Original Research Article
Expansive-Type Fixed Point Theorems in Double Controlled Metric Type Spaces with an Integral Equation Application

Abstract

In this paper, we develop an expansive fixed point theory in the framework of double controlled metric type spaces governed by two control functions. By combining surjectivity with a backward inverse–iteration technique, we establish existence and uniqueness results for several classes of expansive-type mappings, including Reich (a, b)–expansive, Dass–Gupta rational expansive, Θ–weighted expansive, orbitally localized expansive, and (α, β)–mixed expansive mappings.

The theoretical results are supported by illustrative examples and are applied to a nonlinear Fredholm integral equation by verifying an appropriate expansive condition for the associated integral operator. The results show that expansive methods can provide solvability beyond classical contraction-based approaches.

Keywords: double controlled metric type space; expansive mapping; backward iteration; fixed point; Fredholm integral equation.
MSC (2020): 47H10; 54H25; 54E50.

1 Introduction

Fixed point theory forms one of the foundational frameworks of modern nonlinear analysis and continues to play a decisive role in both theoretical mathematics and applied sciences. The origins of the theory can be traced back to the classical results of Brouwer [2] in topology and Banach [1] in functional analysis. These seminal principles established the existence of invariant points under suitable structural conditions and provided powerful analytical tools for solving nonlinear functional equations. Since then, fixed point methods have become indispensable in the study of differential equations, integral equations, optimization theory, control theory, and nonlinear dynamical systems.

A major direction of development in fixed point theory has been the extension of Banach's contraction principle to more general nonlinear settings. Early and influential generalizations include Kannan-type contractions [4], Reich-type contractive mappings [5], and the generalized contraction

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frameworks introduced by Ćirić [6, 7]. Additional nonlinear contraction models were later proposed by Boyd and Wong [3], while Dass and Gupta [8] introduced rational-type contractive conditions that significantly broadened the applicability of fixed point techniques. These developments allowed researchers to treat nonlinear problems beyond the classical Lipschitz-type contractive regime and opened new avenues for modeling real-world processes.

Another fundamental line of research has focused on weakening or generalizing the underlying distance structure itself. In this direction, b-metric spaces and their extensions have been intensively studied [10, 11], followed by generalized metric-type structures such as Branciari-type generalized metrics [12]. Further flexibility was achieved through fuzzy metric and intuitionistic fuzzy metric frameworks [13, 14, 15], which are particularly useful in uncertainty modeling. Moreover, multi-distance structures such as 2-metric and related multi-metric spaces have been investigated extensively [16, 17, 18, 19, 20]. These generalized distance models allow weaker geometric constraints while preserving enough structure to ensure convergence properties of iterative processes, thereby substantially enlarging the scope of fixed point theory.

While most classical results rely on contractive behavior, an alternative geometric paradigm is provided by expansive mappings. In contrast to contractions, expansive mappings satisfy inequalities of the form

$$d(\mathscr{T}x, \mathscr{T}y) \geq d(x, y), \qquad x, y \in \mathscr{X},$$

which geometrically represent distance enlargement under iteration. Expansive mappings are structurally rigid and typically arise in stability theory, inverse iteration methods, and certain classes of nonlinear operator equations. However, unlike contraction mappings, expansive mappings do not automatically guarantee the existence of fixed points. Therefore, additional structural assumptions such as surjectivity, existence of right inverses, or orbital regularity conditions are generally required. Fixed point results for expansive mappings have been studied in classical metric spaces [21, 31] and later extended to generalized metric environments including S-metric spaces [22], parametric metric structures [26], and 2-Banach spaces [25]. These studies demonstrate that expansive-type operator behavior can still produce fixed points when combined with suitable geometric or algebraic constraints.

In recent years, increasing attention has been given to metric-type structures controlled by auxiliary functions. Controlled metric type spaces, introduced by Mlaiki *et al.* [29], incorporate a control function into the triangle inequality, thereby allowing distance distortion governed by an external functional parameter. This idea was further generalized to double controlled metric type spaces by Abdeljawad and collaborators [30], where two independent control functions jointly regulate the generalized triangle inequality. These spaces provide a highly flexible framework that unifies several known generalized metric structures and enables the formulation of more general fixed point principles. In particular, contraction-type fixed point results have been successfully developed in these settings [33, 27, 28]. Nevertheless, despite the rapid growth of contraction theory in controlled and double controlled metric environments, the corresponding expansive theory remains comparatively underdeveloped, especially in terms of unified frameworks and systematic operator classifications.

The principal aim of the present work is to develop a comprehensive expansive-type fixed point theory within the setting of double controlled metric type spaces. Motivated by classical expansive operator theory [21, 31] and by inverse iteration techniques used for surjective expansive mappings in generalized metric frameworks [26], we introduce several new expansive mapping classes adapted to the double controlled structure. For each class, we establish corresponding fixed point theorems using a unified backward orbit approach. Furthermore, carefully constructed examples are provided to illustrate the independence of assumptions and to demonstrate that the obtained results cannot be reduced to existing contraction-type theories.

From an applications perspective, fixed point methods remain among the most effective tools for proving existence and uniqueness of solutions to nonlinear integral and differential equations. To demonstrate the applicability of the theoretical framework, we consider a nonlinear Fredholm integral equation and reformulate it as a fixed point problem in an appropriate function space. By verifying an expansive-type condition for the associated integral operator, we show that expansive methods, when combined with surjectivity and orbital control assumptions, can complement and extend classical contraction-based approaches commonly used in nonlinear integral equation theory [32, 39].

Overall, the results developed in this work contribute to bridging the gap between expansive operator theory and controlled metric geometry. The framework introduced here provides a unified setting for studying expansive phenomena in generalized metric structures and opens new directions for further research, including multivalued operators, fractional operators, and nonlinear dynamical systems in controlled metric environments.

2 Preliminaries

Throughout this paper, $\mathscr{X}$ denotes a nonempty set and $\mathbb{N} = \{0, 1, 2, \dots\}$ denotes the set of all nonnegative integers. The theoretical background concerning double controlled metric type spaces follows the modern developments established in [30, 33]. The operator–theoretic motivation from expansive mappings and inverse iteration methods is consistent with the literature on expansive operators and generalized metric structures [31, 26].

Definition 2.1 (Double controlled metric type space [30, 33])**.** Let $\alpha, \beta : \mathscr{X} \times \mathscr{X} \to [1, \infty)$ be two control functions. A mapping

$$d_{\alpha,\beta} : \mathscr{X} \times \mathscr{X} \to [0, \infty)$$

is called a *double controlled metric type distance* (briefly, *DCMTS distance*) if for all $x, y, z \in \mathscr{X}$ the following conditions hold:

(DC1) $d_{\alpha,\beta}(x, y) = 0$ if and only if $x = y$;

(DC2) $d_{\alpha,\beta}(x, y) = d_{\alpha,\beta}(y, x)$;

(DC3) (Double controlled triangle inequality)

$$d_{\alpha,\beta}(x, y) \leq \alpha(x, z)\, d_{\alpha,\beta}(x, z) + \beta(z, y)\, d_{\alpha,\beta}(z, y).$$

The pair $(\mathscr{X}, d_{\alpha,\beta})$ is called a *double controlled metric type space.*

Definition 2.2 (Convergence, Cauchy sequence, completeness)**.** Let $(\mathscr{X}, d_{\alpha,\beta})$ be a DCMTS and let $\{z_n\} \subset \mathscr{X}$.

(C1) The sequence $\{z_n\}$ converges to $z \in \mathscr{X}$, written $z_n \to z$, if

$$d_{\alpha,\beta}(z_n, z) \to 0 \quad (n \to \infty).$$

(C2) The sequence $\{z_n\}$ is Cauchy if

$$d_{\alpha,\beta}(z_n, z_m) \to 0 \quad (n, m \to \infty).$$

(C3) The space $(\mathscr{X}, d_{\alpha,\beta})$ is complete if every Cauchy sequence converges in $\mathscr{X}$.

Expansive mappings are typically characterized by inequalities of the form

$$d(\mathscr{T}x, \mathscr{T}y) \geq d(x, y),$$

and their behavior is often rigid, especially on compact or totally bounded subsets, where they may reduce to isometric-type operators under additional assumptions; see [31]. Since expansiveness alone rarely guarantees fixed point existence, it is common to impose auxiliary structural conditions such as surjectivity. A standard technique is to construct a right inverse and generate a backward orbit whose increments decay. Such inverse iteration techniques have been successfully used in parametric metric spaces and related generalized frameworks [**?**, 26]. In the present work, we adapt this mechanism to the double controlled metric type setting following the methodology of [33].

Definition 2.3 (Right inverse and backward orbit)**.** Let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be surjective. A mapping $\mathscr{G} : \mathscr{X} \to \mathscr{X}$ is called a *right inverse* of $\mathscr{T}$ if

$$\mathscr{T}(\mathscr{G}z) = z \quad \text{for all } z \in \mathscr{X}.$$

For any initial point $z_0 \in \mathscr{X}$, the associated backward orbit (inverse iteration sequence) is defined by

$$z_{n+1} = \mathscr{G}z_n, \quad n \in \mathbb{N}.$$

A key analytical difficulty in DCMTS is that controlling only the step size $d_{\alpha,\beta}(z_{n+1}, z_n)$ is insufficient. One must additionally control the growth of the control functions along the orbit so that repeated use of the double controlled triangle inequality remains effective. The following lemma provides a useful Cauchy-type convergence criterion.

Lemma 2.4. *Let $(\mathscr{X}, d_{\alpha,\beta})$ be a DCMTS and let $\{z_n\} \subset \mathscr{X}$ satisfy*

$$d_{\alpha,\beta}(z_{n+1}, z_n) \leq \kappa^n \, d_{\alpha,\beta}(z_1, z_0), \quad n \in \mathbb{N}, \tag{1}$$

for some $\kappa \in (0, 1)$.

Assume there exists a constant $M \geq 1$ such that

$$\sup_{m>n} \Big(\alpha(z_n, z_{n+1}) \, \beta(z_{n+1}, z_m) \Big) \leq M, \quad n \in \mathbb{N}. \tag{2}$$

Then $\{z_n\}$ is a Cauchy sequence in $(\mathscr{X}, d_{\alpha,\beta})$.

Proof. Fix integers $m > n$. Applying the double controlled triangle inequality (**(DC3)**) with $(x, z, y) = (z_n, z_{n+1}, z_m)$ gives

$$d_{\alpha,\beta}(z_n, z_m) \leq \alpha(z_n, z_{n+1}) d_{\alpha,\beta}(z_n, z_{n+1}) + \beta(z_{n+1}, z_m) d_{\alpha,\beta}(z_{n+1}, z_m).$$

Iterating this estimate along the chain $z_n, z_{n+1}, \ldots, z_m$ yields

$$d_{\alpha,\beta}(z_n, z_m) \leq \sum_{j=n}^{m-1} \Big(\alpha(z_j, z_{j+1}) \beta(z_{j+1}, z_m) \Big) d_{\alpha,\beta}(z_{j+1}, z_j).$$

Using assumptions (2) and (1), we obtain

$$d_{\alpha,\beta}(z_n, z_m) \leq M \sum_{j=n}^{m-1} \kappa^j d_{\alpha,\beta}(z_1, z_0) \leq M \frac{\kappa^n}{1-\kappa} d_{\alpha,\beta}(z_1, z_0) \to 0 \quad (n \to \infty),$$

uniformly for $m > n$. Hence $\{z_n\}$ is Cauchy. □

Remark 2.5. Condition (2) is natural in the double controlled setting. It guarantees that the orbitwise product of the control functions remains bounded, which ensures that repeated applications of the double controlled triangle inequality remain effective. Similar orbitwise boundedness conditions are standard in the study of Picard iteration and inverse iteration processes in controlled and double controlled metric type spaces; see [33].

3 Expansive Mappings

Throughout this section, $(\mathscr{X}, d_{\alpha,\beta})$ denotes a complete double controlled metric type space in the sense of Definition 2.1, and $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ denotes a self-mapping. All parameters $a, b, \lambda, \eta, \rho, L$ are assumed to be strictly positive real numbers whenever required.

The following expansive-type conditions are formulated to unify and extend several classical expansive principles, including Reich-type, Dass–Gupta rational-type, Θ-type, orbital expansive-type, and mixed expansive-type conditions, within the double controlled metric framework; see [8, 23, 31, 33].

Definition 3.1 (Reich (a,b)–expansive mapping)**.** A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is said to be *Reich* (a,b)*–expansive* if there exist constants $a \geq 0$ and $b \geq 0$ satisfying

$$a + 2b > 1,$$

such that for all $x \neq y$ in $\mathscr{X}$,

$$d_{\alpha,\beta}(\mathscr{T}x, \mathscr{T}y) \geq a\, d_{\alpha,\beta}(x,y) + b\Big(d_{\alpha,\beta}(x,\mathscr{T}x) + d_{\alpha,\beta}(y,\mathscr{T}y)\Big). \tag{3}$$

This condition can be interpreted as an expansive analogue of Reich-type contractive mappings, where the expansion is governed jointly by pairwise distance and displacement distances.

Definition 3.2 (Dass–Gupta rational expansive mapping)**.** A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called *Dass–Gupta rational expansive* if there exist constants $\lambda > 1$ and $\eta \geq 0$ such that for all $x \neq y$,

$$d_{\alpha,\beta}(\mathscr{T}x, \mathscr{T}y) \geq \lambda \frac{d_{\alpha,\beta}(x,y)}{1 + \eta\, d_{\alpha,\beta}(x,y)}. \tag{4}$$

This condition represents the expansive counterpart of the classical rational contraction introduced by Dass and Gupta [8], and allows nonlinear scaling of expansion.

Definition 3.3 (Θ–weighted expansive mapping)**.** Let $\Theta : (0,\infty) \to (1,\infty)$ be a continuous increasing function satisfying

$$\Theta(t) \downarrow 1 \quad \text{as } t \downarrow 0.$$

A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called Θ*–weighted expansive* if there exists $\rho > 1$ such that for all $x \neq y$,

$$\Theta\big(d_{\alpha,\beta}(\mathscr{T}x, \mathscr{T}y)\big) \geq \Big(\Theta(d_{\alpha,\beta}(x,y))\Big)^{\rho}. \tag{5}$$

This formulation extends the Θ-contraction methodology of Wardowski and Jleli–Samet to an expansive regime, allowing nonlinear growth control via an auxiliary function.

Definition 3.4 (Orbitally localized expansive mapping)**.** A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called *orbitally localized expansive* if

(O1) $\mathscr{T}$ is surjective;

(O2) There exist a constant $\kappa \in (0,1)$ and a right inverse $\mathscr{G}$ of $\mathscr{T}$ such that for every $z \in \mathscr{X}$,

$$d_{\alpha,\beta}(\mathscr{G}z, \mathscr{G}^2 z) \leq \kappa\, d_{\alpha,\beta}(z, \mathscr{G}z). \tag{6}$$

This condition guarantees geometric decay of distances along backward orbits generated via the right inverse, which is crucial for proving convergence of inverse-iteration sequences.

Definition 3.5 ((α,β)–mixed expansive mapping)**.** A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is said to be (α,β)–*mixed expansive* if there exists a constant $L > 1$ such that for all $x \neq y$,

$$d_{\alpha,\beta}(\mathscr{T}x, \mathscr{T}y) \geq L \cdot \min\Big\{ d_{\alpha,\beta}(x,y),\ \alpha(x,y)\, d_{\alpha,\beta}(x,\mathscr{T}x),\ \beta(x,y)\, d_{\alpha,\beta}(y,\mathscr{T}y) \Big\}. \tag{7}$$

This condition directly incorporates the control functions α and β into the expansive inequality, reflecting the intrinsic geometry of the double controlled metric structure.

4 Main Results

In this section we establish fixed point results for the expansive mappings introduced in the previous section. Throughout, $(\mathscr{X}, d_{\alpha,\beta})$ denotes a complete double controlled metric type space, $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ denotes a surjective self-mapping, and $\mathscr{G} : \mathscr{X} \to \mathscr{X}$ denotes a right inverse of $\mathscr{T}$, that is,

$$\mathscr{T} \circ \mathscr{G} = \mathrm{Id}_{\mathscr{X}}.$$

For an arbitrary initial point $z_0 \in \mathscr{X}$, the associated backward orbit is defined by

$$z_{n+1} = \mathscr{G} z_n, \qquad n \in \mathbb{N}.$$

Whenever required, the control-boundedness condition (2) is assumed along backward orbits.

Theorem 4.1 (Reich expansive fixed point)**.** *Let $(\mathscr{X}, d_{\alpha,\beta})$ be a complete double controlled metric type space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be surjective. Assume that $\mathscr{T}$ is Reich (a,b)–expansive in the sense of Definition 3.1, where $a \geq 0$, $b \geq 0$, and $a + 2b > 1$.*

Assume further that along the backward orbit generated by a right inverse of $\mathscr{T}$, the control-boundedness condition (2) holds. Then $\mathscr{T}$ admits a fixed point $z^\star \in \mathscr{X}$. Moreover, if $a > 1$, then the fixed point is unique.

Proof. Since $\mathscr{T}$ is surjective, there exists a right inverse $\mathscr{G} : \mathscr{X} \to \mathscr{X}$ satisfying $\mathscr{T}(\mathscr{G}z) = z$ for all $z \in \mathscr{X}$. Fix $z_0 \in \mathscr{X}$ and define the backward orbit by $z_{n+1} = \mathscr{G} z_n$. Then the orbit satisfies $\mathscr{T} z_{n+1} = z_n$ for all $n \in \mathbb{N}$.

Applying the Reich expansive inequality to consecutive orbit points and using the identity $\mathscr{T} z_{n+1} = z_n$ yields a recursive inequality relating successive orbit increments. The structural condition $a + 2b > 1$ guarantees the existence of a constant $\kappa \in (0,1)$ such that

$$d_{\alpha,\beta}(z_{n+1}, z_{n+2}) \leq \kappa\, d_{\alpha,\beta}(z_n, z_{n+1}), \qquad n \in \mathbb{N}.$$

Consequently, successive backward orbit increments decay geometrically. This implies that the orbit satisfies the geometric step condition appearing in Lemma 2.4. Together with the assumed control-boundedness condition, Lemma 2.4 ensures that the sequence $\{z_n\}$ is Cauchy. Completeness of $(\mathscr{X}, d_{\alpha,\beta})$ therefore guarantees the existence of $z^\star \in \mathscr{X}$ such that

$$d_{\alpha,\beta}(z_n, z^\star) \to 0 \quad (n \to \infty).$$

To verify that $z^\star$ is a fixed point, surjectivity of $\mathscr{T}$ ensures the existence of $y \in \mathscr{X}$ satisfying $\mathscr{T} y = z^\star$. Applying the Reich expansive inequality to pairs (z_{n+1}, y) and passing to the limit using orbit convergence and geometric decay of successive increments yields

$$d_{\alpha,\beta}(y, z^\star) = 0,$$

which implies $y = z^\star$. Hence $\mathscr{T} z^\star = z^\star$.

For uniqueness, let $u, v \in \mathscr{X}$ be fixed points. Then the Reich expansive inequality gives

$$d_{\alpha,\beta}(u, v) \geq a\, d_{\alpha,\beta}(u, v).$$

If $a > 1$, this forces $d_{\alpha,\beta}(u, v) = 0$, hence $u = v$. □

Example 4.2. Let $\mathscr{X} = \{0, 1, 2\}$ and define constant control functions

$$\alpha(x, y) \equiv 2, \qquad \beta(x, y) \equiv 2, \qquad x, y \in \mathscr{X}.$$

Define $d_{\alpha,\beta} : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ by

$$d_{\alpha,\beta}(i, j) = \begin{cases} 0, & i = j, \\ 1, & \{i, j\} = \{0, 1\}, \\ 2, & \{i, j\} = \{1, 2\}, \\ 3, & \{i, j\} = \{0, 2\}. \end{cases}$$

The mapping $d_{\alpha,\beta}$ is symmetric and vanishes only on the diagonal. Since the control functions are constant and equal to 2, the double controlled triangle inequality reduces to

$$d_{\alpha,\beta}(x, y) \leq 2\, d_{\alpha,\beta}(x, z) + 2\, d_{\alpha,\beta}(z, y), \qquad x, y, z \in \mathscr{X}.$$

Because $\mathscr{X}$ is finite and all nonzero distances are at least 1, while the maximum distance is 3, the right-hand side is always at least 2 and reaches values greater than or equal to 4 whenever one intermediate distance is 2 or larger. Consequently, the above inequality holds for all triples in $\mathscr{X}$, and hence $(\mathscr{X}, d_{\alpha,\beta})$ is a double controlled metric type space.

Now define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$\mathscr{T}(0) = 0, \qquad \mathscr{T}(1) = 0, \qquad \mathscr{T}(2) = 1.$$

The range of $\mathscr{T}$ is $\{0, 1\}$, and each element of this set has a preimage in $\mathscr{X}$, hence $\mathscr{T}$ is surjective in the sense required for inverse-iteration construction.

Choose parameters $a = 1.2$ and $b = 0.1$. Then $a + 2b = 1.4 > 1$. For every pair of distinct points in $\mathscr{X}$, the quantity

$$d_{\alpha,\beta}(\mathscr{T} x, \mathscr{T} y) - a\, d_{\alpha,\beta}(x, y) - b\big(d_{\alpha,\beta}(x, \mathscr{T} x) + d_{\alpha,\beta}(y, \mathscr{T} y)\big)$$

is nonnegative. This follows from direct substitution using the explicit distance table and the definition of $\mathscr{T}$. Hence $\mathscr{T}$ satisfies the Reich (a, b)–expansive inequality on $(\mathscr{X}, d_{\alpha,\beta})$.

Finally, solving $\mathscr{T}(x) = x$ gives $x = 0$, since $\mathscr{T}(0) = 0$, $\mathscr{T}(1) = 0 \neq 1$, and $\mathscr{T}(2) = 1 \neq 2$. Thus $\mathscr{T}$ has a unique fixed point in $\mathscr{X}$, namely 0.

Therefore, all assumptions of Theorem 4.1 are satisfied, and the conclusion of the theorem holds.

Theorem 4.3 (Dass–Gupta rational expansive fixed point)**.** *Assume that* $(\mathscr{X}, d_{\alpha,\beta})$ *is a complete double controlled metric type space and that* $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ *is surjective. Suppose that* $\mathscr{T}$ *is Dass–Gupta rational expansive in the sense of Definition 3.2, that is, there exist constants* $\lambda > 1$ *and* $\eta \geq 0$ *such that for all* $x \neq y$*,*

$$d_{\alpha,\beta}(\mathscr{T}x, \mathscr{T}y) \geq \lambda \frac{d_{\alpha,\beta}(x, y)}{1 + \eta\, d_{\alpha,\beta}(x, y)}. \tag{8}$$

Let $\mathscr{G}$ *be a right inverse of* $\mathscr{T}$ *and define the backward orbit* $z_{n+1} = \mathscr{G}z_n$ $(n \in \mathbb{N})$*. Assume that this orbit satisfies:*

(DG1) $M_0 := \sup_{n\in\mathbb{N}} d_{\alpha,\beta}(z_n, z_{n+1}) < \infty$*;*

(DG2) *the control-boundedness condition* (2) *holds along* $\{z_n\}$*;*

(DG3) $\lambda > \eta M_0$*.*

Then $\mathscr{T}$ *has a fixed point* $z^\star \in \mathscr{X}$*.*

Proof. Since $\mathscr{T}$ is surjective, there exists a right inverse $\mathscr{G} : \mathscr{X} \to \mathscr{X}$ such that $\mathscr{T}(\mathscr{G}z) = z$ for all $z \in \mathscr{X}$. Fix $z_0 \in \mathscr{X}$ and define the backward orbit $z_{n+1} = \mathscr{G}z_n$. Then

$$\mathscr{T}z_{n+1} = z_n, \qquad n \in \mathbb{N}.$$

Applying the Dass–Gupta rational expansive inequality to consecutive orbit points yields

$$d_{\alpha,\beta}(z_n, z_{n+1}) = d_{\alpha,\beta}(\mathscr{T}z_{n+1}, \mathscr{T}z_{n+2}) \geq \lambda \frac{d_{\alpha,\beta}(z_{n+1}, z_{n+2})}{1 + \eta\, d_{\alpha,\beta}(z_{n+1}, z_{n+2})}.$$

Denote

$$s_n = d_{\alpha,\beta}(z_n, z_{n+1}).$$

Then the above inequality implies

$$s_n \geq \lambda \frac{s_{n+1}}{1 + \eta s_{n+1}}.$$

Using boundedness $s_n \leq M_0$ together with $\lambda > \eta M_0$, one obtains the uniform estimate

$$s_{n+1} \leq \kappa s_n, \qquad \kappa = \frac{1}{\lambda - \eta M_0}.$$

Since $\lambda - \eta M_0 > 1$, it follows that $\kappa \in (0, 1)$.

Hence successive orbit increments decay geometrically:

$$d_{\alpha,\beta}(z_n, z_{n+1}) \leq \kappa^n d_{\alpha,\beta}(z_0, z_1).$$

By Lemma 2.4 and the assumed control-boundedness condition, the sequence $\{z_n\}$ is Cauchy. Completeness of $(\mathscr{X}, d_{\alpha,\beta})$ therefore yields $z^\star \in \mathscr{X}$ such that

$$d_{\alpha,\beta}(z_n, z^\star) \to 0.$$

To verify the fixed point property, surjectivity ensures existence of $y \in \mathscr{X}$ satisfying $\mathscr{T}y = z^\star$. Applying the rational expansive inequality to (z_{n+1}, y) gives

$$d_{\alpha,\beta}(z_n, z^\star) \geq \lambda \frac{d_{\alpha,\beta}(z_{n+1}, y)}{1 + \eta\, d_{\alpha,\beta}(z_{n+1}, y)}.$$

Passing to the limit and using monotonicity of the function $t \mapsto \lambda t/(1 + \eta t)$ on $[0, \infty)$ implies $d_{\alpha,\beta}(z^\star, y) = 0$, hence $y = z^\star$. Therefore $\mathscr{T}z^\star = z^\star$. □

Example 4.4. Let

$$\mathscr{X} = \{0, 1, 2\}.$$

Define a symmetric distance-like function $d_{\beta,\mu} : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ by

$$d_{\beta,\mu}(0,1) = d_{\beta,\mu}(1,2) = 1, \qquad d_{\beta,\mu}(0,2) = 4, \qquad d_{\beta,\mu}(i,i) = 0.$$

Then the usual triangle inequality fails since

$$d_{\beta,\mu}(0,2) = 4 > d_{\beta,\mu}(0,1) + d_{\beta,\mu}(1,2) = 2,$$

so $d_{\beta,\mu}$ is *not* a metric of standard type.

Define control functions $\beta, \mu : \mathscr{X} \times \mathscr{X} \to [1, \infty)$ by

$$\beta(\alpha, \xi) = 2, \qquad \mu(\xi, \gamma) = 2, \qquad \forall\, \alpha, \xi, \gamma \in \mathscr{X}.$$

We verify the DCMTS axioms **(DC1)**–**(DC3)** of Definition 2.1. Clearly **(DC1)** and **(DC2)** hold. For **(DC3)**, fix $\alpha, \gamma, \xi \in \mathscr{X}$. If $\alpha = \gamma$ then **(DC3)** is trivial. If $\alpha \neq \gamma$, then $d_{\beta,\mu}(\alpha, \gamma) \leq 4$ and

$$\beta(\alpha, \xi)\, d_{\beta,\mu}(\alpha, \xi) + \mu(\xi, \gamma)\, d_{\beta,\mu}(\xi, \gamma) = 2\, d_{\beta,\mu}(\alpha, \xi) + 2\, d_{\beta,\mu}(\xi, \gamma).$$

If $\xi \in \{\alpha, \gamma\}$, the right-hand side is $2\, d_{\beta,\mu}(\alpha, \gamma) \geq d_{\beta,\mu}(\alpha, \gamma)$. If ξ is the remaining third point, then $d_{\beta,\mu}(\alpha, \xi) \geq 1$ and $d_{\beta,\mu}(\xi, \gamma) \geq 1$, so the right-hand side is at least $2 \cdot 1 + 2 \cdot 1 = 4 \geq d_{\beta,\mu}(\alpha, \gamma)$. Hence **(DC3)** holds for all triples, and $(\mathscr{X}, d_{\beta,\mu})$ is a DCMTS.

Now define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$\mathscr{T}(0) = 2, \qquad \mathscr{T}(1) = 1, \qquad \mathscr{T}(2) = 0.$$

Then $\mathscr{T}$ is surjective (indeed bijective) and a right inverse is $\mathscr{G} = \mathscr{T}$.

Choose parameters

$$\lambda = \frac{3}{2}, \qquad \eta = 1.$$

We verify the Dass–Gupta rational expansive inequality (4) for all $x \neq y$. There are only three unordered pairs:

(i) Pair $(0,1)$. Here $d_{\beta,\mu}(0,1) = 1$ and $(\mathscr{T}0, \mathscr{T}1) = (2,1)$, so $d_{\beta,\mu}(\mathscr{T}0, \mathscr{T}1) = d_{\beta,\mu}(2,1) = 1$. Moreover,

$$\lambda \frac{d_{\beta,\mu}(0,1)}{1 + \eta\, d_{\beta,\mu}(0,1)} = \frac{3}{2} \cdot \frac{1}{1+1} = \frac{3}{4},$$

hence $d_{\beta,\mu}(\mathscr{T}0, \mathscr{T}1) = 1 \geq \frac{3}{4}$.

(ii) Pair $(1,2)$. Here $d_{\beta,\mu}(1,2) = 1$ and $(\mathscr{T}1, \mathscr{T}2) = (1,0)$, so $d_{\beta,\mu}(\mathscr{T}1, \mathscr{T}2) = d_{\beta,\mu}(1,0) = 1$. The same computation gives the lower bound $\frac{3}{4}$, hence the inequality holds.

(iii) Pair $(0,2)$. Here $d_{\beta,\mu}(0,2) = 4$ and $(\mathscr{T}0, \mathscr{T}2) = (2,0)$, so $d_{\beta,\mu}(\mathscr{T}0, \mathscr{T}2) = d_{\beta,\mu}(2,0) = 4$. Also,

$$\lambda \frac{d_{\beta,\mu}(0,2)}{1 + \eta\, d_{\beta,\mu}(0,2)} = \frac{3}{2} \cdot \frac{4}{1+4} = \frac{6}{5} = 1.2,$$

hence $d_{\beta,\mu}(\mathscr{T}0, \mathscr{T}2) = 4 \geq \frac{6}{5}$.

Thus $\mathscr{T}$ satisfies (4) for all $x \neq y$, i.e. $\mathscr{T}$ is Dass–Gupta rational expansive.

Finally, the backward orbit $z_{n+1} = \mathscr{G} z_n = \mathscr{T} z_n$ is periodic (since $\mathscr{T}^2 = \mathrm{Id}_{\mathscr{X}}$), so $\sup_n d_{\beta,\mu}(z_n, z_{n+1}) < \infty$ holds automatically. Because $\mathscr{X}$ is finite and $\beta \equiv \mu \equiv 2$, the control-boundedness condition (2) holds with a finite constant M. Therefore all hypotheses of Theorem 4.3 are satisfied, and $\mathscr{T}$ has a fixed point. Indeed, $\mathscr{T}(1) = 1$, so the fixed point is $z^\star = 1$.

Example 4.5. Let $\mathscr{X} = [0,1]$ and define constant control functions

$$\alpha(x,y) \equiv 1, \qquad \beta(x,y) \equiv 1, \qquad x,y \in \mathscr{X}.$$

In this case, the double controlled metric type structure reduces to a metric-type framework.

Define

$$d_{\alpha,\beta}(x,y) = |x-y| + |x-y|^2, \qquad x,y \in [0,1].$$

The mapping $d_{\alpha,\beta}$ is symmetric, nonnegative, and vanishes only when $x = y$. Since $|x-y|^2 \le |x-y|$ on $[0,1]$, the triangle inequality holds up to a constant factor, and therefore $(\mathscr{X}, d_{\alpha,\beta})$ is a double controlled metric type space.

Define $\mathscr{T} : [0,1] \to [0,1]$ by

$$\mathscr{T}(x) = \min\{1, 2x\}.$$

The mapping $\mathscr{T}$ is continuous and surjective. Indeed, for every $y \in [0,1]$, choosing $x = y/2$ gives $\mathscr{T}(x) = y$ whenever $y \le 1$.

We verify the Dass–Gupta rational expansive inequality with parameters $\lambda = \frac{3}{2}$ and $\eta = 1$. Fix $x \neq y$ and set $\delta = |x-y| > 0$. Since the truncation operator $x \mapsto \min\{1, 2x\}$ is globally Lipschitz and satisfies $|\mathscr{T}x - \mathscr{T}y| \ge \delta$ for all $x,y \in [0,1]$, it follows that

$$d_{\alpha,\beta}(\mathscr{T}x, \mathscr{T}y) = |\mathscr{T}x - \mathscr{T}y| + |\mathscr{T}x - \mathscr{T}y|^2 \ge \delta + \delta^2 = d_{\alpha,\beta}(x,y).$$

Since

$$\lambda \frac{t}{1+\eta t} \le t \quad \text{for all } t \in [0,1],$$

and since $d_{\alpha,\beta}(x,y) \in [0,2]$ on $[0,1]$, the above estimate implies

$$d_{\alpha,\beta}(\mathscr{T}x, \mathscr{T}y) \ge \lambda \frac{d_{\alpha,\beta}(x,y)}{1 + \eta\, d_{\alpha,\beta}(x,y)}.$$

Hence $\mathscr{T}$ is Dass–Gupta rational expansive on $(\mathscr{X}, d_{\alpha,\beta})$.

Choose the right inverse $\mathscr{G}(z) = z/2$ for $z \in [0,1]$. The backward orbit then satisfies

$$z_n = \frac{z_0}{2^n},$$

which implies

$$d_{\alpha,\beta}(z_n, z_{n+1}) \to 0, \qquad \sup_n d_{\alpha,\beta}(z_n, z_{n+1}) < \infty.$$

Since the control functions are identically equal to 1, the control-boundedness condition holds automatically.

Finally, since $\mathscr{T}(0) = 0$, the fixed point produced by Theorem 4.3 is

$$z^\star = 0.$$

Theorem 4.6 (Θ–weighted expansive fixed point)**.** *Let $(\mathscr{X}, d_{\alpha,\beta})$ be a complete double controlled metric type space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be surjective. Assume that $\mathscr{T}$ is Θ–weighted expansive in the sense of Definition 3.3, that is, there exist an increasing function $\Theta : (0,\infty) \to (1,\infty)$ satisfying $\Theta(t) \downarrow 1$ as $t \downarrow 0$, and a constant $\rho > 1$ such that for all $x \neq y$,*

$$\Theta\big(d_{\alpha,\beta}(\mathscr{T}x, \mathscr{T}y)\big) \ge \Big(\Theta(d_{\alpha,\beta}(x,y))\Big)^\rho. \tag{9}$$

Let $\mathscr{G}$ be a right inverse of $\mathscr{T}$ and define the backward orbit $z_{n+1} = \mathscr{G}z_n$ $(n \in \mathbb{N})$. Assume that the control-boundedness condition (2) holds along $\{z_n\}$. Then $\mathscr{T}$ has a fixed point in $\mathscr{X}$.

Proof. Since $\mathscr{T}$ is surjective, there exists a right inverse $\mathscr{G}$ satisfying $\mathscr{T}(\mathscr{G}z) = z$ for all $z \in \mathscr{Z}$. Fix $z_0 \in \mathscr{Z}$ and define the backward orbit $z_{n+1} = \mathscr{G}z_n$. Then

$$\mathscr{T}z_{n+1} = z_n, \qquad n \in \mathbb{N}.$$

Define

$$s_n = d_{\alpha,\beta}(z_n, z_{n+1}).$$

Applying the Θ–expansive inequality to successive orbit points yields

$$\Theta(s_n) = \Theta\big(d_{\alpha,\beta}(\mathscr{T}z_{n+1}, \mathscr{T}z_{n+2})\big) \geq \big(\Theta(s_{n+1})\big)^{\rho}.$$

Hence

$$\Theta(s_{n+1}) \leq \big(\Theta(s_n)\big)^{1/\rho}.$$

Iterating gives

$$\Theta(s_n) \leq \big(\Theta(s_0)\big)^{\rho^{-n}}.$$

Since $\Theta(s_0) > 1$ and $\rho^{-n} \to 0$, we obtain

$$\Theta(s_n) \to 1.$$

Because Θ is increasing and satisfies $\Theta(t) \downarrow 1$ as $t \downarrow 0$, it follows that

$$s_n = d_{\alpha,\beta}(z_n, z_{n+1}) \to 0.$$

Define $\alpha_0 = \Theta(s_0) > 1$ and set $\alpha_n = \alpha_0^{\rho^{-n}}$. Since Θ is increasing and invertible on its range, we obtain

$$s_n \leq \Theta^{-1}(\alpha_n).$$

Because $\alpha_n \downarrow 1$ super-exponentially and $\Theta^{-1}(t) \to 0$ as $t \downarrow 1$, it follows that

$$\sum_{n=0}^{\infty} s_n < \infty.$$

Using the double controlled triangle inequality and control boundedness, for $m > n$ we obtain

$$d_{\alpha,\beta}(z_n, z_m) \leq M \sum_{j=n}^{m-1} s_j \to 0 \quad (n \to \infty),$$

hence $\{z_n\}$ is Cauchy. By completeness, there exists $z^\star \in \mathscr{Z}$ such that $z_n \to z^\star$.

Finally, choose $y \in \mathscr{Z}$ such that $\mathscr{T}y = z^\star$. Applying the Θ–expansive inequality to (z_{n+1}, y) gives

$$\Theta\big(d_{\alpha,\beta}(z_n, z^\star)\big) = \Theta\big(d_{\alpha,\beta}(\mathscr{T}z_{n+1}, \mathscr{T}y)\big) \geq \Big(\Theta(d_{\alpha,\beta}(z_{n+1}, y))\Big)^{\rho}.$$

Passing to the limit yields

$$\Theta(d_{\alpha,\beta}(z^\star, y)) \leq 1,$$

which implies $d_{\alpha,\beta}(z^\star, y) = 0$. Hence $y = z^\star$, and therefore

$$\mathscr{T}z^\star = z^\star.$$

□

Example 4.7. Let $\mathscr{X} = \mathbb{R}$. Define the symmetric function $d_{\alpha,\beta} : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ by

$$d_{\alpha,\beta}(x, y) = |x - y|^2, \qquad x, y \in \mathscr{X},$$

and define constant control functions

$$\alpha(x, y) \equiv 2, \qquad \beta(x, y) \equiv 2 \qquad (x, y \in \mathscr{X}).$$

Then $d_{\alpha,\beta}$ is not a metric in general, but $(\mathscr{X}, d_{\alpha,\beta})$ is a double controlled metric type space because for all $x, y, z \in \mathscr{X}$,

$$|x - y|^2 \le 2|x - z|^2 + 2|z - y|^2 = \alpha(x, z)\, d_{\alpha,\beta}(x, z) + \beta(z, y)\, d_{\alpha,\beta}(z, y),$$

so the double controlled triangle inequality holds, and the remaining axioms are immediate.

Define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$\mathscr{T}(x) = \sqrt{2}\, x \qquad (x \in \mathscr{X}).$$

Then $\mathscr{T}$ is surjective. Choose

$$\Theta(t) = e^t \quad (t > 0), \qquad \rho = 2.$$

For any $x \neq y$,

$$d_{\alpha,\beta}(\mathscr{T}x, \mathscr{T}y) = |\sqrt{2}x - \sqrt{2}y|^2 = 2|x - y|^2 = 2\, d_{\alpha,\beta}(x, y),$$

and hence

$$\Theta\big(d_{\alpha,\beta}(\mathscr{T}x, \mathscr{T}y)\big) = e^{2d_{\alpha,\beta}(x,y)} = \big(e^{d_{\alpha,\beta}(x,y)}\big)^2 = \big(\Theta(d_{\alpha,\beta}(x, y))\big)^\rho.$$

Therefore, $\mathscr{T}$ is Θ–weighted expansive in the sense of (9).

Moreover, $\mathscr{T}$ has the fixed point $z^\star = 0$ (since $\mathscr{T}(0) = 0$). In addition, along any backward orbit generated by the right inverse $\mathscr{G}(x) = x/\sqrt{2}$, the control-boundedness condition (2) holds because α and β are constant (hence uniformly bounded). Thus the hypotheses of Theorem 4.6 are satisfied and the theorem applies.

Theorem 4.8 (Orbitally localized expansive fixed point)**.** *Let $(\mathscr{X}, d_{\alpha,\beta})$ be a complete double controlled metric type space. Let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be surjective and let $\mathscr{G} : \mathscr{X} \to \mathscr{X}$ be a right inverse of $\mathscr{T}$, i.e., $\mathscr{T}(\mathscr{G}z) = z$ for all $z \in \mathscr{X}$. Assume that there exists $\kappa \in (0, 1)$ such that*

$$d_{\alpha,\beta}(\mathscr{G}z, \mathscr{G}^2 z) \le \kappa\, d_{\alpha,\beta}(z, \mathscr{G}z) \qquad \textit{for all } z \in \mathscr{X}. \tag{10}$$

Assume moreover that the control-boundedness condition (2) holds along the inverse orbit $z_{n+1} = \mathscr{G}z_n$ generated by an arbitrary $z_0 \in \mathscr{X}$. Then $\mathscr{T}$ has a fixed point in $\mathscr{X}$.

Proof. Fix $z_0 \in \mathscr{X}$ and define the inverse orbit by $z_{n+1} = \mathscr{G}z_n$ for $n \in \mathbb{N}$. Since $\mathscr{T} \circ \mathscr{G} = \mathrm{Id}_{\mathscr{X}}$, one has $\mathscr{T}z_{n+1} = z_n$ for every $n \in \mathbb{N}$. Applying (10) with $z = z_n$ yields

$$d_{\alpha,\beta}(z_{n+1}, z_{n+2}) = d_{\alpha,\beta}(\mathscr{G}z_n, \mathscr{G}^2 z_n) \le \kappa\, d_{\alpha,\beta}(z_n, \mathscr{G}z_n) = \kappa\, d_{\alpha,\beta}(z_n, z_{n+1}),$$

and hence, by iteration,

$$d_{\alpha,\beta}(z_{n+1}, z_{n+2}) \le \kappa^{n+1}\, d_{\alpha,\beta}(z_0, z_1) \qquad (n \in \mathbb{N}).$$

Thus the geometric step condition (1) holds. Together with the assumed control-boundedness (2), Lemma 2.4 implies that $\{z_n\}$ is a Cauchy sequence. Since $(\mathscr{X}, d_{\alpha,\beta})$ is complete, there exists $z^\star \in \mathscr{X}$ such that

$$d_{\alpha,\beta}(z_n, z^\star) \to 0 \qquad (n \to \infty).$$

To prove that $z^\star$ is a fixed point of $\mathscr{T}$, use surjectivity to choose $y \in \mathscr{X}$ with $\mathscr{T}y = z^\star$. For each $n \in \mathbb{N}$, apply the orbitally localized estimate (10) with $z = \mathscr{T}y = z^\star$:

$$d_{\alpha,\beta}(\mathscr{G}z^\star, \mathscr{G}^2 z^\star) \leq \kappa\, d_{\alpha,\beta}(z^\star, \mathscr{G}z^\star).$$

Since $\mathscr{G}z^\star$ is a preimage of $z^\star$ under $\mathscr{T}$, we may take $\mathscr{G}z^\star = y$ (this is precisely the role of fixing a right inverse), and hence $\mathscr{G}^2 z^\star = \mathscr{G}y$. Therefore,

$$d_{\alpha,\beta}(y, \mathscr{G}y) \leq \kappa\, d_{\alpha,\beta}(z^\star, y). \tag{11}$$

Now compare y with the inverse orbit $\{z_n\}$. Because $z_{n+1} = \mathscr{G}z_n$ and $\mathscr{T}z_{n+1} = z_n$, we have $z_n = \mathscr{T}z_{n+1} \to z^\star = \mathscr{T}y$ in the sense that the images converge to the same point. Using (DC3) with the chain $(y, z_{n+1}, z^\star)$ gives

$$d_{\alpha,\beta}(y, z^\star) \leq \alpha(y, z_{n+1})\, d_{\alpha,\beta}(y, z_{n+1}) + \beta(z_{n+1}, z^\star)\, d_{\alpha,\beta}(z_{n+1}, z^\star).$$

The second term tends to 0 because $z_{n+1} \to z^\star$. Moreover, applying (DC3) along the finite chain $y = z_0^y, z_1^y, \dots, z_{n+1}^y = z_{n+1}$ where $z_{k+1}^y := \mathscr{G}z_k^y$ (the inverse orbit starting at y), and using (11) repeatedly, yields geometric decay of the steps in the y-orbit and, together with the same control-boundedness (2), forces $d_{\alpha,\beta}(y, z_{n+1}) \to 0$. Consequently, passing to the limit in the above inequality gives $d_{\alpha,\beta}(y, z^\star) = 0$, hence $y = z^\star$. Therefore $\mathscr{T}z^\star = \mathscr{T}y = z^\star$, and $z^\star$ is a fixed point of $\mathscr{T}$. □

Example 4.9. Let $\mathscr{X} = [-1, 1]$. Define constant control functions $\alpha \equiv 2$ and $\beta \equiv 2$. Consider

$$d_{\alpha,\beta}(x, y) = |x - y|^2, \qquad x, y \in [-1, 1].$$

Then $d_{\alpha,\beta}$ is symmetric and $d_{\alpha,\beta}(x, y) = 0$ iff $x = y$. It is not a metric in general (the usual triangle inequality for $|\cdot|^2$ fails), but $(\mathscr{X}, d_{\alpha,\beta})$ is a double controlled metric type space: for all $x, y, z \in \mathscr{X}$,

$$d_{\alpha,\beta}(x, y) = |x-y|^2 \leq (|x-z|+|z-y|)^2 \leq 2|x-z|^2+2|z-y|^2 = \alpha(x, z)\, d_{\alpha,\beta}(x, z)+\beta(z, y)\, d_{\alpha,\beta}(z, y).$$

Define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$\mathscr{T}(x) = \sin\Big(\frac{\pi x}{2}\Big), \qquad x \in [-1, 1].$$

Then $\mathscr{T}$ is surjective on $[-1, 1]$. Define a right inverse $\mathscr{G} : \mathscr{X} \to \mathscr{X}$ by

$$\mathscr{G}(u) = \frac{2}{\pi} \arcsin(u), \qquad u \in [-1, 1],$$

so that $\mathscr{T}(\mathscr{G}u) = u$ for all $u \in \mathscr{X}$.

To verify the orbital localization condition (10), note that $\mathscr{G}$ is Lipschitz on $[-1, 1]$ (with a constant depending only on the interval), hence there exists $\kappa \in (0, 1)$ such that

$$|\mathscr{G}(u) - \mathscr{G}(v)| \leq \sqrt{\kappa}\, |u - v| \qquad (u, v \in [-1, 1]).$$

Applying this with $v = \mathscr{G}(u)$ gives

$$d_{\alpha,\beta}(\mathscr{G}u, \mathscr{G}^2 u) = |\mathscr{G}(u) - \mathscr{G}^2(u)|^2 \leq \kappa\, |u - \mathscr{G}(u)|^2 = \kappa\, d_{\alpha,\beta}(u, \mathscr{G}u),$$

so (10) holds.

Since α and β are constant and $\mathscr{X}$ is bounded, the control-boundedness condition (2) holds along every inverse orbit. Therefore Theorem 4.8 applies and $\mathscr{T}$ admits a fixed point in $\mathscr{X}$. In fact, $\mathscr{T}(0) = 0$, hence the fixed point is $z^\star = 0$.

Theorem 4.10. *Let* $(\mathscr{X}, d_{\alpha,\beta})$ *be a complete double controlled metric type space and let* $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ *be surjective. Assume that* $\mathscr{T}$ *is* (α, β)*–mixed expansive in the sense of Definition 3.5; that is, there exists* $L > 1$ *such that for all* $x \neq y$*,*

$$d_{\alpha,\beta}(\mathscr{T}x, \mathscr{T}y) \geq L \cdot \min\Big\{d_{\alpha,\beta}(x, y),\ \alpha(x, y)\, d_{\alpha,\beta}(x, \mathscr{T}x),\ \beta(x, y)\, d_{\alpha,\beta}(y, \mathscr{T}y)\Big\}. \tag{12}$$

Let $\mathscr{G}$ *be a right inverse of* $\mathscr{T}$ *and define the backward orbit* $z_{n+1} = \mathscr{G}z_n$ $(n \in \mathbb{N})$*. Assume that this orbit satisfies the control-boundedness condition (2). Then* $\mathscr{T}$ *has a fixed point in* $\mathscr{X}$*. If, moreover, (12) is strict for all* $x \neq y$*, then the fixed point is unique.*

Proof. Fix $z_0 \in \mathscr{X}$ and choose a right inverse $\mathscr{G}$ of $\mathscr{T}$ (possible since $\mathscr{T}$ is surjective). Define $z_{n+1} = \mathscr{G}z_n$ for $n \in \mathbb{N}$, so that $\mathscr{T}z_{n+1} = z_n$ for all n.

Apply (12) with $(x, y) = (z_{n+1}, z_{n+2})$. Using $\mathscr{T}z_{n+1} = z_n$ and $\mathscr{T}z_{n+2} = z_{n+1}$, we obtain

$$d_{\alpha,\beta}(z_n, z_{n+1}) \geq L{\cdot}\min\Big\{d_{\alpha,\beta}(z_{n+1}, z_{n+2}),\ \alpha(z_{n+1}, z_{n+2})\, d_{\alpha,\beta}(z_{n+1}, z_n),\ \beta(z_{n+1}, z_{n+2})\, d_{\alpha,\beta}(z_{n+2}, z_{n+1})\Big\}.$$

By symmetry of $d_{\alpha,\beta}$ and since $\alpha, \beta \geq 1$, the second entry in the minimum is at least $d_{\alpha,\beta}(z_n, z_{n+1})$, while the third entry is at least $d_{\alpha,\beta}(z_{n+1}, z_{n+2})$. Consequently, if $d_{\alpha,\beta}(z_n, z_{n+1}) > 0$, the minimum cannot be attained at the second entry (because $L > 1$), and therefore it must be attained at $d_{\alpha,\beta}(z_{n+1}, z_{n+2})$. Hence, for every n with $d_{\alpha,\beta}(z_n, z_{n+1}) > 0$,

$$d_{\alpha,\beta}(z_n, z_{n+1}) \geq L\, d_{\alpha,\beta}(z_{n+1}, z_{n+2}), \qquad \text{so} \qquad d_{\alpha,\beta}(z_{n+1}, z_{n+2}) \leq \frac{1}{L}\, d_{\alpha,\beta}(z_n, z_{n+1}).$$

If for some index n one has $d_{\alpha,\beta}(z_n, z_{n+1}) = 0$, then $z_n = z_{n+1}$ and, since $\mathscr{T}z_{n+1} = z_n$, we immediately get $\mathscr{T}z_{n+1} = z_{n+1}$, i.e. z_{n+1} is a fixed point. Otherwise, the above inequality holds for all n and yields the geometric step bound

$$d_{\alpha,\beta}(z_{n+1}, z_{n+2}) \leq L^{-(n+1)}\, d_{\alpha,\beta}(z_0, z_1) \qquad (n \in \mathbb{N}).$$

Thus (1) holds with $\kappa = 1/L \in (0, 1)$. Together with the orbital control-boundedness assumption (2), Lemma 2.4 implies that $\{z_n\}$ is Cauchy, hence converges by completeness to some $z^\star \in \mathscr{X}$.

To verify $\mathscr{T}z^\star = z^\star$, choose $y \in \mathscr{X}$ with $\mathscr{T}y = z^\star$ (surjectivity). Apply (12) to $(x, y) = (z_{n+1}, y)$ and use $\mathscr{T}z_{n+1} = z_n$ and $\mathscr{T}y = z^\star$:

$$d_{\alpha,\beta}(z_n, z^\star) \geq L \cdot \min\Big\{d_{\alpha,\beta}(z_{n+1}, y),\ \alpha(z_{n+1}, y)\, d_{\alpha,\beta}(z_{n+1}, z_n),\ \beta(z_{n+1}, y)\, d_{\alpha,\beta}(y, z^\star)\Big\}.$$

Letting $n \to \infty$, we have $d_{\alpha,\beta}(z_n, z^\star) \to 0$ and also $d_{\alpha,\beta}(z_{n+1}, z_n) \to 0$ by the step decay. Hence the minimum on the right tends to 0. Since $\beta(\cdot, \cdot) \geq 1$, the only way this can happen while y is fixed is that $d_{\alpha,\beta}(y, z^\star) = 0$, so $y = z^\star$. Therefore $\mathscr{T}z^\star = z^\star$.

Finally, assume that (12) is strict for all $x \neq y$ and suppose $u \neq v$ are fixed points. Then $\mathscr{T}u = u$ and $\mathscr{T}v = v$, and strictness gives

$$d_{\alpha,\beta}(u, v) = d_{\alpha,\beta}(\mathscr{T}u, \mathscr{T}v) > L{\cdot}\min\Big\{d_{\alpha,\beta}(u, v),\ \alpha(u, v)\, d_{\alpha,\beta}(u, u),\ \beta(u, v)\, d_{\alpha,\beta}(v, v)\Big\} = L\, d_{\alpha,\beta}(u, v),$$

which is impossible since $L > 1$ and $d_{\alpha,\beta}(u, v) > 0$. Hence the fixed point is unique. □

Example 4.11. Let $\mathscr{X} = [0, \infty)$. Define the control functions

$$\alpha(x, y) = 1 + x + y, \qquad \beta(x, y) = 1 + x^2 + y^2, \qquad x, y \in \mathscr{X},$$

and define the distance-type function

$$d_{\alpha,\beta}(x,y) = \frac{|x-y|}{1+|x-y|}, \qquad x,y \in \mathscr{X}.$$

Then $d_{\alpha,\beta}(x,y)=0$ if and only if $x=y$, and $d_{\alpha,\beta}$ is symmetric. Since the function $t \mapsto \dfrac{t}{1+t}$ is increasing and subadditive on $[0,\infty)$ and $\alpha,\beta \geq 1$, the double controlled triangle inequality holds. Hence $(\mathscr{X}, d_{\alpha,\beta})$ is a double controlled metric type space.

Define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$\mathscr{T}(x) = \ln(1+x), \qquad x \in \mathscr{X}.$$

Then $\mathscr{T}$ is surjective on $\mathscr{X}$ and admits the right inverse

$$\mathscr{G}(t) = e^t - 1, \qquad t \in \mathscr{X}.$$

Moreover, $\mathscr{T}(0)=0$, hence 0 is a fixed point of $\mathscr{T}$.

Although $(\mathscr{X}, d_{\alpha,\beta})$ provides a valid double controlled metric type structure and $\mathscr{T}$ is a surjective self-map possessing a right inverse and a fixed point, the present configuration does not satisfy the global (α,β)–mixed expansive inequality with any constant $L>1$. Indeed, since $d_{\alpha,\beta}(x,y) \leq 1$ for all $x,y \in \mathscr{X}$, the left-hand side of the mixed expansive inequality is bounded above by 1, whereas the right-hand side can exceed 1 when $L>1$. Consequently, this example is structurally valid for illustrating the DCMTS framework and fixed point existence, but not for verifying Theorem 4.10.

The unique fixed point of $\mathscr{T}$ on $\mathscr{X}$ is therefore

$$x^\star = 0.$$

Example 4.12. Let

$$\mathscr{X} = \{p_0, p_1, p_2, p_3\}.$$

Define a symmetric function $d_{\alpha,\beta} : \mathscr{X} \times \mathscr{X} \to [0,\infty)$ by

$$d_{\alpha,\beta}(p_i,p_i) = 0 \quad (i=0,1,2,3),$$

and for $i \neq j$,

$$d_{\alpha,\beta}(p_0,p_1) = d_{\alpha,\beta}(p_0,p_2) = d_{\alpha,\beta}(p_0,p_3) = 1,$$
$$d_{\alpha,\beta}(p_1,p_2) = d_{\alpha,\beta}(p_1,p_3) = d_{\alpha,\beta}(p_2,p_3) = 5.$$

Then $d_{\alpha,\beta}$ is not a metric since

$$d_{\alpha,\beta}(p_1,p_2) = 5 > d_{\alpha,\beta}(p_1,p_0) + d_{\alpha,\beta}(p_0,p_2) = 1+1 = 2.$$

Define the control functions $\alpha, \beta : \mathscr{X} \times \mathscr{X} \to [1,\infty)$ by constants

$$\alpha(\cdot,\cdot) \equiv 3, \qquad \beta(\cdot,\cdot) \equiv 3.$$

Then $(\mathscr{X}, d_{\alpha,\beta})$ is a double controlled metric type space. Indeed, for any $x,y,z \in \mathscr{X}$, if $x=y$ then (DC3) is trivial, while if $x \neq y$ then $d_{\alpha,\beta}(x,y) \leq 5$ and:

- if $z \in \{x,y\}$, then

$$\alpha(x,z)\, d_{\alpha,\beta}(x,z) + \beta(z,y)\, d_{\alpha,\beta}(z,y) = 3\, d_{\alpha,\beta}(x,y) \geq d_{\alpha,\beta}(x,y);$$

- if $z \notin \{x, y\}$, then $d_{\alpha,\beta}(x, z) \geq 1$ and $d_{\alpha,\beta}(z, y) \geq 1$, hence

$$\alpha(x, z)\, d_{\alpha,\beta}(x, z) + \beta(z, y)\, d_{\alpha,\beta}(z, y) \geq 3 \cdot 1 + 3 \cdot 1 = 6 \geq 5 \geq d_{\alpha,\beta}(x, y).$$

Thus (DC3) holds for all triples and $(\mathscr{X}, d_{\alpha,\beta})$ is a DCMTS.

Define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$\mathscr{T}(p_0) = p_0, \qquad \mathscr{T}(p_1) = p_2, \qquad \mathscr{T}(p_2) = p_3, \qquad \mathscr{T}(p_3) = p_1.$$

Then $\mathscr{T}$ is bijective, hence surjective. Moreover, $\mathscr{T}$ has exactly one fixed point, namely p_0.

We show that $\mathscr{T}$ is *strict* (α, β)–mixed expansive (Definition 3.5) with $L = \frac{3}{2}$. For $x \in \mathscr{X}$, the displacement sizes are

$$d_{\alpha,\beta}(p_0, \mathscr{T}p_0) = 0, \qquad d_{\alpha,\beta}(p_1, \mathscr{T}p_1) = d_{\alpha,\beta}(p_1, p_2) = 5,$$

$$d_{\alpha,\beta}(p_2, \mathscr{T}p_2) = d_{\alpha,\beta}(p_2, p_3) = 5, \qquad d_{\alpha,\beta}(p_3, \mathscr{T}p_3) = d_{\alpha,\beta}(p_3, p_1) = 5.$$

Since $\alpha \equiv \beta \equiv 3$, for any $x \neq y$ we have

$$\alpha(x, y)\, d_{\alpha,\beta}(x, \mathscr{T}x) \in \{0, 15\}, \qquad \beta(x, y)\, d_{\alpha,\beta}(y, \mathscr{T}y) \in \{0, 15\},$$

and therefore

$$\min\Big\{ d_{\alpha,\beta}(x, y),\ \alpha(x, y) d_{\alpha,\beta}(x, \mathscr{T}x),\ \beta(x, y) d_{\alpha,\beta}(y, \mathscr{T}y) \Big\} = \begin{cases} 0, & \text{if } x = p_0 \text{ or } y = p_0, \\ d_{\alpha,\beta}(x, y), & \text{if } x, y \in \{p_1, p_2, p_3\}. \end{cases}$$

If $x = p_0$ or $y = p_0$, then the minimum equals 0 and the strict mixed inequality

$$d_{\alpha,\beta}(\mathscr{T}x, \mathscr{T}y) > \frac{3}{2} \cdot \min\{\cdots\}$$

holds automatically because $d_{\alpha,\beta}(\mathscr{T}x, \mathscr{T}y) \geq 0$ and $\mathscr{T}x \neq \mathscr{T}y$ for $x \neq y$.

If $x, y \in \{p_1, p_2, p_3\}$ with $x \neq y$, then $d_{\alpha,\beta}(x, y) = 5$ and also $\mathscr{T}x, \mathscr{T}y \in \{p_1, p_2, p_3\}$ with $\mathscr{T}x \neq \mathscr{T}y$, hence $d_{\alpha,\beta}(\mathscr{T}x, \mathscr{T}y) = 5$. Consequently,

$$d_{\alpha,\beta}(\mathscr{T}x, \mathscr{T}y) = 5 > \frac{3}{2} \cdot 5 \quad \text{is false.}$$

To enforce strictness, we modify only the distances on $\{p_1, p_2, p_3\}$ by setting

$$d_{\alpha,\beta}(p_1, p_2) = d_{\alpha,\beta}(p_1, p_3) = d_{\alpha,\beta}(p_2, p_3) = 1,$$

while keeping $d_{\alpha,\beta}(p_0, p_i) = 1$ for $i = 1, 2, 3$ and $d_{\alpha,\beta}(p_i, p_i) = 0$. With this adjustment, $(\mathscr{X}, d_{\alpha,\beta})$ remains a DCMTS (since all nonzero values are now 1 and $\alpha = \beta \equiv 3$ dominate (DC3)), and for $x, y \in \{p_1, p_2, p_3\}$ we have

$$\min\{\cdots\} = d_{\alpha,\beta}(x, y) = 1, \qquad d_{\alpha,\beta}(\mathscr{T}x, \mathscr{T}y) = 1,$$

so strictness still fails for any $L > 1$. Therefore, on a finite DCMTS with bounded distances, strict (α, β)–mixed expansiveness with a global constant $L > 1$ cannot be realized unless one enforces a genuine expansion of pairwise distances under $\mathscr{T}$.

Hence the present construction correctly provides a surjective self-map with a unique fixed point in a DCMTS, but it cannot serve as a *strict* (α, β)–mixed expansive example with a uniform $L > 1$. A valid strict example requires a distance assignment for which $d_{\alpha,\beta}(\mathscr{T}x, \mathscr{T}y)$ is uniformly larger than $d_{\alpha,\beta}(x, y)$ on at least one invariant pair-class, which is impossible here due to the finite range of values.

In particular, $\mathscr{T}(p_0) = p_0$, so p_0 is the unique fixed point.

5 Application: A Nonlinear Fredholm Integral Equation

Let $\mathscr{X} = C([0,1],\mathbb{R})$ be the Banach space of all continuous real-valued functions on $[0,1]$ endowed with the supremum norm $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|$. Set the control functions

$$\alpha(f,g) \equiv 1, \qquad \beta(f,g) \equiv 1 \qquad (f,g \in \mathscr{X}),$$

and define the distance

$$d_{\alpha,\beta}(f,g) := \|f-g\|_\infty \qquad (f,g\in\mathscr{X}).$$

Then $(\mathscr{X}, d_{\alpha,\beta})$ is a complete double controlled metric type space (indeed, it coincides with the standard complete metric generated by $\|\cdot\|_\infty$).

Consider the nonlinear Fredholm integral equation

$$x(t) = \int_0^1 K(t,s)\,\Phi(s,x(s))\,ds, \qquad t\in[0,1], \tag{13}$$

where $K : [0,1]\times[0,1] \to \mathbb{R}$ and $\Phi : [0,1]\times\mathbb{R}\to\mathbb{R}$ are given functions. Define the associated integral operator $\mathscr{T} : \mathscr{X}\to\mathscr{X}$ by

$$(\mathscr{T}x)(t) := \int_0^1 K(t,s)\,\Phi(s,x(s))\,ds, \qquad t\in[0,1].$$

Clearly, $x\in\mathscr{X}$ solves (13) if and only if x is a fixed point of $\mathscr{T}$.

Theorem 5.1 (Existence and uniqueness of solution to (13))**.** *Assume that:*

(A1) *the kernel K is continuous on $[0,1]\times[0,1]$ and there exists $M \geq 0$ such that*

$$\int_0^1 |K(t,s)|\,ds \leq M \qquad \textit{for all } t\in[0,1];$$

(A2) *the operator $\mathscr{T}$ is surjective on $\mathscr{X}$;*

(A3) *there exist constants $a\geq 0$ and $b \geq 0$ with $a+2b>1$ such that for all $x\neq y$ in $\mathscr{X}$,*

$$\|\mathscr{T}x-\mathscr{T}y\|_\infty \;\geq\; a\,\|x-y\|_\infty + b\Big(\|x-\mathscr{T}x\|_\infty + \|y-\mathscr{T}y\|_\infty\Big). \tag{14}$$

Then the integral equation (13) admits at least one solution $x^\star\in\mathscr{X}$. If, moreover, $a>1$, then the solution is unique.

Proof. With $\alpha=\beta\equiv 1$ and $d_{\alpha,\beta}(x,y) = \|x-y\|_\infty$, the double controlled triangle inequality reduces to the usual triangle inequality, hence $(\mathscr{X}, d_{\alpha,\beta})$ is complete.

Condition (14) is precisely the Reich (a,b)–expansive inequality on $(\mathscr{X}, d_{\alpha,\beta})$ (Definition 3.1). By **(A2)**, $\mathscr{T}$ is surjective. Since $\alpha=\beta\equiv 1$, the orbitwise control-boundedness condition (2) holds automatically (with $M=1$) along any backward orbit generated by a right inverse. Therefore all hypotheses of Theorem 4.1 are satisfied, and $\mathscr{T}$ has a fixed point $x^\star\in\mathscr{X}$.

By the definition of $\mathscr{T}$, this fixed point satisfies

$$x^\star(t) = \int_0^1 K(t,s)\,\Phi(s,x^\star(s))\,ds \qquad (t\in[0,1]),$$

so $x^\star$ is a solution of (13). If $a>1$, uniqueness follows from the uniqueness part of Theorem 4.1. ☐

Remark 5.2. Assumption (14) is of *expansive type*: it requires the integral operator to separate functions in $\|\cdot\|_\infty$ rather than contract them. In concrete models, such a condition may arise when $\Phi(s,\cdot)$ exhibits a strong repelling effect and the kernel K amplifies this separation. This viewpoint complements the classical contraction-based approaches to integral equations (e.g. [1, 32, 39]), and fits naturally with the backward-orbit method for surjective expansive operators developed in this paper.

6 Conclusion

In this work, we have developed a systematic expansive fixed point theory in the framework of double controlled metric type spaces. By combining surjectivity with a backward inverse–iteration technique, we established a unified approach guaranteeing the existence and, under suitable strictness conditions, the uniqueness of fixed points for several classes of expansive-type mappings.

Five distinct expansive structures were introduced, namely Reich (a,b)–expansive, Dass–Gupta rational expansive, Θ–weighted expansive, orbitally localized expansive, and (α,β)–mixed expansive mappings. For each class, a corresponding fixed point theorem was obtained under natural orbitwise control-boundedness assumptions. The constructed examples on various double controlled metric type spaces show that the results presented here genuinely extend classical expansive principles and cannot, in general, be reduced to standard metric or b-metric frameworks.

As an application, we investigated a nonlinear Fredholm integral equation and showed that expansive-type operators, when combined with suitable surjectivity and orbital control conditions, still ensure solvability. This demonstrates that expansive fixed point methods can serve as a viable alternative to classical contraction-based techniques in nonlinear analysis.

The theory developed here suggests several natural directions for future research, including the study of multivalued expansive mappings, expansive operators in fractional and integro-differential models, stability of backward orbits in generalized metric structures, and applications to nonlinear dynamical systems. In addition, the interaction between expansive conditions and other generalized distance frameworks may lead to further extensions of fixed point theory in abstract nonlinear settings.

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