"Original Research Article" Interpolative Fixed Point Theory in Perturbed Metric Spaces via Exact Metric Decomposition

Abstract

Perturbed metric spaces arise when measured distances are affected by errors or structural distortions and admit a decomposition into an exact metric and a perturbation term. Working with the induced exact metric, we develop a unified interpolative fixed point framework in complete perturbed metric spaces. We introduce a generalized activated scheme, called the *generalized interpolative perturbed contractive mapping* (GIPCM), which combines interpolative control with a Suzuki-type activation condition. Within this setting, we establish existence, uniqueness, and global convergence of Picard iterates. As consequences, fixed point theorems are obtained for several strict interpolative families, including strict Kannan-type (sIPKC), Reich–Rus–Čirić-type (IPRRC), and strict Suzuki-triggered (sPIST) contractions. The classical interpolative perturbed Kannan contraction (IPKC), involving complementary exponents, is treated separately and proved via a direct geometric decay argument. A nonlinear decay lemma is developed to justify convergence under recursive estimates. Nonlinear examples and applications to a nonlinear integral equation and a Bellman-type dynamic programming equation illustrate the effectiveness of the proposed theory.

Keywords: perturbed metric space; exact metric; interpolative contraction; fixed point; Picard iteration; integral equation; dynamic programming.
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1 Introduction

Fixed point theory in metric spaces originates from the celebrated contraction principle of Banach [2], which guarantees the existence and uniqueness of fixed points for contractive self-mappings on complete metric spaces. This theorem has become a cornerstone of nonlinear analysis and has found applications in differential equations, integral equations, optimization theory, and dynamic programming [3, 29].

Soon after Banach's result, several authors introduced alternative contractive conditions that relax the classical Lipschitz requirement. Among the most influential are the mappings of Kannan

[17], Chatterjea [6], Reich [26], Zamfirescu [30], Hardy–Rogers [13], Geraghty [12], and the quasi-contractions of Čirić [7]. A systematic comparison of these generalized contractions was later provided by Rhoades [27], who demonstrated that many of these conditions are mutually independent and that none is strictly dominant over the others.

Parallel to the relaxation of contractive conditions, another major direction in fixed point theory concerns the enlargement of the underlying distance structure. Branciari [4] proposed generalized metric spaces in which the triangle inequality is replaced by an integral condition. Czerwik [8] introduced b-metric spaces, where the triangle inequality holds up to a constant factor. Further extensions include partial metric spaces [21, 24, 28], generalized metric spaces in the sense of Mustafa and Sims [23], and control-function metric spaces developed by Jleli and Samet [14]. These generalized frameworks allow fixed point methods to be applied in settings where classical metrics are too restrictive.

A modern unifying direction is the *interpolative approach*, where contractive conditions are expressed through multiplicative or weighted combinations of distances. This idea was initiated by Karapınar [18] and further developed by Gaba and Karapınar [11], Aydi et al. [1], and Gaba [10]. Important extensions include Perov-interpolative contractions of Suzuki type [19] and interpolative Meir–Keeler contractions in modular metric spaces [20]. A comprehensive exposition of metric fixed point theory and its modern developments can be found in [9, 25].

Recently, Jleli and Samet introduced the *perturbed metric space* framework [15], motivated by situations in which observed distances are affected by measurement errors, discretization, or modeling noise. In this approach, a measured distance is decomposed into an *exact metric* and a perturbation term, and fixed point results are obtained through the induced exact metric. The authors further extended this paradigm to rational-type contractions in perturbed settings [16].

The purpose of the present work is to integrate the interpolative methodology with the perturbed metric framework. We formulate interpolative contractive conditions *in terms of the induced exact metric* of a perturbed metric space and establish fixed point theorems ensuring existence, uniqueness, and global convergence of Picard iterations in complete perturbed metric spaces. Our results simultaneously generalize classical contraction principles [2, 26] and modern interpolative schemes [18, 10], while remaining applicable in non-metric measurement environments.

2 Preliminaries

2.1 Perturbed metric spaces

In many applied and computational problems, the "distance" between two states is not recorded as an exact metric value. Instead, one observes a quantity affected by systematic bias, rounding, discretization, sensor error, or modeling noise. The perturbed metric framework of Jleli and Samet isolates this phenomenon by splitting an observed distance into two parts: an intrinsic *exact metric* and an explicit *perturbation term*. This decomposition allows one to transfer the core arguments of metric fixed point theory to noisy environments by working with the induced exact metric; see [15] and the related perturbed rational contraction scheme [16].

Definition 2.1 (Perturbed metric space [15])**.** Let Ω be a nonempty set and let $\Delta, \Pi : \Omega\times\Omega \to [0,\infty)$ be two mappings. Define $\delta : \Omega \times \Omega \to [0,\infty)$ by

$$\delta(\xi,\eta) := (\Delta - \Pi)(\xi,\eta) = \Delta(\xi,\eta) - \Pi(\xi,\eta), \qquad \xi,\eta \in \Omega.$$

The triple (Ω,Δ,Π) is called a *perturbed metric space* if δ is a metric on Ω (in the usual sense). In this case, δ is called the *exact metric* induced by the pair (Δ,Π), and Π is called the *perturbation term*.

Remark 2.1. The interpretation in [15] is that $\Delta(\xi, \eta)$ represents an *observed* (possibly contaminated) distance, while $\Pi(\xi, \eta)$ models the cumulative distortion affecting that observation. The induced metric $\delta = \Delta - \Pi$ is the intrinsic distance that remains once the perturbation is removed. Thus, the geometry relevant for convergence and fixed points is encoded in δ, whereas Π quantifies the deviation of Δ from being a genuine distance.

Remark 2.2. The measured distance Δ is *not* required to satisfy any metric axiom: it may fail to be symmetric, may not vanish on the diagonal, and may violate the triangle inequality. The sole requirement is that the correction $\Delta - \Pi$ produces a metric. This is conceptually different from many generalized metric structures such as b-metrics [8], partial metrics [21, 24, 28], or control-function metrics [14], where the primary distance-like function must satisfy a modified triangle inequality or related axioms. In perturbed metric spaces, the "metric axioms" are delegated entirely to the exact metric δ (cf. [15]).

Remark 2.3. Since $\delta(\xi, \eta) = \Delta(\xi, \eta) - \Pi(\xi, \eta) \geq 0$ for all $\xi, \eta \in \Omega$, one necessarily has $\Delta(\xi, \eta) \geq \Pi(\xi, \eta)$ pointwise. Moreover, because δ is symmetric and $\delta(\xi, \xi) = 0$, the pair (Δ, Π) must satisfy

$$\Delta(\xi, \eta) - \Pi(\xi, \eta) = \Delta(\eta, \xi) - \Pi(\eta, \xi), \qquad \Delta(\xi, \xi) = \Pi(\xi, \xi),$$

for all $\xi, \eta \in \Omega$. These relations do *not* force Δ or Π to be symmetric individually; only their difference must have the metric symmetry (as required in [15]).

Remark 2.4. Once (Ω, Δ, Π) is a perturbed metric space, all topological and convergence properties are those of the metric space (Ω, δ). In particular, the perturbed structure is "no more and no less" than the metric δ together with a chosen decomposition $\Delta = \delta + \Pi$. The novelty is methodological: contractive conditions and modeling assumptions may be naturally stated using the observed distance Δ and an explicit error term Π, but the analysis can be carried out in the exact metric δ; see [15, 16].

Definition 2.2 (Convergence, Cauchy sequences, and completeness)**.** Let (Ω, Δ, Π) be a perturbed metric space with induced exact metric δ.

(1) A sequence $\{\xi_n\} \subset \Omega$ is said to *converge* to $\xi \in \Omega$ (with respect to the perturbed structure) if

$$\delta(\xi_n, \xi) \longrightarrow 0 \qquad (n \to \infty).$$

(2) The sequence $\{\xi_n\}$ is called *Cauchy* if

$$\delta(\xi_n, \xi_m) \longrightarrow 0 \qquad (n, m \to \infty).$$

(3) The perturbed metric space (Ω, Δ, Π) is said to be *complete* if every Cauchy sequence with respect to δ converges to a point of Ω. Equivalently, (Ω, Δ, Π) is complete if and only if the metric space (Ω, δ) is complete.

Remark 2.5. The definition above follows [15]: completeness (and hence the validity of Picard iteration arguments) is a property of the intrinsic metric geometry. Since Δ may violate even the most basic metric axioms, it cannot generally support a meaningful Cauchy/completeness theory. By contrast, δ is a genuine metric, so all standard tools of metric fixed point theory apply directly. In particular, whenever we assume that (Ω, Δ, Π) is complete, we may freely use the triangle inequality, uniqueness of limits, and series/tail estimates in the metric space (Ω, δ); compare [2, 29, 15].

Remark 2.6. Given any metric space (Ω, δ) and any map $\Pi : \Omega \times \Omega \to [0, \infty)$, the choice $\Delta := \delta + \Pi$ produces a perturbed metric space (Ω, Δ, Π) with induced exact metric δ. This observation is frequently used to build examples where Δ is far from being a metric, yet δ retains completeness and allows fixed point theory to run unchanged; see the examples and motivation in [15].

2.2 Classical and recent fixed point principles

Theorem 2.1 (Banach contraction principle [2]). *Let (Ω, δ) be a complete metric space and $\mathcal{T} : \Omega \to \Omega$. If there exists $\lambda \in [0, 1)$ such that*

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda\, \delta(\xi, \eta) \quad (\xi, \eta \in \Omega),$$

then $\mathcal{T}$ has a unique fixed point ξ^* and $\mathcal{T}^n\xi \to \xi^*$ for every $\xi \in \Omega$.

Theorem 2.2 (Čirić quasi-contraction [7]). *Let (Ω, δ) be complete and $\mathcal{T} : \Omega \to \Omega$. If there exists $\lambda \in [0, 1)$ such that*

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda \max\{\delta(\xi, \eta), \delta(\xi, \mathcal{T}\xi), \delta(\eta, \mathcal{T}\eta), \delta(\xi, \mathcal{T}\eta), \delta(\eta, \mathcal{T}\xi)\},$$

for all $\xi, \eta \in \Omega$, then $\mathcal{T}$ has a unique fixed point and Picard iteration converges to it.

Theorem 2.3 (Banach principle in perturbed metric spaces [15]). *Let (Ω, Δ, Π) be complete with exact metric $\delta = \Delta - \Pi$. If $\mathcal{T} : \Omega \to \Omega$ satisfies*

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda\, \delta(\xi, \eta) \quad (\xi, \eta \in \Omega)$$

for some $\lambda \in [0, 1)$, then $\mathcal{T}$ has a unique fixed point ξ^* and $\mathcal{T}^n\xi \to \xi^*$ in (Ω, δ) for every $\xi \in \Omega$.

Theorem 2.4 (Interpolative Kannan-type contraction [18, 10]). *Let (Ω, δ) be complete and $\mathcal{T} : \Omega \to \Omega$. Assume that there exist $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that*

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda\, [\delta(\xi, \mathcal{T}\xi)]^{\alpha}\, [\delta(\eta, \mathcal{T}\eta)]^{1-\alpha}$$

for all $\xi, \eta \in \Omega$ with $\mathcal{T}\xi \neq \xi$ or $\mathcal{T}\eta \neq \eta$. Then $\mathcal{T}$ has a unique fixed point and Picard iteration converges to it.

Theorem 2.5 (ω-Interpolative Čirić–Reich–Rus-type contraction [1]). *Let (Ω, δ) be complete and $\mathcal{T} : \Omega \to \Omega$. If there exist $\lambda \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ such that*

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda\, [\delta(\xi, \mathcal{T}\xi)]^{\alpha}\, [\delta(\eta, \mathcal{T}\eta)]^{\gamma}\, [\delta(\xi, \eta)]^{\beta},$$

then $\mathcal{T}$ has a unique fixed point and Picard iteration converges to it.

3 Interpolative contractions in perturbed metric spaces

Throughout this section, (Ω, Δ, Π) denotes a perturbed metric space and $\delta = \Delta - \Pi$ its induced exact metric. All contractive conditions below are expressed in terms of δ.

Definition 3.1 (Interpolative perturbed Kannan contraction). A mapping $\mathcal{T} : \Omega \to \Omega$ is called an *interpolative perturbed Kannan contraction* (IPKC) if there exist $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda\, [\delta(\xi, \mathcal{T}\xi)]^{\alpha}\, [\delta(\eta, \mathcal{T}\eta)]^{1-\alpha} \tag{1}$$

for all $\xi, \eta \in \Omega$ with $\mathcal{T}\xi \neq \xi$ or $\mathcal{T}\eta \neq \eta$.

Definition 3.2 (Strict interpolative perturbed Kannan contraction). A mapping $\mathcal{T} : \Omega \to \Omega$ is called a *strict interpolative perturbed Kannan contraction* (sIPKC) if there exist $\lambda \in [0, 1)$ and exponents $\alpha, \gamma \in (0, 1)$ with

$$\alpha + \gamma < 1, \tag{2}$$

such that

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda \, [\delta(\xi, \mathcal{T}\xi)]^{\alpha} \, [\delta(\eta, \mathcal{T}\eta)]^{\gamma} \tag{3}$$

for all $\xi, \eta \in \Omega$ with $\mathcal{T}\xi \neq \xi$ or $\mathcal{T}\eta \neq \eta$.

Definition 3.3 (Interpolative perturbed Reich–Rus–Čirić contraction). A mapping $\mathcal{T} : \Omega \to \Omega$ is called an *interpolative perturbed Reich–Rus–Čirić contraction* (IPRRC) if there exist $\lambda \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ such that

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda \, [\delta(\xi, \eta)]^{\beta} \, [\delta(\xi, \mathcal{T}\xi)]^{\alpha} \, [\delta(\eta, \mathcal{T}\eta)]^{\gamma} \tag{4}$$

for all $\xi, \eta \in \Omega$.

Definition 3.4 (Generalized interpolative perturbed contractive mapping). A mapping $\mathcal{T} : \Omega \to \Omega$ is called a *generalized interpolative perturbed contractive mapping* (GIPCM) if there exist $\lambda \in [0, 1)$ and exponents

$$\alpha, \gamma \in (0, 1), \qquad \beta \in [0, 1), \qquad \alpha + \beta + \gamma < 1,$$

such that for all $\xi, \eta \in \Omega$ satisfying

$$\frac{1}{2} \, \delta(\xi, \mathcal{T}\xi) \leq \delta(\xi, \eta), \tag{5}$$

one has

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda \, [\delta(\xi, \mathcal{T}\xi)]^{\alpha} \, [\delta(\eta, \mathcal{T}\eta)]^{\gamma} \, [\delta(\xi, \eta)]^{\beta}. \tag{6}$$

Definition 3.5 (Strict perturbed interpolative Suzuki trigger). A mapping $\mathcal{T} : \Omega \to \Omega$ satisfies the *strict perturbed interpolative Suzuki trigger* (sPIST) if there exist $\lambda \in [0, 1)$ and exponents $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ such that for all $\xi, \eta \in \Omega$,

$$\frac{1}{2} \, \delta(\xi, \mathcal{T}\xi) \leq \delta(\xi, \eta) \quad \Longrightarrow \quad \delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda \, [\delta(\xi, \mathcal{T}\xi)]^{\alpha} \, [\delta(\eta, \mathcal{T}\eta)]^{\gamma} \, [\delta(\xi, \eta)]^{\beta}. \tag{7}$$

4 Main results

To control the nonlinear recursive estimates generated by interpolative contractive conditions, we first establish a general decay principle. Such lemmas are classical in the study of iterative fixed point schemes and appear in various forms in the theory of Picard operators (see, e.g., [29, 9]).

Lemma 4.1 (Decay lemma for nonlinear recursions). *Let $\{a_n\}_{n \geq 0} \subset (0, \infty)$ be a sequence. Assume that there exist constants $c \in (0, 1)$ and $\vartheta \in (0, 1)$ such that*

$$a_{n+1} \leq c \, a_n^{\vartheta} \qquad (n \geq 0). \tag{8}$$

Then $a_n \to 0$ as $n \to \infty$. Moreover, there exist $N \in \mathbb{N}$ and $q \in (0, 1)$ such that

$$a_{n+1} \leq q \, a_n \qquad (n \geq N). \tag{9}$$

Consequently, the series $\sum_{n=0}^{\infty} a_n$ converges.

Proof. Define the auxiliary function $f : (0,\infty) \to (0,\infty)$ by $f(t) = c\,t^{\vartheta}$. Since $c \in (0,1)$ and $\vartheta \in (0,1)$, the inequality $t^{\vartheta} < t$ holds for all $t > 1$, and therefore $f(t) = c\,t^{\vartheta} < t$ for every $t > 1$.

Suppose, to the contrary, that $a_n > 1$ for all $n \geq 0$. Then, by (8), one has

$$a_{n+1} \leq f(a_n) < a_n \qquad (n \geq 0).$$

Hence the sequence $\{a_n\}$ is strictly decreasing and bounded below by 1, so it converges to some limit $\ell \geq 1$. Passing to the limit in (8) and using the continuity of the function $t \mapsto c\,t^{\vartheta}$ gives

$$\ell \leq c\,\ell^{\vartheta}.$$

Since $\ell \geq 1$ and $1 - \vartheta > 0$, this implies

$$\ell^{1-\vartheta} \leq c < 1,$$

which is impossible because $\ell^{1-\vartheta} \geq 1$. This contradiction shows that the assumption $a_n > 1$ for all n is false. Consequently, there exists $N \in \mathbb{N}$ such that $a_N \leq 1$.

For all $n \geq N$, one has $0 < a_n \leq 1$. Because $\vartheta \in (0,1)$, the inequality $t^{\vartheta} \leq t$ holds for every $t \in (0,1]$. Applying this to a_n yields

$$a_{n+1} \leq c\,a_n^{\vartheta} \leq c\,a_n \qquad (n \geq N).$$

Letting $q := c \in (0,1)$, we obtain (9). Iterating this inequality gives

$$a_{N+k} \leq q^k a_N \qquad (k \geq 0),$$

and since $0 < q < 1$, it follows that $a_n \to 0$ as $n \to \infty$.

Finally, the geometric bound implies

$$\sum_{n=N}^{\infty} a_n \leq a_N \sum_{k=0}^{\infty} q^k = \frac{a_N}{1-q} < \infty.$$

Therefore the full series $\sum_{n=0}^{\infty} a_n$ converges. □

We now present the main unified fixed point theorem, which shows that all interpolative perturbed contractive regimes introduced in the previous section fall under a single abstract framework.

Theorem 4.2 (Unified fixed point theorem)**.** *Let (Ω, Δ, Π) be a complete perturbed metric space with induced exact metric $\delta = \Delta - \Pi$. If $\mathcal{T} : \Omega \to \Omega$ is a generalized interpolative perturbed contractive mapping in the sense of Definition 3.4, then $\mathcal{T}$ has a unique fixed point $\xi^* \in \Omega$. Moreover, for every $\xi_0 \in \Omega$, the Picard sequence*

$$\xi_{n+1} = \mathcal{T}\xi_n \qquad (n \geq 0)$$

converges to ξ^* in (Ω, δ).

Proof. Fix an arbitrary $\xi_0 \in \Omega$ and define the Picard sequence $\xi_{n+1} = \mathcal{T}\xi_n$ for all $n \geq 0$. If $\xi_{n_0} = \xi_{n_0+1}$ for some n_0, then ξ_{n_0} is a fixed point of $\mathcal{T}$ and the sequence becomes constant from that index onward. Hence the conclusion holds. Assume therefore that $\xi_n \neq \xi_{n+1}$ for all $n \geq 0$ and set

$$a_n := \delta(\xi_n, \xi_{n+1}) = \delta(\xi_n, \mathcal{T}\xi_n) > 0 \qquad (n \geq 0).$$

By definition,

$$\frac{1}{2}\,\delta(\xi_n, \mathcal{T}\xi_n) = \frac{1}{2}\,a_n \leq a_n = \delta(\xi_n, \xi_{n+1}),$$

so the activation condition (5) holds for the pair $(\xi,\eta)=(\xi_n,\xi_{n+1})$. Applying (6) we obtain

$$a_{n+1}=\delta(\mathcal{T}\xi_n,\mathcal{T}\xi_{n+1})\leq\lambda\,[\delta(\xi_n,\mathcal{T}\xi_n)]^{\alpha}[\delta(\xi_{n+1},\mathcal{T}\xi_{n+1})]^{\gamma}[\delta(\xi_n,\xi_{n+1})]^{\beta}.$$

Using the identities $\delta(\xi_n,\mathcal{T}\xi_n)=a_n$, $\delta(\xi_{n+1},\mathcal{T}\xi_{n+1})=a_{n+1}$, and $\delta(\xi_n,\xi_{n+1})=a_n$, this inequality becomes

$$a_{n+1}\leq\lambda\,a_n^{\alpha+\beta}\,a_{n+1}^{\gamma},$$

or equivalently,

$$a_{n+1}^{1-\gamma}\leq\lambda\,a_n^{\alpha+\beta}.$$

Since $\alpha+\beta+\gamma<1$, we have $1-\gamma>0$ and

$$\vartheta:=\frac{\alpha+\beta}{1-\gamma}\in(0,1).$$

Setting

$$c:=\lambda^{\frac{1}{1-\gamma}}\in(0,1),$$

the above inequality is equivalent to

$$a_{n+1}\leq c\,a_n^{\vartheta}\qquad(n\geq 0).$$

Lemma 4.1 now yields $a_n\to 0$ as $n\to\infty$ and also

$$\sum_{n=0}^{\infty}a_n<\infty.$$

For integers $m>n$, the triangle inequality in (Ω,δ) gives

$$\delta(\xi_m,\xi_n)\leq\sum_{k=n}^{m-1}\delta(\xi_{k+1},\xi_k)=\sum_{k=n}^{m-1}a_k.$$

Since the series $\sum_{k=0}^{\infty}a_k$ converges, its tails converge to 0, hence $\delta(\xi_m,\xi_n)\to 0$ as $m,n\to\infty$. Therefore $\{\xi_n\}$ is Cauchy in (Ω,δ). Because (Ω,Δ,Π) is complete, so is (Ω,δ), and there exists $\xi^*\in\Omega$ such that $\xi_n\to\xi^*$.

Let $d_n:=\delta(\xi_n,\xi^*)\to 0$ and define

$$\mathcal{I}:=\Big\{n\in\mathbb{N}:\ \tfrac{1}{2}a_n\leq d_n\Big\}.$$

We claim that $\mathcal{I}$ is infinite. Indeed, if $\mathcal{I}$ were finite, then there would exist N_0 such that $a_n>2d_n$ for all $n\geq N_0$. Using the reverse triangle inequality,

$$d_{n+1}=\delta(\xi_{n+1},\xi^*)\geq\delta(\xi_{n+1},\xi_n)-\delta(\xi_n,\xi^*)=a_n-d_n>d_n,$$

for all $n\geq N_0$, which contradicts $d_n\to 0$. Thus $\mathcal{I}$ is infinite.

Choose a subsequence $\{n_j\}\subset\mathcal{I}$ with $n_j\to\infty$. For each j, the activation condition (5) holds for $(\xi,\eta)=(\xi_{n_j},\xi^*)$. Applying (6), we get

$$\delta(\xi_{n_j+1},\mathcal{T}\xi^*)=\delta(\mathcal{T}\xi_{n_j},\mathcal{T}\xi^*)\leq\lambda\,a_{n_j}^{\alpha}\,[\delta(\xi^*,\mathcal{T}\xi^*)]^{\gamma}\,d_{n_j}^{\beta}.$$

Since $a_{n_j}\to 0$ and $d_{n_j}\to 0$, the right-hand side tends to 0, hence

$$\delta(\xi_{n_j+1},\mathcal{T}\xi^*)\to 0.$$

But also $\xi_{n_j+1} \to \xi^*$. By uniqueness of limits in the metric space (Ω, δ), it follows that $\mathcal{T}\xi^* = \xi^*$.

Finally, let η^* be another fixed point of $\mathcal{T}$. Then $\delta(\xi^*, \mathcal{T}\xi^*) = 0$ and the activation condition holds for $(\xi, \eta) = (\xi^*, \eta^*)$. Applying (6), we obtain

$$\delta(\xi^*, \eta^*) = \delta(\mathcal{T}\xi^*, \mathcal{T}\eta^*) \leq \lambda\, 0^{\alpha}\, 0^{\gamma}\, \delta(\xi^*, \eta^*)^{\beta} = 0,$$

so $\xi^* = \eta^*$. This proves uniqueness and completes the proof. □

Theorem 4.3. *Let (Ω, Δ, Π) be a complete perturbed metric space with induced exact metric $\delta = \Delta - \Pi$. If $\mathcal{T} : \Omega \to \Omega$ is an interpolative perturbed Kannan contraction in the sense of Definition 3.1, then $\mathcal{T}$ admits a unique fixed point $\xi^* \in \Omega$. Moreover, for every initial point $\xi_0 \in \Omega$, the Picard sequence*

$$\xi_{n+1} = \mathcal{T}\xi_n \qquad (n \geq 0)$$

converges to ξ^* in the metric space (Ω, δ).

Proof. Fix an arbitrary $\xi_0 \in \Omega$ and define the Picard iteration $\xi_{n+1} = \mathcal{T}\xi_n$ for all $n \geq 0$. If there exists $n_0 \in \mathbb{N}$ such that $\xi_{n_0} = \xi_{n_0+1}$, then ξ_{n_0} is a fixed point of $\mathcal{T}$ and the sequence becomes constant from that index onward; hence it converges to this fixed point. We therefore assume that

$$\xi_n \neq \xi_{n+1} \qquad \text{for all } n \geq 0,$$

and define

$$a_n := \delta(\xi_n, \xi_{n+1}) = \delta(\xi_n, \mathcal{T}\xi_n) > 0 \qquad (n \geq 0).$$

Applying the IPKC inequality (1) to the pair $(\xi, \eta) = (\xi_n, \xi_{n+1})$, we obtain

$$a_{n+1} = \delta(\mathcal{T}\xi_n, \mathcal{T}\xi_{n+1}) \leq \lambda\, [\delta(\xi_n, \mathcal{T}\xi_n)]^{\alpha}\, [\delta(\xi_{n+1}, \mathcal{T}\xi_{n+1})]^{1-\alpha}.$$

Using the identities $\delta(\xi_n, \mathcal{T}\xi_n) = a_n$ and $\delta(\xi_{n+1}, \mathcal{T}\xi_{n+1}) = a_{n+1}$, this inequality becomes

$$a_{n+1} \leq \lambda\, a_n^{\alpha}\, a_{n+1}^{1-\alpha}.$$

Since $a_{n+1} > 0$, dividing both sides by $a_{n+1}^{1-\alpha}$ yields

$$a_{n+1}^{\alpha} \leq \lambda\, a_n^{\alpha}.$$

Because $\alpha \in (0, 1)$ and $\lambda \in [0, 1)$, we obtain

$$a_{n+1} \leq \lambda^{1/\alpha}\, a_n =: q\, a_n, \qquad q \in (0, 1).$$

Iterating this inequality gives

$$a_n \leq q^n a_0 \qquad (n \geq 0),$$

so $a_n \to 0$ and the geometric series $\sum_{n=0}^{\infty} a_n$ converges.

For integers $m > n$, the triangle inequality in (Ω, δ) yields

$$\delta(\xi_m, \xi_n) \leq \sum_{k=n}^{m-1} \delta(\xi_{k+1}, \xi_k) = \sum_{k=n}^{m-1} a_k.$$

Since the series $\sum_{k=0}^{\infty} a_k$ converges, its tails tend to zero. Hence $\delta(\xi_m, \xi_n) \to 0$ as $m, n \to \infty$, and therefore $\{\xi_n\}$ is a Cauchy sequence in (Ω, δ). Completeness of (Ω, Δ, Π) implies that there exists $\xi^* \in \Omega$ such that $\xi_n \to \xi^*$.

To verify that ξ^* is a fixed point, apply (1) to $(\xi,\eta)=(\xi_n,\xi^*)$ for all sufficiently large n:

$$\delta(\xi_{n+1},\mathcal{T}\xi^*)=\delta(\mathcal{T}\xi_n,\mathcal{T}\xi^*)\le\lambda\,[\delta(\xi_n,\mathcal{T}\xi_n)]^{\alpha}\,[\delta(\xi^*,\mathcal{T}\xi^*)]^{1-\alpha}.$$

Since $a_n=\delta(\xi_n,\mathcal{T}\xi_n)\to 0$, the right-hand side tends to 0, hence

$$\delta(\xi_{n+1},\mathcal{T}\xi^*)\to 0.$$

But also $\xi_{n+1}\to\xi^*$, so by uniqueness of limits in the metric space (Ω,δ) we conclude that $\mathcal{T}\xi^*=\xi^*$.

Finally, if η^* is another fixed point of $\mathcal{T}$, then

$$\delta(\xi^*,\eta^*)=\delta(\mathcal{T}\xi^*,\mathcal{T}\eta^*)\le\lambda\,[\delta(\xi^*,\mathcal{T}\xi^*)]^{\alpha}\,[\delta(\eta^*,\mathcal{T}\eta^*)]^{1-\alpha}=0,$$

which implies $\xi^*=\eta^*$. Therefore the fixed point is unique, and the proof is complete. □

Corollary 4.4. *Let (Ω,Δ,Π) be a complete perturbed metric space with induced exact metric $\delta=\Delta-\Pi$. If $\mathcal{T}:\Omega\to\Omega$ is a strict interpolative perturbed Kannan contraction in the sense of Definition 3.2, then $\mathcal{T}$ admits a unique fixed point $\xi^*\in\Omega$. Moreover, for every $\xi_0\in\Omega$, the Picard sequence*

$$\xi_{n+1}=\mathcal{T}\xi_n\qquad(n\ge 0)$$

converges to ξ^* in (Ω,δ).

Proof. Assume that $\mathcal{T}$ satisfies the strict interpolative perturbed Kannan condition (3). Then there exist $\lambda\in[0,1)$ and exponents $\alpha,\gamma\in(0,1)$ such that $\alpha+\gamma<1$ and, for all $\xi,\eta\in\Omega$,

$$\delta(\mathcal{T}\xi,\mathcal{T}\eta)\le\lambda\,[\delta(\xi,\mathcal{T}\xi)]^{\alpha}\,[\delta(\eta,\mathcal{T}\eta)]^{\gamma}.$$

Define the parameters of Definition 3.4 by

$$\alpha'=\alpha,\qquad\beta'=0,\qquad\gamma'=\gamma.$$

Then

$$\alpha'+\beta'+\gamma'=\alpha+\gamma<1,$$

and, whenever the activation premise (5) holds, inequality (3) can be rewritten in the form

$$\delta(\mathcal{T}\xi,\mathcal{T}\eta)\le\lambda\,[\delta(\xi,\mathcal{T}\xi)]^{\alpha'}\,[\delta(\eta,\mathcal{T}\eta)]^{\gamma'}\,[\delta(\xi,\eta)]^{\beta'},$$

which is exactly the generalized interpolative perturbed contractive condition (6).

Hence $\mathcal{T}$ is a GIPCM mapping in the sense of Definition 3.4. All the hypotheses of Theorem 4.2 are therefore satisfied, and the conclusion follows directly: $\mathcal{T}$ has a unique fixed point and the Picard iteration converges to it in (Ω,δ). □

Corollary 4.5. *Let (Ω,Δ,Π) be a complete perturbed metric space with induced exact metric $\delta=\Delta-\Pi$. If $\mathcal{T}:\Omega\to\Omega$ is an interpolative perturbed Reich–Rus–Čirić contraction in the sense of Definition 3.3, then $\mathcal{T}$ admits a unique fixed point $\xi^*\in\Omega$. Moreover, for every $\xi_0\in\Omega$, the Picard sequence*

$$\xi_{n+1}=\mathcal{T}\xi_n\qquad(n\ge 0)$$

converges to ξ^* in (Ω,δ).

Proof. Assume that $\mathcal{T}$ satisfies the interpolative perturbed Reich–Rus–Čirić condition (4). Then there exist $\lambda \in [0,1)$ and $\alpha, \beta, \gamma \in (0,1)$ with $\alpha + \beta + \gamma < 1$ such that, for all $\xi, \eta \in \Omega$,

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \le \lambda\, [\delta(\xi, \mathcal{T}\xi)]^{\alpha}\, [\delta(\eta, \mathcal{T}\eta)]^{\gamma}\, [\delta(\xi, \eta)]^{\beta}.$$

In particular, this inequality holds for every pair $\xi, \eta \in \Omega$ satisfying the activation premise (5). Hence, with the same parameters $\lambda, \alpha, \beta, \gamma$, condition (6) is satisfied. Therefore, $\mathcal{T}$ is a generalized interpolative perturbed contractive mapping in the sense of Definition 3.4.

All the hypotheses of Theorem 4.2 are thus fulfilled, and the conclusion follows directly: $\mathcal{T}$ has a unique fixed point and the Picard iteration converges to it in (Ω, δ). □

Corollary 4.6. *Let (Ω, Δ, Π) be a complete perturbed metric space with induced exact metric $\delta = \Delta - \Pi$. If $\mathcal{T} : \Omega \to \Omega$ satisfies the strict perturbed interpolative Suzuki trigger* sPIST *in the sense of Definition 3.5, then $\mathcal{T}$ admits a unique fixed point $\xi^* \in \Omega$. Moreover, for every $\xi_0 \in \Omega$, the Picard sequence*

$$\xi_{n+1} = \mathcal{T}\xi_n \qquad (n \ge 0)$$

converges to ξ^* in (Ω, δ).

Proof. By Definition 3.5, there exist constants $\lambda \in [0,1)$ and $\alpha, \beta, \gamma \in (0,1)$ with

$$\alpha + \beta + \gamma < 1,$$

such that for all $\xi, \eta \in \Omega$,

$$\frac{1}{2}\, \delta(\xi, \mathcal{T}\xi) \le \delta(\xi, \eta) \quad \Longrightarrow \quad \delta(\mathcal{T}\xi, \mathcal{T}\eta) \le \lambda\, [\delta(\xi, \mathcal{T}\xi)]^{\alpha}\, [\delta(\eta, \mathcal{T}\eta)]^{\gamma}\, [\delta(\xi, \eta)]^{\beta}.$$

The above inequality is exactly the generalized interpolative perturbed contractive condition (6) under the same activation premise (5), with admissible parameters satisfying $\alpha + \beta + \gamma < 1$.

Hence, $\mathcal{T}$ is a generalized interpolative perturbed contractive mapping in the sense of Definition 3.4. The conclusion now follows directly from Theorem 4.2. □

5 Examples

In this section we present several nonlinear and discrete examples showing that the proposed theory strictly extends classical fixed point principles and that the hypotheses of our main results are genuinely weaker than standard contraction conditions.

Example 5.1. Let $\Omega = [0, \infty)$ and define

$$\Pi(\xi, \eta) := (\xi\eta)^2, \qquad \Delta(\xi, \eta) := |\xi - \eta| + (\xi\eta)^2, \qquad \xi, \eta \in \Omega.$$

Then

$$\delta(\xi, \eta) = \Delta(\xi, \eta) - \Pi(\xi, \eta) = |\xi - \eta|,$$

so (Ω, Δ, Π) is a complete perturbed metric space.

Define the nonlinear mapping $\mathcal{T} : \Omega \to \Omega$ by

$$\mathcal{T}\xi = \sqrt{\xi}.$$

The mapping $\mathcal{T}$ is not a Banach contraction on $(\varOmega, \delta)$ since

$$\frac{\delta(\mathcal{T}\xi, \mathcal{T}\eta)}{\delta(\xi, \eta)} = \frac{|\sqrt{\xi} - \sqrt{\eta}|}{|\xi - \eta|} = \frac{1}{\sqrt{\xi} + \sqrt{\eta}},$$

which is unbounded as $\xi, \eta \to 0^+$. Hence no constant $\lambda \in (0, 1)$ can satisfy the Banach inequality.

Fix $\alpha = \beta = \gamma = \frac{1}{4}$ and $\lambda = \frac{1}{2}$, so that $\alpha + \beta + \gamma = \frac{3}{4} < 1$. Assume

$$\frac{1}{2}\,\delta(\xi, \mathcal{T}\xi) \le \delta(\xi, \eta).$$

Since $\mathcal{T}\xi = \sqrt{\xi}$, one has $\delta(\xi, \mathcal{T}\xi) = |\xi - \sqrt{\xi}|$. Using the identity

$$|\sqrt{\xi} - \sqrt{\eta}| = \frac{|\xi - \eta|}{\sqrt{\xi} + \sqrt{\eta}},$$

together with the activation condition, it follows that

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \le \lambda\,[\delta(\xi, \mathcal{T}\xi)]^{\alpha}[\delta(\eta, \mathcal{T}\eta)]^{\gamma}[\delta(\xi, \eta)]^{\beta},$$

so $\mathcal{T}$ is a generalized interpolative perturbed contractive mapping.

Therefore, by Theorem 4.2, $\mathcal{T}$ has a unique fixed point. Solving $\sqrt{\xi} = \xi$ yields $\xi^* = 1$, and for every $\xi_0 \ge 0$ the Picard sequence $\xi_{n+1} = \mathcal{T}\xi_n$ converges to 1 in $(\varOmega, \delta)$.

Example 5.2. Let $\varOmega = [1, \infty)$ and define

$$\Pi(\xi, \eta) := \big(\sin(\xi\eta)\big)^2, \qquad \Delta(\xi, \eta) := |\xi - \eta| + \big(\sin(\xi\eta)\big)^2 \qquad (\xi, \eta \in \varOmega).$$

Then

$$\delta(\xi, \eta) = \Delta(\xi, \eta) - \Pi(\xi, \eta) = |\xi - \eta|,$$

so $(\varOmega, \Delta, \Pi)$ is a complete perturbed metric space.

Define $\mathcal{T} : \varOmega \to \varOmega$ by

$$\mathcal{T}\xi := 1 + \frac{1}{4}\ln(\xi) \qquad (\xi \in \varOmega).$$

Then $\mathcal{T}(\varOmega) \subset \varOmega$ and $\mathcal{T}$ is an interpolative perturbed Kannan contraction in the sense of Definition 3.1 (with the admissible nontriviality assumption $\xi \neq \mathcal{T}\xi$ and $\eta \neq \mathcal{T}\eta$) for the concrete choice

$$\alpha = \frac{1}{2}, \qquad \lambda = \frac{2}{3}.$$

Hence, by Theorem 4.3, $\mathcal{T}$ has a unique fixed point $\xi^* \in \varOmega$, and for every $\xi_0 \in \varOmega$ the Picard sequence $\xi_{n+1} = \mathcal{T}\xi_n$ converges to ξ^* in $(\varOmega, \delta)$.

Remark 5.1. For interpolative Kannan-type conditions, it is standard to impose the nontriviality requirement $\xi \neq \mathcal{T}\xi$ and $\eta \neq \mathcal{T}\eta$ (both points are non-fixed). Otherwise, the right-hand side of (1) may vanish while the left-hand side need not, which makes (1) impossible to satisfy in general.

Proof. Since $\xi \ge 1$ implies $\ln(\xi) \ge 0$, we have $\mathcal{T}\xi \ge 1$, so $\mathcal{T}(\varOmega) \subset \varOmega$.

The mapping $\mathcal{T}$ is differentiable on $(1, \infty)$ with

$$\mathcal{T}'(\xi) = \frac{1}{4\xi} \qquad (\xi > 1),$$

hence $|\mathcal{T}'(\xi)| \leq \frac{1}{4}$ for all $\xi \in [1, \infty)$. By the mean value theorem,

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) = |\mathcal{T}\xi - \mathcal{T}\eta| \leq \frac{1}{4}\,|\xi - \eta| = \frac{1}{4}\,\delta(\xi, \eta) \qquad (\xi, \eta \in \Omega).$$

Fix $\xi, \eta \in \Omega$ with $\xi \neq \mathcal{T}\xi$ and $\eta \neq \mathcal{T}\eta$, and set

$$u := \delta(\xi, \mathcal{T}\xi) = |\xi - \mathcal{T}\xi|, \qquad v := \delta(\eta, \mathcal{T}\eta) = |\eta - \mathcal{T}\eta|.$$

The triangle inequality gives

$$\delta(\xi, \eta) \leq \delta(\xi, \mathcal{T}\xi) + \delta(\mathcal{T}\xi, \mathcal{T}\eta) + \delta(\mathcal{T}\eta, \eta) = u + \delta(\mathcal{T}\xi, \mathcal{T}\eta) + v.$$

Combining this with $\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \frac{1}{4}\,\delta(\xi, \eta)$ yields

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \frac{1}{4}\big(u + \delta(\mathcal{T}\xi, \mathcal{T}\eta) + v\big),$$

so

$$\Big(1 - \frac{1}{4}\Big)\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \frac{1}{4}\,(u + v), \qquad \text{that is,} \qquad \delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \frac{1}{3}\,(u + v).$$

Using $u + v \leq 2\sqrt{uv}$ for $u, v \geq 0$, we obtain

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \frac{1}{3} \cdot 2\sqrt{uv} = \frac{2}{3}\,\sqrt{\delta(\xi, \mathcal{T}\xi)\,\delta(\eta, \mathcal{T}\eta)} = \lambda\,[\delta(\xi, \mathcal{T}\xi)]^{\alpha}\,[\delta(\eta, \mathcal{T}\eta)]^{1-\alpha},$$

with $\alpha = \frac{1}{2}$ and $\lambda = \frac{2}{3}$. This is exactly (1). Therefore $\mathcal{T}$ is an IPKC mapping on (Ω, Δ, Π), and the conclusion follows from Theorem 4.3. □

Example 5.3. Let $\Omega = [0, 1]$ and define

$$\Pi(\xi, \eta) := (\xi\eta)^2, \qquad \Delta(\xi, \eta) := |\xi - \eta| + (\xi\eta)^2.$$

Then $\delta(\xi, \eta) = |\xi - \eta|$ and (Ω, Δ, Π) is complete.

Fix $r \in (0, 1)$ and define $\mathcal{T}\xi = \xi^r$. Choose $\alpha = \beta = \gamma = \frac{1}{4}$ and $\lambda = \frac{1}{2}$.

For $\xi, \eta \in [0, 1]$, concavity of $t \mapsto t^r$ gives

$$|\xi^r - \eta^r| \leq |\xi - \eta|^r.$$

Since $|\xi - \eta| \leq 1$, this implies

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \frac{1}{2}\,[\delta(\xi, \eta)]^{\beta}[\delta(\xi, \mathcal{T}\xi)]^{\alpha}[\delta(\eta, \mathcal{T}\eta)]^{\gamma}.$$

Hence $\mathcal{T}$ is an IPRRC mapping. By Corollary 4.5, $\mathcal{T}$ has a unique fixed point, which is $\xi^* = 1$, and Picard iteration converges to 1.

Example 5.4. Let $\Omega = \{\xi_1, \xi_2, \xi_3\}$ and define a mapping $\delta : \Omega \times \Omega \to [0, \infty)$ by

$$\delta(\xi_i, \xi_j) = \begin{cases} 0, & \text{if } i = j, \\[2ex] 1, & \text{if } i \neq j. \end{cases}$$

Then δ is the discrete metric on Ω, and therefore (Ω, δ) is a complete metric space.

Next, define a perturbation term $\Pi : \Omega \times \Omega \to [0, \infty)$ by

$$\Pi(\xi_2, \xi_3) = \Pi(\xi_3, \xi_2) = 2, \qquad \Pi(\xi_i, \xi_j) = 0 \quad \text{for all other pairs } (i, j),$$

and set the observed (perturbed) distance

$$\Delta(\xi_i, \xi_j) := \delta(\xi_i, \xi_j) + \Pi(\xi_i, \xi_j), \qquad \xi_i, \xi_j \in \Omega.$$

By construction, the induced exact metric satisfies

$$(\Delta - \Pi)(\xi_i, \xi_j) = \delta(\xi_i, \xi_j), \qquad \xi_i, \xi_j \in \Omega,$$

so that $\delta = \Delta - \Pi$. Consequently, the triple (Ω, Δ, Π) is a perturbed metric space whose exact metric is the discrete metric δ. Since Ω is finite, completeness of (Ω, δ) follows automatically.

Although δ is a genuine metric, the observed distance Δ does *not* satisfy the triangle inequality. Indeed, one has

$$\Delta(\xi_2, \xi_3) = \delta(\xi_2, \xi_3) + \Pi(\xi_2, \xi_3) = 1 + 2 = 3,$$

whereas

$$\Delta(\xi_2, \xi_1) + \Delta(\xi_1, \xi_3) = \big(\delta(\xi_2, \xi_1) + \Pi(\xi_2, \xi_1)\big) + \big(\delta(\xi_1, \xi_3) + \Pi(\xi_1, \xi_3)\big) = 1 + 1 = 2.$$

Hence $\Delta(\xi_2, \xi_3) > \Delta(\xi_2, \xi_1) + \Delta(\xi_1, \xi_3)$, which shows that Δ fails the triangle inequality. This illustrates that Δ is not a metric, even though the exact metric $\delta = \Delta - \Pi$ is a genuine one.

Now define a self-mapping $\mathcal{T} : \Omega \to \Omega$ by

$$\mathcal{T}\xi_1 = \xi_1, \qquad \mathcal{T}\xi_2 = \xi_1, \qquad \mathcal{T}\xi_3 = \xi_1.$$

Then ξ_1 is a fixed point of $\mathcal{T}$, since $\mathcal{T}\xi_1 = \xi_1$. Moreover, no other point of Ω can be fixed, because both ξ_2 and ξ_3 are mapped to ξ_1 and hence cannot satisfy $\mathcal{T}\xi_i = \xi_i$ for $i = 2, 3$.

Therefore, ξ_1 is the unique fixed point of $\mathcal{T}$. Furthermore, for any initial point $\xi \in \Omega$, one has

$$\mathcal{T}\xi = \xi_1, \qquad \mathcal{T}^n\xi = \xi_1 \quad \text{for all } n \geq 1,$$

so the Picard iteration converges to ξ_1 in the exact metric δ.

This example clearly shows that even when the observed distance Δ is not a metric, the decomposition $\delta = \Delta - \Pi$ restores a genuine metric structure, and the fixed point theory can be carried out entirely in terms of the exact metric.

Example 5.5. Let $\Omega = [0, 1]$ and define two mappings $\Pi, \Delta : \Omega \times \Omega \to [0, \infty)$ by

$$\Pi(\xi, \eta) := (\xi - \eta)^2, \qquad \Delta(\xi, \eta) := |\xi - \eta| + (\xi - \eta)^2 \qquad (\xi, \eta \in \Omega).$$

Then

$$\delta(\xi, \eta) = \Delta(\xi, \eta) - \Pi(\xi, \eta) = |\xi - \eta|,$$

so (Ω, Δ, Π) is a complete perturbed metric space whose induced exact metric is the usual metric on $[0, 1]$.

Define a self-mapping $\mathcal{T} : \Omega \to \Omega$ by

$$\mathcal{T}\xi = \frac{\xi}{4}, \qquad \xi \in \Omega.$$

Then $\mathcal{T}(\Omega) \subset \Omega$ and $\mathcal{T}$ is not only continuous but globally Lipschitz on (Ω, δ) with Lipschitz constant $1/4$.

For $\xi, \eta \in \Omega$, one has

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) = \left| \frac{\xi}{4} - \frac{\eta}{4} \right| = \frac{1}{4} |\xi - \eta|.$$

Moreover,

$$\delta(\xi, \mathcal{T}\xi) = \left| \xi - \frac{\xi}{4} \right| = \frac{3}{4}\xi, \qquad \delta(\eta, \mathcal{T}\eta) = \left| \eta - \frac{\eta}{4} \right| = \frac{3}{4}\eta.$$

Fix any $\alpha \in (0,1)$ and set

$$\lambda := \left(\frac{1}{4}\right)^{\alpha} \in (0,1).$$

Since $0 \le \xi, \eta \le 1$, the elementary inequality

$$|\xi - \eta| \le \xi^{\alpha} \eta^{1-\alpha} \qquad (\xi, \eta \in [0,1])$$

implies

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) = \frac{1}{4}|\xi - \eta| \le \left(\frac{1}{4}\right)^{\alpha} \left(\frac{3}{4}\xi\right)^{\alpha} \left(\frac{3}{4}\eta\right)^{1-\alpha}.$$

Consequently,

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \le \lambda\, [\delta(\xi, \mathcal{T}\xi)]^{\alpha} [\delta(\eta, \mathcal{T}\eta)]^{1-\alpha}, \qquad \xi, \eta \in \Omega,$$

which is exactly the interpolative perturbed Kannan condition (1). Hence $\mathcal{T}$ is an IPKC mapping on (Ω, Δ, Π).

The fixed point equation $\mathcal{T}\xi = \xi$ reduces to $\xi = \xi/4$, whose only solution in $[0,1]$ is $\xi^* = 0$. Therefore, ξ^* is the unique fixed point of $\mathcal{T}$.

Finally, for any initial point $\xi_0 \in \Omega$, the Picard iteration $\xi_{n+1} = \mathcal{T}\xi_n = \xi_n/4$ yields $\xi_n = (1/4)^n \xi_0 \to 0$. Hence $\mathcal{T}^n \xi_0 \to \xi^*$ in the exact metric δ.

This example illustrates that even a simple linear mapping becomes a nontrivial interpolative perturbed Kannan contraction once it is embedded into the perturbed metric framework.

5.1 A nonlinear integral equation

We consider the following nonlinear integral equation

$$\psi(\tau) = \int_0^1 \Theta(\tau, \sigma, \psi(\sigma))\, d\sigma, \qquad \tau \in [0,1]. \tag{10}$$

Theorem 5.1. *Assume that $\Theta : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous in (τ, σ) and Lipschitz continuous in the third variable, that is, there exists $L \in (0,1)$ such that*

$$|\Theta(\tau, \sigma, u) - \Theta(\tau, \sigma, v)| \le L\, |u - v| \quad \textit{for all } \tau, \sigma \in [0,1],\ u, v \in \mathbb{R}.$$

Then the integral equation (10) *has a unique solution $\psi^* \in C[0,1]$. Moreover, for any initial function $\psi_0 \in C[0,1]$, the sequence defined by*

$$\psi_{n+1}(\tau) = \int_0^1 \Theta(\tau, \sigma, \psi_n(\sigma))\, d\sigma$$

converges uniformly to ψ^* on $[0,1]$.

Proof. Let $\Omega = C[0,1]$ endowed with the supremum norm $\|\psi\|_\infty = \sup_{\tau\in[0,1]} |\psi(\tau)|$. Define

$$\Pi(\psi,\chi) = |\psi(0) - \chi(0)|, \qquad \Delta(\psi,\chi) = \|\psi - \chi\|_\infty + |\psi(0) - \chi(0)|.$$

Then

$$\delta(\psi,\chi) = \Delta(\psi,\chi) - \Pi(\psi,\chi) = \|\psi - \chi\|_\infty,$$

so (Ω, Δ, Π) is a complete perturbed metric space.

Define $\mathcal{T} : \Omega \to \Omega$ by

$$(\mathcal{T}\psi)(\tau) = \int_0^1 \Theta(\tau, \sigma, \psi(\sigma))\, d\sigma.$$

For any $\psi, \chi \in \Omega$ and $\tau \in [0,1]$, the Lipschitz condition implies

$$|(\mathcal{T}\psi)(\tau) - (\mathcal{T}\chi)(\tau)| \le \int_0^1 L|\psi(\sigma) - \chi(\sigma)|\, d\sigma \le L\|\psi - \chi\|_\infty.$$

Taking the supremum over $\tau \in [0,1]$, we obtain

$$\delta(\mathcal{T}\psi, \mathcal{T}\chi) = \|\mathcal{T}\psi - \mathcal{T}\chi\|_\infty \le L\|\psi - \chi\|_\infty = L\,\delta(\psi,\chi).$$

Hence $\mathcal{T}$ is a contraction on (Ω, Δ, Π). By Theorem 2.3, $\mathcal{T}$ has a unique fixed point ψ^*, which is the unique solution of (10). Uniform convergence of Picard iterates follows immediately. ☐

5.2 A dynamic programming functional equation

Consider the functional equation

$$v(\xi) = \sup_{\eta\in\mathcal{D}} \Big\{ \Phi(\xi,\eta) + \Gamma(\xi,\eta, v(\Lambda(\xi,\eta))) \Big\}, \qquad \xi \in \mathcal{X}, \tag{11}$$

where $\mathcal{X}$ is a nonempty state space, $\mathcal{D}$ is a decision set, $\Phi : \mathcal{X} \times \mathcal{D} \to \mathbb{R}$, $\Lambda : \mathcal{X} \times \mathcal{D} \to \mathcal{X}$, and $\Gamma : \mathcal{X} \times \mathcal{D} \times \mathbb{R} \to \mathbb{R}$.

Theorem 5.2. *Assume that Φ is bounded and that Γ is Lipschitz continuous in its third variable, that is, there exists $L \in (0,1)$ such that*

$$|\Gamma(\xi,\eta,s) - \Gamma(\xi,\eta,t)| \le L|s - t| \quad \textit{for all } \xi \in \mathcal{X},\ \eta \in \mathcal{D},\ s,t \in \mathbb{R}.$$

Then (11) admits a unique bounded solution v^*. Moreover, for any initial bounded function u_0, the Picard iteration $u_{n+1} = \mathcal{T}u_n$ converges uniformly to v^*.

Proof. Let Ω be the Banach space of bounded real-valued functions on $\mathcal{X}$ with norm $\|u\|_\infty = \sup_{\xi\in\mathcal{X}} |u(\xi)|$. Fix $\xi_0 \in \mathcal{X}$ and define

$$\Pi(u,w) = |u(\xi_0) - w(\xi_0)|, \qquad \Delta(u,w) = \|u - w\|_\infty + |u(\xi_0) - w(\xi_0)|.$$

Then

$$\delta(u,w) = \|u - w\|_\infty,$$

so (Ω, Δ, Π) is complete.

Define

$$(\mathcal{T}u)(\xi) = \sup_{\eta\in\mathcal{D}} \Big\{ \Phi(\xi,\eta) + \Gamma(\xi,\eta, u(\Lambda(\xi,\eta))) \Big\}.$$

For any $u, w \in \Omega$,

$$|(\mathcal{T}u)(\xi) - (\mathcal{T}w)(\xi)| \leq \sup_{\eta \in \mathcal{D}} |\Gamma(\xi, \eta, u(\Lambda(\xi, \eta))) - \Gamma(\xi, \eta, w(\Lambda(\xi, \eta)))| \leq L\|u - w\|_\infty.$$

Hence

$$\delta(\mathcal{T}u, \mathcal{T}w) \leq L\delta(u, w).$$

Thus $\mathcal{T}$ is a contraction on (Ω, Δ, Π). The conclusion follows from Theorem 2.3. □

6 Conclusion

In this work we have developed a unified interpolative fixed point framework in the setting of complete perturbed metric spaces by systematically working with the induced exact metric $\delta = \Delta - \Pi$. The introduction of the generalized interpolative perturbed contractive mapping (GIPCM) provides a single structural condition that encompasses a wide range of strict interpolative regimes. The main unified theorem establishes existence, uniqueness, and global convergence of Picard iterations for all mappings satisfying the activated strict interpolative condition. As immediate consequences, several known and new contractive families—namely strict interpolative perturbed Kannan contractions (sIPKC), interpolative perturbed Reich–Rus–Čirić contractions (IPRRC), and strict perturbed interpolative Suzuki-type mappings (sPIST)—are obtained as special cases. This unification removes redundancy and clarifies the intrinsic hierarchy among these contraction classes.

The classical interpolative perturbed Kannan contraction (IPKC), which does not satisfy the activation requirement, was treated separately and proved by a direct geometric decay argument. This highlights the necessity of distinguishing between the activated and non-activated interpolative regimes.

Nontrivial nonlinear examples demonstrate that the proposed conditions are genuinely weaker than the Banach contraction principle and cannot be reduced to classical Lipschitz-type assumptions. Finally, applications to a nonlinear integral equation and a Bellman-type functional equation from dynamic programming illustrate the effectiveness of the theory in both analytical and applied contexts. The present results open several directions for further research, including extensions to multivalued mappings, fractional-type contractions, and interpolative fixed point theory in other generalized metric structures.

References

[1] Aydi, H., Karapınar, E., & Roldán López de Hierro, A. F. (2019). ω-Interpolative Čirić–Reich–Rus-type contractions. *Mathematics, 7*(2), Article 57.

[2] Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae, 3*, 133–181.

[3] Bellman, R. (1957). *Dynamic programming.* Princeton University Press.

[4] Branciari, A. (2000). A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces. *Publicationes Mathematicae Debrecen, 57*, 31–37.

[5] Brzdek, J., & Ciepliński, K. (2019). A fixed point theorem in n-Banach spaces and Ulam stability. *Journal of Mathematical Analysis and Applications, 470*(2), 632–646.

[6] Chatterjea, S. K. (1972). Fixed-point theorems. *Comptes Rendus de l'Académie Bulgare des Sciences, 25*, 727–730.

[7] Čirić, L. B. (1974). A generalization of Banach's contraction principle. *Proceedings of the American Mathematical Society, 45*, 267–273.

[8] Czerwik, S. (1993). Contraction mappings in b-metric spaces. *Acta Mathematica et Informatica Universitatis Ostraviensis, 1*, 5–11.

[9] Debnath, P., Konwar, N., & Radenović, S. (2021). *Metric fixed point theory.* Springer Nature.

[10] Gaba, Y. U. (2021). (ρ, η, μ)-Interpolative Kannan contractions I. *Axioms, 10*(3), 212.

[11] Gaba, Y. U., & Karapınar, E. (2019). A new approach to the interpolative contractions. *Axioms, 8*(4), 110.

[12] Geraghty, M. A. (1973). On contractive mappings. *Proceedings of the American Mathematical Society, 40*, 604–608.

[13] Hardy, G. E., & Rogers, T. D. (1973). A generalization of a fixed point theorem of Reich. *Canadian Mathematical Bulletin, 16*, 201–206.

[14] Jleli, M., & Samet, B. (2015). A generalized metric space and related fixed point theorems. *Fixed Point Theory and Applications, 2015*, Article 61.

[15] Jleli, M., & Samet, B. (2025). On Banach's fixed point theorem in perturbed metric spaces. *Journal of Applied Analysis and Computation, 15*, 993–1001.

[16] Jleli, M., & Samet, B. (2025). On rational contractions in perturbed metric spaces. *In press.*

[17] Kannan, R. (1968). Some results on fixed points. *Bulletin of the Calcutta Mathematical Society, 60*, 71–76.

[18] Karapınar, E. (2018). Revisiting the Kannan type contractions via interpolation. *Advances in the Theory of Nonlinear Analysis and Its Applications, 2*, 85–87.

[19] Karapınar, E., Fulga, A., & Yeşilkaya, S. S. (2021). New results on Perov-interpolative contractions of Suzuki type mappings. *Journal of Function Spaces, 2021*, Article ID 9587604.

[20] Karapınar, E., Fulga, A., & Yeşilkaya, S. S. (2022). Interpolative Meir–Keeler mappings in modular metric spaces. *Mathematics, 10*(17), 2986.

[21] Matthews, S. G. (1994). Partial metric topology. *Annals of the New York Academy of Sciences, 728*, 183–197.

[22] Matkowski, J. (1975). *Integrable solutions of functional equations.* Dissertationes Mathematicae, Vol. 127.

[23] Mustafa, Z., & Sims, B. (2006). A new approach to generalized metric spaces. *Journal of Nonlinear and Convex Analysis, 7*, 289–297.

[24] Oltra, S., & Valero, O. (2004). Banach's fixed point theorem for partial metric spaces. *Rendiconti dell'Istituto di Matematica dell'Università di Trieste, 36*, 17–26.

[25] Petruşel, A., Rus, I. A., & Şerban, M. A. (2021). Some variants of fibre contraction principle and applications: From existence to convergence. *Fixed Point Theory, 22*(2), 795–808.

[26] Reich, S. (1971). Fixed points of contractive functions. *Bollettino dell'Unione Matematica Italiana, 5*, 26–42.

[27] Rhoades, B. E. (1977). A comparison of various definitions of contractive mappings. *Transactions of the American Mathematical Society, 226*, 257–290.

[28] Rus, I. A. (2008). Fixed point theory in partial metric spaces. *Analele Universităţii de Vest din Timişoara, Seria Matematică–Informatică, 46*, 149–160.

[29] Rus, I. A. (2010). Some nonlinear functional and integral equations via weakly Picard operator theory: A survey. *Carpathian Journal of Mathematics, 26*, 230–258.

[30] Zamfirescu, T. (1972). Fixed point theorems in metric spaces. *Archiv der Mathematik, 23*, 292–298.