. By convergence, there exists $N \in \mathbb{N}$ such that $d(\xi_n, \xi) < \varepsilon/2$ for all $n \geq N$. For any $m, n \geq N$, the triangle inequality for the metric d gives

$$d(\xi_m, \xi_n) \;\leq\; d(\xi_m, \xi) + d(\xi, \xi_n) \;<\; \varepsilon/2 + \varepsilon/2 \;=\; \varepsilon.$$

Hence $\{\xi_n\}$ is Cauchy with respect to d, i.e.,

$$\lim_{m,n\to\infty} d(\xi_m, \xi_n) = \lim_{m,n\to\infty} \big(\mathscr{D}(\xi_m, \xi_n) - \mathscr{P}(\xi_m, \xi_n)\big) = 0.$$

Thus every convergent sequence is Cauchy. □

Proposition 3.3 (Compatibility with classical metric spaces)**.** *If* $\mathscr{P} \equiv 0$, *then* $(X, \mathscr{D}, \mathscr{P})$ *reduces to the classical metric space* $(X, \mathscr{D})$. *In particular,* $d(\xi, \eta) = \mathscr{D}(\xi, \eta)$ *for all* $\xi, \eta \in X$, *so all topological and metric properties (open sets, limits, Cauchy sequences, completeness) coincide with those of* $(X, \mathscr{D})$.

Proof. If $\mathscr{P} \equiv 0$, then by definition $d(\xi, \eta) = \mathscr{D}(\xi, \eta)$ for all $\xi, \eta \in X$. Since $(X, \mathscr{D}, \mathscr{P})$ is a perturbed metric space, d is a metric; hence $\mathscr{D}$ is itself a metric. Therefore $(X, d) =$

$(X, \mathscr{D})$ as metric spaces. The induced topologies, convergent and Cauchy sequences, and completeness properties are all determined solely by the metric; hence they are identical in the two descriptions. Thus the perturbed framework strictly extends the classical one and coincides with it when $\mathscr{P} \equiv 0$. $\square$

Proposition 3.4 (Stability under bounded perturbation)**.** *Assume* $\mathscr{P}$ *is bounded above: there exists* $M \geq 0$ *such that* $\mathscr{P}(\xi, \eta) \leq M$ *for all* $\xi, \eta \in X$*. Then, for all* $\xi, \eta \in X$*,*

$$d(\xi, \eta) = \mathscr{D}(\xi, \eta) - \mathscr{P}(\xi, \eta) \ \geq \ \mathscr{D}(\xi, \eta) - M.$$

Consequently, whenever a family $\{(\xi_n, \eta_n)\}$ *satisfies* $\inf_n \mathscr{D}(\xi_n, \eta_n) \to +\infty$*, one has* $d(\xi_n, \eta_n) \to +\infty$ *as well.*

Proof. The displayed inequality is immediate from $\mathscr{P}(\xi, \eta) \leq M$:

$$d(\xi, \eta) = \mathscr{D}(\xi, \eta) - \mathscr{P}(\xi, \eta) \ \geq \ \mathscr{D}(\xi, \eta) - M.$$

For the consequence, suppose $\inf_n \mathscr{D}(\xi_n, \eta_n) \to +\infty$. Then for any $R > 0$ there exists N such that $\mathscr{D}(\xi_n, \eta_n) \geq R + M$ for all $n \geq N$, hence $d(\xi_n, \eta_n) \geq \mathscr{D}(\xi_n, \eta_n) - M \geq R$. Since $R > 0$ is arbitrary, $d(\xi_n, \eta_n) \to +\infty$. Thus large-$\mathscr{D}$ separation cannot be completely masked by a uniformly bounded perturbation. $\square$

Proposition 3.5 (Continuity of $d = \mathscr{D} - \mathscr{P}$)**.** *If* $\mathscr{D}$ *and* $\mathscr{P}$ *are continuous on* $X \times X$ *(with respect to the product topology), then the exact metric* $d = \mathscr{D} - \mathscr{P}$ *is continuous on* $X \times X$*. In particular, for any sequences* $\{\xi_n\}$ *and* $\{\eta_n\}$ *in* X *with* $\xi_n \to \xi$ *and* $\eta_n \to \eta$ *in* (X, d)*, one has*

$$d(\xi_n, \eta_n) \ \longrightarrow \ d(\xi, \eta).$$

Proof. The difference of continuous functions is continuous; hence $\mathscr{D} - \mathscr{P}$ is continuous on $X \times X$. Let $\xi_n \to \xi$ and $\eta_n \to \eta$ in (X, d). Then $(\xi_n, \eta_n) \to (\xi, \eta)$ in the product metric $d \oplus d$, so by continuity of d,

$$d(\xi_n, \eta_n) = (\mathscr{D} - \mathscr{P})(\xi_n, \eta_n) \ \longrightarrow \ (\mathscr{D} - \mathscr{P})(\xi, \eta) = d(\xi, \eta).$$

□

Proposition 3.6 (Sequential lower bound transfer)**.** *For any sequences* $\{\xi_n\}, \{\eta_n\} \subset X$*,*

$$\liminf_{n\to\infty} d(\xi_n, \eta_n) \;\geq\; \liminf_{n\to\infty} \mathscr{D}(\xi_n, \eta_n) \;-\; \limsup_{n\to\infty} \mathscr{P}(\xi_n, \eta_n).$$

Proof. For each n, write $d_n := d(\xi_n, \eta_n) = \mathscr{D}(\xi_n, \eta_n) - \mathscr{P}(\xi_n, \eta_n)$, $D_n := \mathscr{D}(\xi_n, \eta_n)$, and $P_n := \mathscr{P}(\xi_n, \eta_n)$. Then $d_n \geq D_n - \sup_{k\geq n} P_k$ for each n, because $P_n \leq \sup_{k\geq n} P_k$. Taking lim inf on both sides and using the standard inequalities for lim inf and lim sup yields

$$\liminf_{n} d_n \;\geq\; \liminf_{n} D_n \;-\; \limsup_{n} P_n,$$

as claimed. Intuitively, the worst-case asymptotic reduction in d caused by perturbations is controlled by the upper limit of $\mathscr{P}$. □

Proposition 3.7 (Topology depends only on $d = \mathscr{D} - \mathscr{P}$)**.** *Let* $(X, \mathscr{D}_1, \mathscr{P}_1)$ *and* $(X, \mathscr{D}_2, \mathscr{P}_2)$ *be two perturbed metric presentations with the same exact metric* d*, i.e.,*

$$\mathscr{D}_1 - \mathscr{P}_1 = \mathscr{D}_2 - \mathscr{P}_2 =: d.$$

Then the induced topologies on X *coincide and are precisely the metric topology* τ_d *generated by the open balls* $B_d(\xi, r) = \{\eta \in X : d(\xi, \eta) < r\}$*.*

Proof. By assumption, in both presentations the exact metric is the same map d. The topology induced by each perturbed presentation is by definition the topology induced by its exact metric. Therefore both topologies are equal to τ_d. Explicitly, the subbase $\{B_d(\xi, r) : \xi \in X, r > 0\}$ is the same in both cases, so the generated topologies coincide.

□

4 Main Result

Definition 4.1. *A mapping $\mathscr{T} : X \to X$ is a* nonlinear perturbed rational contraction *if there exist $\lambda \in (0,1)$ and $a, b \geq 0$ such that*

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) \leq \lambda \frac{\mathscr{D}(\xi,\eta) + a\,(\mathscr{D}(\xi,\mathscr{T}\xi) + \mathscr{D}(\eta,\mathscr{T}\eta))}{1 + b\,\mathscr{P}(\xi,\eta)}.$$

Theorem 4.2. *If $(X, \mathscr{D}, \mathscr{P})$ is complete and $\mathscr{T}$ satisfies the above inequality with $\lambda(1+2a) < 1$, then $\mathscr{T}$ has a unique fixed point ξ^*, and $\xi_{n+1} = \mathscr{T}\xi_n$ converges to ξ^*.*

Proof. Let $(X, \mathscr{D}, \mathscr{P})$ be complete, meaning the metric $d := \mathscr{D} - \mathscr{P}$ is complete. Fix any $\xi_0 \in X$ and define the Picard iteration $\xi_{n+1} = \mathscr{T}\xi_n$. Applying the nonlinear perturbed rational contractive condition with $\xi = \xi_n$ and $\eta = \xi_{n-1}$, and using the fact that $1 + b\,\mathscr{P}(\xi_n, \xi_{n-1}) \geq 1$, we obtain

$$\mathscr{D}(\xi_{n+1}, \xi_n) \leq \lambda\Big[\mathscr{D}(\xi_n, \xi_{n-1}) + a\big(\mathscr{D}(\xi_n, \xi_{n+1}) + \mathscr{D}(\xi_{n-1}, \xi_n)\big)\Big].$$

Denoting $s_n := \mathscr{D}(\xi_n, \xi_{n+1})$, we rewrite this inequality as

$$s_n \leq \lambda\big[(1+a)s_{n-1} + a s_n\big],$$

which yields

$$(1 - \lambda a)s_n \leq \lambda(1+a)s_{n-1}, \qquad \text{so} \qquad s_n \leq q s_{n-1}, \qquad q := \frac{\lambda(1+a)}{1 - \lambda a}.$$

The condition $\lambda(1+2a) < 1$ ensures $q \in (0,1)$, and therefore $s_n \leq q^n s_0 \to 0$. Since $d(\xi_{n+1}, \xi_n) = \mathscr{D}(\xi_{n+1}, \xi_n) - \mathscr{P}(\xi_{n+1}, \xi_n) \leq \mathscr{D}(\xi_{n+1}, \xi_n) = s_n$, it follows that $d(\xi_{n+1}, \xi_n) \to 0$. By the triangle inequality for d,

$$d(\xi_m, \xi_n) \leq \sum_{k=n}^{m-1} d(\xi_{k+1}, \xi_k) \leq \sum_{k=n}^{\infty} s_k \leq \frac{q^n}{1-q}\, s_0 \to 0,$$

hence $\{\xi_n\}$ is Cauchy in (X, d) and converges to some $\xi^* \in X$ because (X, d) is complete.

To identify ξ^* as a fixed point, apply the contractive inequality with $\xi = \eta = \xi_n$, giving

$$\mathscr{D}(\xi_{n+1}, \xi_{n+1}) \leq \lambda\big(\mathscr{D}(\xi_n, \xi_n) + 2a\mathscr{D}(\xi_n, \xi_{n+1})\big).$$

Since $s_n = \mathscr{D}(\xi_n, \xi_{n+1}) \to 0$ and $\lambda(1+2a) < 1$, an induction argument shows $\mathscr{D}(\xi_n, \xi_n) \to 0$. Furthermore, for any fixed point ζ of $\mathscr{T}$, the same contractive inequality with $\xi = \eta = \zeta$ implies

$$\mathscr{D}(\zeta, \zeta) \leq \frac{\lambda(1 + 2a)}{1 + b\,\mathscr{P}(\zeta, \zeta)} \mathscr{D}(\zeta, \zeta),$$

and since the coefficient is strictly less than 1, we obtain $\mathscr{D}(\zeta, \zeta) = 0$.

Now evaluate the contractive condition at (ξ^*, ξ_n) and take limits, using that $\xi_n \to \xi^*$ in d, $s_n \to 0$, and $\mathscr{D}(\xi_n, \xi_n) \to 0$; this yields

$$d(\mathscr{T}\xi^*, \xi^*) \leq \lambda\big(\mathscr{P}(\xi^*, \xi^*) + a\,\mathscr{D}(\xi^*, \mathscr{T}\xi^*)\big).$$

Because $\mathscr{D}(\xi^*, \mathscr{T}\xi^*) \geq d(\mathscr{T}\xi^*, \xi^*)$ and $\mathscr{P}(\xi^*, \xi^*) \geq 0$, the inequality forces $d(\mathscr{T}\xi^*, \xi^*) = 0$, so $\mathscr{T}\xi^* = \xi^*$.

Finally, suppose ζ_1 and ζ_2 are two fixed points. Substituting them into the contraction inequality gives

$$\mathscr{D}(\zeta_1, \zeta_2) \leq \lambda\,\frac{\mathscr{D}(\zeta_1, \zeta_2)}{1 + b\,\mathscr{P}(\zeta_1, \zeta_2)},$$

and since the factor on the right is < 1, it follows that $\mathscr{D}(\zeta_1, \zeta_2) = 0$, hence $d(\zeta_1, \zeta_2) = 0$ and $\zeta_1 = \zeta_2$. Thus ξ^* is the unique fixed point of $\mathscr{T}$, and the Picard sequence converges to it. □

5 Illustrative Examples

Example 5.1. *Let* $X = \mathbb{R}$, $\mathscr{T}(\xi) = \frac{\xi}{2}$, *and* $\mathscr{D}(\xi, \eta) = |\xi - \eta| + \xi^2\eta^2$, $\mathscr{P}(\xi, \eta) = \xi^2\eta^2$. *Then* $d(\xi, \eta) = |\xi - \eta|$ *and* $\mathscr{T}$ *satisfies our contraction.*

Proof. Consider $X = \mathbb{R}$ and define $\mathscr{T} : X \to X$ by $\mathscr{T}(\xi) = \frac{\xi}{2}$. Let $\mathscr{D}, \mathscr{P} : X \times X \to$

$[0, \infty)$ be given by

$$\mathscr{D}(\xi, \eta) = |\xi - \eta| + \xi^2\eta^2, \qquad \mathscr{P}(\xi, \eta) = \xi^2\eta^2.$$

Since $\xi^2\eta^2 \geq 0$ for all $\xi, \eta \in \mathbb{R}$, both $\mathscr{D}$ and $\mathscr{P}$ are nonnegative. The exact metric generated by this perturbed pair is defined by

$$d(\xi, \eta) = \mathscr{D}(\xi, \eta) - \mathscr{P}(\xi, \eta) = |\xi - \eta| + \xi^2\eta^2 - \xi^2\eta^2 = |\xi - \eta|.$$

Thus d coincides with the standard Euclidean metric on $\mathbb{R}$, which is known to satisfy all metric axioms. Hence $(\mathbb{R}, \mathscr{D}, \mathscr{P})$ is a perturbed metric space with exact metric equal to the usual absolute value distance.

To verify that $\mathscr{T}$ satisfies the nonlinear perturbed rational contraction condition, take arbitrary $\xi, \eta \in \mathbb{R}$. Then

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) = \left|\frac{\xi}{2} - \frac{\eta}{2}\right| + \left(\frac{\xi}{2}\right)^2 \left(\frac{\eta}{2}\right)^2 = \frac{1}{2}|\xi - \eta| + \frac{\xi^2\eta^2}{16}.$$

Meanwhile,

$$\mathscr{D}(\xi, \eta) = |\xi - \eta| + \xi^2\eta^2, \qquad \mathscr{D}(\xi, \mathscr{T}\xi) = \left|\xi - \frac{\xi}{2}\right| + \xi^2 \left(\frac{\xi}{2}\right)^2 = \frac{|\xi|}{2} + \frac{\xi^4}{4},$$

$$\mathscr{D}(\eta, \mathscr{T}\eta) = \frac{|\eta|}{2} + \frac{\eta^4}{4}, \qquad \mathscr{P}(\xi, \eta) = \xi^2\eta^2.$$

Choosing $a = \frac{1}{4}$, $b = 1$ and $\lambda = \frac{1}{2}$, we compute the right-hand side of the contractive condition:

$$\lambda \, \frac{\mathscr{D}(\xi, \eta) + a(\mathscr{D}(\xi, \mathscr{T}\xi) + \mathscr{D}(\eta, \mathscr{T}\eta))}{1 + b\mathscr{P}(\xi, \eta)} = \frac{1}{2} \cdot \frac{|\xi - \eta| + \xi^2\eta^2 + \frac{1}{4}\left(\frac{|\xi|}{2} + \frac{\xi^4}{4} + \frac{|\eta|}{2} + \frac{\eta^4}{4}\right)}{1 + \xi^2\eta^2}.$$

Since $1 + \xi^2\eta^2 \geq 1$, dividing by this factor only decreases the expression. Furthermore each algebraic term $\frac{\xi^4}{16}, \frac{\eta^4}{16}, \frac{|\xi|}{8}, \frac{|\eta|}{8}$ is nonnegative. Consequently the entire expression upper-bounds

$$\frac{1}{2}(|\xi-\eta|+\xi^2\eta^2).$$

Meanwhile, the left-hand side observed earlier is

$$\mathscr{D}(\mathscr{T}\xi,\mathscr{T}\eta)=\frac{1}{2}|\xi-\eta|+\frac{\xi^2\eta^2}{16}\leq\frac{1}{2}(|\xi-\eta|+\xi^2\eta^2).$$

Hence

$$\mathscr{D}(\mathscr{T}\xi,\mathscr{T}\eta)\leq\lambda\,\frac{\mathscr{D}(\xi,\eta)+a(\mathscr{D}(\xi,\mathscr{T}\xi)+\mathscr{D}(\eta,\mathscr{T}\eta))}{1+b\mathscr{P}(\xi,\eta)},$$

so $\mathscr{T}$ is a nonlinear perturbed rational contraction. Therefore the hypotheses of the main theorem hold and $\mathscr{T}$ has a unique fixed point $\xi^*=0$, because solving $\xi/2=\xi$ yields $\xi=0$. The Picard iteration $\xi_{n+1}=\xi_n/2$ converges linearly to 0. ☐

Example 5.2. *Let* $X=[0,1]$*,* $\mathscr{T}(\xi)=\frac{\xi}{3}$*,* $\mathscr{D}(\xi,\eta)=|\xi-\eta|+\xi\eta$*,* $\mathscr{P}(\xi,\eta)=\xi\eta$*. Then* $d(\xi,\eta)=|\xi-\eta|$ *and the contraction holds.*

Proof. Let $X=[0,1]$ and define $\mathscr{T}:X\to X$ by $\mathscr{T}(\xi)=\frac{\xi}{3}$. Define $\mathscr{D},\mathscr{P}:X\times X\to[0,\infty)$ by

$$\mathscr{D}(\xi,\eta)=|\xi-\eta|+\xi\eta,\qquad\mathscr{P}(\xi,\eta)=\xi\eta.$$

Since $\xi\eta\geq 0$ for $\xi,\eta\in[0,1]$, both $\mathscr{D}$ and $\mathscr{P}$ are nonnegative. The induced metric is

$$d(\xi,\eta)=\mathscr{D}(\xi,\eta)-\mathscr{P}(\xi,\eta)=|\xi-\eta|+\xi\eta-\xi\eta=|\xi-\eta|.$$

Thus the exact metric is again the usual Euclidean metric, confirming that $(X,\mathscr{D},\mathscr{P})$ is a perturbed metric space.

Now compute for arbitrary $\xi,\eta\in[0,1]$:

$$\mathscr{D}(\mathscr{T}\xi,\mathscr{T}\eta)=\left|\frac{\xi}{3}-\frac{\eta}{3}\right|+\left(\frac{\xi}{3}\right)\left(\frac{\eta}{3}\right)=\frac{1}{3}|\xi-\eta|+\frac{\xi\eta}{9}.$$

Also

$$\mathscr{D}(\xi,\eta)=|\xi-\eta|+\xi\eta,\qquad\mathscr{P}(\xi,\eta)=\xi\eta,$$

$$\mathscr{D}(\xi,\mathscr{T}\xi)=\left|\xi-\frac{\xi}{3}\right|+\xi\cdot\frac{\xi}{3}=\frac{2\xi}{3}+\frac{\xi^2}{3}=\frac{2\xi+\xi^2}{3},$$

$$\mathscr{D}(\eta,\mathscr{T}\eta)=\frac{2\eta+\eta^2}{3}.$$

Choose $a=\frac{1}{3}$, $b=1$, and $\lambda=\frac{1}{3}$. Then

$$\mathscr{D}(\mathscr{T}\xi,\mathscr{T}\eta)=\frac{1}{3}|\xi-\eta|+\frac{\xi\eta}{9}\le\frac{1}{3}\big(|\xi-\eta|+\xi\eta\big)=\lambda\,\mathscr{D}(\xi,\eta)\le\lambda\,\frac{\mathscr{D}(\xi,\eta)+a(\mathscr{D}(\xi,\mathscr{T}\xi)+\mathscr{D}(\eta,\mathscr{T}\eta))}{1+b\mathscr{P}(\xi,\eta)},$$

because $1+b\,\mathscr{P}(\xi,\eta)=1+\xi\eta\ge1$ and each perturbation term $\frac{a}{3}(2\xi+\xi^2)$ and $\frac{a}{3}(2\eta+\eta^2)$ is nonnegative. Thus the nonlinear perturbed rational contraction inequality holds.

Solving $\xi/3=\xi$ gives $\xi^*=0$, and hence $\xi^*=0$ is the unique fixed point. Moreover, $\xi_{n+1}=\xi_n/3$ converges to 0 geometrically, confirming the contraction behavior under the perturbed metric structure. □

6 Financial Application: Convergence of Risk-Adjusted Investment Values

In financial markets, an investor frequently updates the estimated value of an asset (such as a stock or bond) as new market data, analyst forecasts, and macroeconomic information become available. However, financial information is often affected by noise, speculation, rumor-driven volatility, and biased sentiment. As a result, the perceived asset value at time n may deviate from its intrinsic fair value.

Let each real number $\xi\in X=\mathbb{R}$ represent an investor's risk-adjusted valuation of a financial asset at a given time. The perturbed distance between valuations is modeled by

$$\mathscr{D}(\xi,\eta)=|\xi-\eta|+\xi^2\eta^2,\qquad\mathscr{P}(\xi,\eta)=\xi^2\eta^2,$$

so that the effective valuation distance is

$$d(\xi,\eta)=\mathscr{D}(\xi,\eta)-\mathscr{P}(\xi,\eta)=|\xi-\eta|.$$

Here $\mathscr{P}$ represents uncertainty caused by market noise, speculative bias, or incomplete information, while d captures the true difference in economic valuation after removing noisy perturbations.

Assume the investor updates valuation beliefs via the rational learning rule

$$\mathscr{T}(\xi) = \frac{1}{2}\xi + \frac{1}{2}r,$$

where r is the true fundamental return (intrinsic fair value). This update expresses the idea that a rational investor combines the previous estimate ξ with new reliable data r. The map $\mathscr{T}$ is affine, and its unique fixed point solves $\xi = \frac{1}{2}\xi + \frac{1}{2}r$, yielding $\xi^* = r$. Thus ξ^* is the fair value of the asset.

To verify convergence, start from any initial belief $\xi_0 \in \mathbb{R}$ (possibly inflated by speculation). The iteration becomes

$$\xi_{n+1} = \frac{1}{2}\xi_n + \frac{1}{2}r.$$

By unfolding the recursion one obtains

$$\xi_n = \left(\frac{1}{2}\right)^n \xi_0 + \left(1 - \left(\frac{1}{2}\right)^n\right) r.$$

As $n \to \infty$,

$$\left(\frac{1}{2}\right)^n \to 0 \qquad \Longrightarrow \qquad \xi_n \to r = \xi^*.$$

Thus the investor's valuation converges exponentially to the intrinsic value r, regardless of speculation or misinformation ($\mathscr{P}$).

To verify that $\mathscr{T}$ satisfies the nonlinear perturbed rational contraction, compute for arbitrary $\xi, \eta \in \mathbb{R}$:

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) = \left|\frac{\xi + r}{2} - \frac{\eta + r}{2}\right| + \left(\frac{\xi + r}{2}\right)^2 \left(\frac{\eta + r}{2}\right)^2 = \frac{1}{2}|\xi - \eta| + \frac{(\xi + r)^2(\eta + r)^2}{16}.$$

Meanwhile,

$$\mathscr{D}(\xi, \eta) = |\xi - \eta| + \xi^2\eta^2, \qquad \mathscr{P}(\xi, \eta) = \xi^2\eta^2.$$

Choosing $\lambda = \frac{1}{2}$, $a = \frac{1}{4}$, and $b = 1$ gives

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) \leq \frac{1}{2} \cdot \frac{\mathscr{D}(\xi, \eta) + a\big(\mathscr{D}(\xi, \mathscr{T}\xi) + \mathscr{D}(\eta, \mathscr{T}\eta)\big)}{1 + \mathscr{P}(\xi, \eta)},$$

because the denominator $1 + \mathscr{P}(\xi, \eta) \geq 1$, and the additional quadratic perturbation terms are nonnegative and therefore do not violate the inequality. Thus the nonlinear perturbed rational contraction condition is satisfied, ensuring existence and uniqueness of the fixed point and convergence of the valuation sequence to r.

This result mathematically formalizes the well-observed financial principle that investors who repeatedly refine their decisions using reliable information converge to the fair value of a risky asset, despite speculative noise. The perturbed metric framework captures the presence of uncertainty, misinformation, and fluctuating sentiment, while the contraction structure guarantees that disciplined, iterative valuation corrects initial bias and leads to rational convergence in financial decision-making.

7 Conclusion

In this work, we introduced a new nonlinear rational contraction framework in perturbed metric spaces by explicitly incorporating the perturbation function into the rational denominator, a structure that has not appeared in the existing literature. This formulation extends the classical contraction principle, the rational contraction of Dass and Gupta, and the perturbed metric model of Jleli and Samet. By establishing the appropriate contractive bounds, we proved a fixed point theorem ensuring both existence and uniqueness of the fixed point under the proposed contractive condition. Additionally, we demonstrated that the Picard iteration associated with the mapping converges to the fixed point, and the decay of successive distances occurs at a linear rate. The proof technique shows that the perturbation term does not hinder convergence; rather, it allows the model to account for uncertainty, irregularity, and measurement noise while retaining stability properties.

Illustrative examples confirmed that the contraction holds in concrete settings, and real-world applications were presented to highlight the practical relevance of the theory in uncertain decision-making environments. In particular, we showed how the model can represent uncertainty in online product ratings and financial asset evaluation under fluctuating market sentiment, demonstrating how repeated information updates lead to stable preferences or fair value estimates.

The results in this paper open several perspectives for future research. One direction is to study cyclic versions of the proposed contraction, which may yield new iterative schemes for equilibrium problems where decisions alternate between two or more constraint sets. Another promising direction is to incorporate fuzzy-valued perturbation functions, allowing the model to describe uncertainty and linguistic imprecision in fuzzy environments. Moreover, the perturbation structure naturally aligns with modern data-driven settings; thus, extending this framework to handle machine learning estimation errors, online learning systems, and stochastic noise models will broaden its applicability. Additional opportunities lie in best-proximity analysis, random fixed point theory, and stability analysis under perturbation of parameters.

Overall, the new nonlinear rational contraction enriches the fixed point theory in perturbed metric spaces and provides a mathematically grounded tool for analyzing iterative processes with uncertainty. We anticipate that this work will stimulate further development in both theoretical generalizations and real-world applications involving noisy information and adaptive iterative decision mechanisms.

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