Star Chromatic Index of Tensor Product of Graphs

Abstract

A star edge coloring of a graph G is a proper edge coloring without bichromatic paths or cycles of length 4. The smallest integer k such that G admits a star-edge-coloring with k colors is the star chromatic index of G and is denoted by $\chi'_{st}(G)$. The tensor product of two graphs G and H, denoted by $G \times H$ is the graph with vertex set $V(G) \times V(H)$ and two vertices $u = (u_1, v_1)$, $v = (u_2, v_2)$ are adjacent in $G \times H$ if u_1 is adjacent to u_2 in G and v_1 is adjacent to v_2 in H. This paper focuses on determining the star chromatic index of $P_m \times C_n$ and $C_m \times C_n$ and the obtained results are listed below.
(i) $\chi'_{st}(P_m \times C_n) = 8$, for $m \geq 7, n \geq 7$.
(ii)$\chi'_{st}(C_m \times C_n) = 8$, for $m \geq 3, n \geq 4$.
The coloring technique is based on finding the star chromatic index of the maximal connected subgraph G_1 in G such that $\chi_{st}(G) \geq \chi_{st}(G_1)$.
Keywords: Paths, Cycles, Tensor Product of Graphs, Star Edge Coloring, Star Chromatic Index.
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1 Introduction

All graphs considered in this paper are finite, simple and undirected. Let P_n and C_n respectively denote a path and a cycle of order n. For a graph G, sG denotes s edge disjoint copies of G. For a graph G and a subgraph H of G, $G \backslash H$ denotes the graph G' in which $V(G') = V(G)$ and $E(G') = E(G) - E(H)$. The tensor product of two graphs G and H, denoted by $G \times H$ is the graph with vertex set $V(G) \times V(H)$ and two vertices $u = (u_1, v_1)$, $v = (u_2, v_2)$ are adjacent in $G \times H$ if $u_1u_2 \in V(G)$ is adjacent to $v_1v_2 \in V(H)$.
An edge coloring of a graph $G = (V, E)$ is a function $f : E \rightarrow C \subseteq \mathcal{N}$ in which any two adjacent edges $e, f \in E$ are assigned different colors. The function C is known as the edge coloring function. A graph G for which there exists an edge coloring which requires k colors is called k-edge colorable. "The smallest number k for which there exists a k-edge coloring of G is called the edge chromatic index of a graph G and is denoted by $\chi'(G)$. A star edge coloring of a graph G is a proper edge coloring where atleast three distinct colors are used on the edges of every induced paths and cycles of length four in G, i.e., there is neither an induced bichromatic path nor an induced bichromatic cycle of length four in G. The minimum number of colors for which G admits a star edge coloring is called the star chromatic index of G and is denoted by $\chi'_{st}(G)$"[17].
Additional graph theoretic terminologies used in this paper can be found in [1]. The

star edge coloring was initiated in 2008 by Liu and Deng [11]. Fraser et al. [6] presented an overview of graph colouring and examined several Ramsey-type numbers. Star chromatic index in product graphs has been done by many researchers. Omooni and Dastjerbi [13] determined the upper bounds of the star chromatic index of the cartesian product of paths with cycles, d-dimensional grids, d-dimensional hypercubes and d-dimensional toroided grids. Kaliraj et al. [9] obtained the star chromatic index of the corona product of path and wheel graph families. Sivakami et al. [12] obtained the star coloring of the lexicographic coloring of graphs. Kavita et al. [10] obtained the star chromatic index of the tensor product of two paths. Deng et al.[3] constructed infinite sequence of cubic graphs G with $\chi'_{st}(G) = 6$ and obtained the star chromatic index of striped maximal outerplanar graph. Yunfeng Tang et al. [16] obtained the star edge coloring of $K_{2,t}$ free planar graphs. Fernando et al. obtained the star chromatic index of some simple graphs such as pan graphs, tadpole graphs, friendship graphs, ladder graphs, flower graphs and umbrella graphs. Xingxing Hu et al. obtained the star edge coloring of cubic Halin graphs with star chromatic index 5.

2 Motivation

Star edge coloring has applications in systems where both direct conflicts and short repetitive interaction patterns must be avoided. In communication and optical networks, it is used to assign frequencies or wavelengths to links while preventing short alternating interference paths that cause signal degradation. In scheduling and timetabling problems, tasks sharing common resources are modeled as edges, and star edge coloring avoids not only immediate clashes but also cyclic conflicts over short time intervals.In VLSI and interconnection network design, tensor products capture complex circuit architectures formed from simpler modules, and star edge coloring reduces short-cycle crosstalk and signal interference. It is also useful in scheduling and resource allocation problems involving multiple constraints or dimensions, where the tensor product encodes simultaneous resource dependencies, and star edge coloring enforces robust local conflict avoidance.Star edge coloring has conceptual and structural applications in cryptography, particularly in the design and analysis of secure communication and key-distribution schemes.
The following section discusses the star chromatic index of the tensor product of graphs such as $P_m \times C_n$ and $C_m \times C_n$ for the given positive integers m and n.

3 Preliminaries

We use the following results in the proof of our main theorem.

Theorem 3.1. *[15] If H is a subgraph of G, then $\chi'_{st}(H) \leq \chi'_{st}(G)$.*

Theorem 3.2. *[8] For the given positive integer n,*

$$\chi'_{st}(P_n) = \begin{cases} 1, & \text{if } n = 2 \\ 3, & \text{if } n \geq 3 \end{cases}$$

$$\chi'_{st}(C_n) = \begin{cases} 3, & \text{if } n \neq 5 \\ 4, & \text{if } n = 5. \end{cases}$$

Theorem 3.3. *[10] For the given positive integers $m \geq 2$ and $n \geq 2$, the star chromatic index of $P_m \times P_n$ is given by*

$$\chi'_{st}(P_m \times P_n) = \begin{cases} 3, & \text{if } m = 2 \text{ and } n \geq 5 \\ 5, & \text{if } m = 3, 4 \text{ and } n \geq 5 \\ 6, & \text{if } m \geq 7 \text{ and } n \geq 7 \\ \geq 5 & \text{if } m = 5, 6 \text{ and } n \geq 5\ . \end{cases}$$

4 Star Chromatic Index of Tensor Product of Paths and Cycles

Theorem 4.1. *For the given positive integers $m \geq 2$ and $n \geq 3$, the star chromatic index of $P_m \times C_n$ is given by*

$$\chi'_{st}(P_m \times C_n) = \begin{cases} 3, & \text{for } m = 2 \text{ and } n \geq 3 \\ 5, & \text{for } m = 3 \text{ and } n = 3 \\ 6, & \text{for } m = 3 \text{ and } n \geq 4 \text{ or } m = 4 \text{ and } n \geq 3 \text{ or } m = 5 \text{ and } n = 3 \\ 7, & \text{for } m = 5 \text{ and } n \geq 4 \text{ or } m \in \{6, 7\} \text{ and } 3 \leq n \leq 6 \\ 8, & \text{for } m \geq 7 \text{ and } n \geq 7. \end{cases}$$

Proof. Let $G(V, E) = P_m \times C_n$, by the definition of tensor product, the vertex set and the edge set of the graph $P_m \times C_n$ is given by $V(P_m \times C_n) = \{(u_i, v_j) : 0 \leq i \leq m-1,\ 0 \leq j \leq n-1\ \ u_i \in V(P_m) \text{ and } v_j \in V(P_m)\}$.
$E(G) = E_1 \cup E_2 \cup E_3 \cup E_4$ where
$E_1 = \bigcup_{i=0}^{m-1} \{(u_i, v_j)(u_{i+1}, v_{j+1}) : 0 \leq j \leq n-2\}$
$E_2 = \bigcup_{i=0}^{m-1} \{(u_i, v_j)(u_{i+1}, v_{j-1}) : 1 \leq j \leq n-1\}$
$E_3 = \bigcup_{i=0}^{m-1} \{(u_i, v_0)(u_{i+1}, v_j) : j = n-1\}$
$E_4 = \bigcup_{i=0}^{m-1} \{(u_i, v_j)(u_{i+1}, v_0) : j = n-1\}$.
Let f be the function defined by $f : E \to C \subseteq \mathcal{N} \cup \{0\}$, and C is the set of colors. The star chromatic index of $P_m \times C_n$ according to the given values of m,n is obtained as follows:
Case 1: $m = 2$, $n \geq 3$

In the case of $G = P_2 \times C_n$, $n \geq 3$, G is isomorphic to C_{2n} and by Theorem 2.2, we have $\chi'_{st}(C_{2n}) = 3$ for $n \geq 3$ and therefore $\chi'_{st}(P_2 \times C_n) = 3$.
Case 2: $m = 3$, $n = 3$

In the case of $G = P_3 \times C_3$, the maximum degree of the graph G is 4 and there exists 3 vertices of maximum degree 4. Hence $\chi'_{st}(G) \geq 4$, first the edges that are incident at one of these vertices of maximum degree say at v are colored using 4 colors $\{0, 1, 2, 3\}$. Then to color the edges that are incident at another maximum degree vertex, any of the three colors from the list $\{0, 1, 2, 3\}$ can be used as one edge is already colored from the previous step. But coloring 4 adjacent edges using only

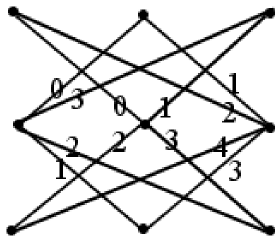

Figure 1: Star edge coloring of $P_3 \times C_3$

3 colors will not yield a proper coloring. Hence fifth color must be introduced. The permutation of 5 colors can be done to color the adjacent edges that are incident at the vertices of degree 4 in G. Thus $\chi'_{st}(G) = 5$.

Case 3: $m = 3$, $n \geq 4$

Subcase (i):

In the case of $G = P_3 \times C_4$, G contains $G_1 = P_3 \times P_4$ as a subgraph, by Theorem 2.3, it is clear that $\chi'_{st}(P_3 \times P_4) = 4$, and it is not possible to color the edges in $G \backslash G_1$ with the same set of colors already used in the coloring of G_1 . So, the fifth color is introduced for the edges in between the first and second layers of $G \backslash G_1$. At the same time the fifth color cannot be used for the new edges in between the second and third layer of $G \backslash G_1$. So, the sixth color can be introduced and hence $\chi'_{st}(P_3 \times C_4) = 6$.

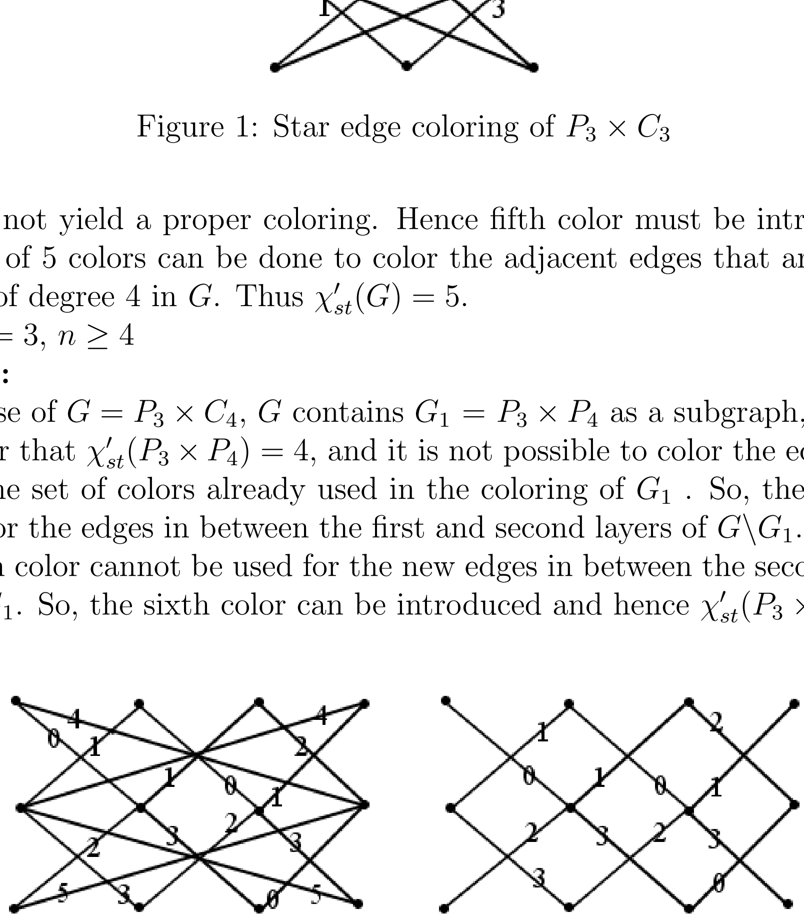

Figure 2: Star edge coloring of the graph $P_3 \times C_4$ and its subgraph $P_3 \times P_4$

Subcase (ii):

In the case of $G = P_3 \times C_n$ for $n \geq 5$ contains $G_2 = P_3 \times P_n$ as a subgraph, by Theorem 2.3, it is clear that $\chi'_{st}(P_3 \times P_n) = 5$, for $n \geq 5$. But, it is not possible to color the edges of $G \backslash G_2$ with the same set of colors already used in the coloring of G_2. The sixth color can be introduced for the edges in between the second and third layers of $G \backslash G_2$. Therefore, $\chi'_{st}(P_3 \times C_n) = 6$, for $n \geq 5$.

Hence $\chi'_{st}(P_m \times C_n) = 6$, for $m = 3$, $n \geq 4$

Case 4: $m = 4$, $n \geq 3$

Subcase (i):

In the case of $G = P_4 \times C_3$ contains $G_3 = P_4 \times P_3$ as a subgraph, by Theorem 2.3, it is clear that $\chi'_{st}(P_4 \times P_3) = 5$. Here every vertex of the graph G is at a distance of one, two or three from the other and hence it is not possible to color the edges of $G \backslash G_3$ with the same set of five colors already used for G_3. So, the sixth color can be introduced. Therefore, $\chi'_{st}(P_4 \times C_3) = 6$. In a similar manner, it can be shown that

$\chi'_{st}(P_4 \times C_4) = 6$.
The proof of other cases are given in the following table as discussed in cases 1 to 4:

m	n	$G = P_m \times C_n$	G_1	$\chi'_{st}(G_1)$	$\chi'_{st}(G)$
4	$n \geq 4$	$P_4 \times C_n$	$P_4 \times P_n$	5	6
5	3	$P_5 \times C_3$	$P_5 \times P_3$	5	6
5	4	$P_5 \times C_4$	$P_5 \times P_4$	5	7
5	5	$P_5 \times C_5$	$P_5 \times P_5$	≥ 5	7
6,7	$3 \leq n \leq 6$	$P_m \times C_n$	$P_5 \times P_3$	5	7
7	7	$P_7 \times C_7$	$P_7 \times P_7$	6	8

Table 1: $\chi'_{st}(P_m \times C_n)$ for $m \geq 4$, $n \geq 3$.

In a similar manner, it can be shown that $\chi'_{st}(P_m \times C_n) = 8$, for $m \geq 7, n \geq 7$. An algorithmic approach for the star edge coloring of $G = P_m \times C_n$, for large values of $m, n \geq 7$ the following coloring pattern is presented. The star edge coloring pattern f of $G = P_m \times C_n$, for $m \geq 7, n \geq 7$ is represented in Tables 2(a) and 2(b) for $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$. Table 2(a) represents the star edge coloring patterns $C_{ij} = f((u_i, v_j)(u_{i+1}, v_{j+1}))$ and $D_{ij} = f((u_i, v_j)(u_{i+1}, v_{j-1}))$ where the rows and columns denote the values of i and j respectively for $i \equiv r_1(mod\ 8)$ where $r_1 \in \{0, 1, 2, ..., 7\}$ and $j \equiv r_2(mod\ 8)$ where $r_2 \in \{0, 1, 2\}$ and the Table 2(b) represents the star edge coloring patterns $E_{ij} = f((u_i, v_0)(u_{i+1}, v_j))$ and $F_{ij} = f((u_i, v_j)(u_{i+1}, v_0))$, for $0 \leq i \leq m-1$ and $j = n-1$.

Table 2 (a):The star edge coloring patterns C_{ij} and D_{ij}

i \ j	0	1	2
0	0	1	2
1	3	4	5
2	6	7	0
3	1	2	3
4	4	5	6
5	7	0	1
6	2	3	4
7	5	6	7

Table 2 (b):The star edge coloring patterns E_{ij} and F_{ij}

	E_{ij}	F_{ij}
For $n \equiv 0(mod\ 3)$	$3i + 2(mod\ 8)$	$3i + 2(mod\ 8)$
For $n \equiv 1(mod\ 3)$	$3i + 1(mod\ 8)$	$3i + 1(mod\ 8)$
For $n \equiv 2(mod\ 3)$	$3i + 1(mod\ 8)$	$3i + 1(mod\ 8)$

This coloring pattern and the following example validate that the star chromatic index of $P_m \times C_n$ is 8.

Example 1: Since f has a repeated star edge coloring pattern for all $m \geq 7$, $n \geq 7$, it sufficies to check that $P_{10} \times C_{10}$ is 8-star edge colorable (see in Figure 3).

□

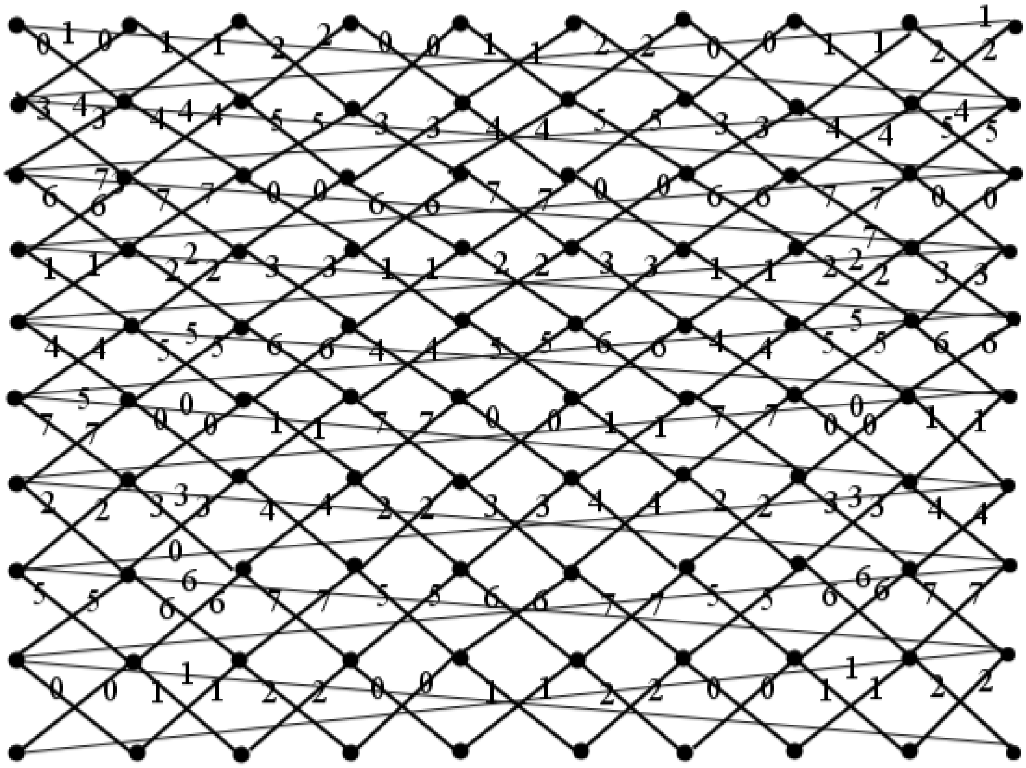

Figure 3: Star edge coloring of $P_{10} \times C_{10}$

5 Star Chromatic Index of Tensor Product of Cycles

Theorem 5.1. *For the given positive integers $m \geq 3$ and $n \geq 3$, the star chromatic index of $C_m \times C_n$ is given by*

$$\chi'_{st}(C_m \times C_n) = \begin{cases} 7, & \text{for } m = 3, n = 3 \\ 8, & \text{otherwise} \end{cases}$$

Proof. Let $G = C_m \times C_n$ and by the definition of tensor product, $V(G) = \{(u_i, v_j) : 0 \leq i \leq m-1,\ 0 \leq j \leq n-1\}$, $E(G) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \cup E_8$, where
$E_1 = \bigcup_{i=1}^{m-2} \{(u_i, v_j)(u_{i+1}, v_{j+1}) : 0 \leq j \leq n-2\}$
$E_2 = \bigcup_{i=1}^{m-2} \{(u_i, v_j)(u_{i+1}, v_{j-1}) : 1 \leq j \leq n-1\}$
$E_3 = \bigcup_{i=0}^{m-1} \{(u_i, v_0)(u_{i+1}, v_j) : j = n-1\}$
$E_4 = \bigcup_{i=0}^{m-1} \{(u_i, v_j)(u_{i+1}, v_0) : j = n-1\}$
$E_5 = \bigcup_{j=1}^{n-1} \{(u_0, v_j)(u_i, v_{j-1}) : i = m-1\}$
$E_6 = \bigcup_{j=0}^{n-2} \{(u_0, v_j)(u_i, v_{j+1}) : i = m-1\}$.
$E_7 = \{(u_0, v_j)(u_i, v_0) : i = m-1, j = n-1\}$.
$E_8 = \{(u_0, v_0)(u_i, v_j) : i = m-1, j = n-1\}$.
and j is taken addition modulo n with residues $0, 1, 2, ..., n-1$. Let f be the function defined by $f : E \to C \subseteq \mathcal{N} \cup \{0\}$ and C is the set of colors. We obtain the star edge

chromatic index of $C_m \times C_n$ according to the given values of m,n as follows:
Case 1: $m = 3$, $n = 3$

In the case of $G = C_3 \times C_3$ is a 4-regular graph and contains $G_1 = P_3 \times C_3$ as a subgraph, and by Theorem 4.1, it is found that $\chi'_{st}(P_3 \times C_3) = 5$. Also, $G_2 = G \backslash G_1$ is a cycle of length six and by Theorem 2.2, $\chi'_{st}(C_6) = 3$. In the graph G, it is not possible to color the edges of G_2 with the same set of colors already used in the coloring of G_1. Since if we choose any vertex v from the first and last layers of G_2 there exists two new adjacent edges at the particular vertex v, which cannot be colored with the colors already used for the edges of G_1 . Therefore, two additional colors are required to color these two edges. Hence, to color the edges of G_2, the sixth and seventh color can be introduced along with one more color which is already used to color G_1. Hence it is found that, $\chi'_{st}(C_3 \times C_3) = 7$.

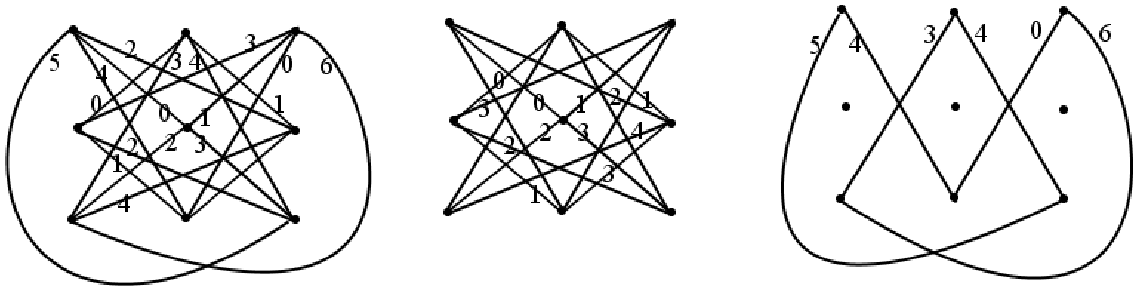

Figure 4: $G = C_3 \times C_3, G_1 = P_3 \times C_3$ and $G_2 = G \backslash G_1$

Case 2: i)$m = 3, n \geq 3$ *and* ii) $m \geq 4, n \geq 3$

When $m = 3$ and $n = 4$, the graph $G = C_3 \times C_4$ is a 4-regular graph that contains $G_3 = P_3 \times C_4$ as a subgraph, and by Theorem 4.1, it is found that $\chi'_{st}(P_3 \times C_4) = 6$. Also, in $G_4 = G \backslash G_3$ the two cycles of length six are found and by Theorem 2.2 $\chi'_{st}(C_6) = 3$. In the graph G_4, it is not possible to color the edges with the same set of colors already used in the coloring of G_3, because if any vertex v in G_4 is chosen , two new adjacent edges are added at the particular vertex v, hence additional two colors are needed to color those two edges. Therefore, seventh and eighth color are introduced for the edges in G_4. Hence it is found that $\chi'_{st}(C_3 \times C_4) = 8$. By using the same argument it can be proved that $\chi'_{st}(C_3 \times C_n) = 8$, for $n \geq 4$.
The proof of other cases are discussed based on the star chromatic index of the maximal induced subgraph G_1 of G for which $\chi'_{st}(G_1)$ is already known and that can be used to fix the lower bound of $\chi'_{st}(G)$ as in the following table:

Table (3):The star edge coloring of $G = C_m \times C_n$, for $m \geq 3$, $n \geq 4$

m	n	$C_m \times C_n$	G_1	$\chi'_{st}(G_1)$	$\chi'_{st}(G)$
4	$n \geq 3$	$C_4 \times C_n$	$P_4 \times C_n$	6	8
5	$n \geq 4$	$C_5 \times C_n$	$P_5 \times C_n$	7	8
6,7	$3 \leq n \leq 6$	$C_m \times C_n$	$P_m \times C_n$	7	8
7	7	$C_7 \times C_7$	$P_7 \times P_7$	8	8

By using a similar argument it can be proved that $\chi'_{st}(C_m \times C_n) = 8$, for $m \geq 7, n \geq 7$. To develop an algorithmic approach for the star edge coloring pattern of $G = C_m \times C_n$, for large values of $m \geq 3$, $n > 3$, the following coloring pattern is discussed. And the star edge coloring pattern f of $G = C_m \times C_n$, for $m \geq 3$, $n > 3$ is given below. As previously mentioned, the edges of the subgraph $G' = P_m \times C_n$ can be colored using the same coloring pattern as given in Theorem 3.1. Here $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$.

Case 1: $m \equiv 0(mod\ 8), n > 3$

$$f((u_0, v_0)(u_{m-1}, v_{1)})) = 5$$

$$f((u_0, v_0)(u_{m-1}, v_{n-1})) = 7$$

$$f((u_0, v_{n-1})(u_{m-1}, v_{n-2})) = \begin{cases} 3, & \text{if } n \equiv 0(mod\ 3) \\ 6, & \text{if } n \equiv 1(mod\ 3) \\ 7, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$$f((u_0, v_{n-1})(u_{m-1}, v_0)) = \begin{cases} 7, & \text{if } n \equiv 0(mod\ 3) \\ 7, & \text{if } n \equiv 1(mod\ 3) \\ 3, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq\ j \leq n-2$,

$$f((u_0, v_j)(u_{m-1}, v_{j-1})) = \begin{cases} 7, & \text{if } n \equiv 0(mod\ 3) \\ 2, & \text{if } n \equiv 1(mod\ 3) \\ 6, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq\ j \leq n-2$,

$$f((u_0, v_j)(u_{m-1}, v_{j+1})) = \begin{cases} 5, & \text{if } n \equiv 0(mod\ 3) \\ 6, & \text{if } n \equiv 1(mod\ 3) \\ 7, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

Case 2: $m \equiv 1(mod\ 8), n > 3$

$$f((u_0, v_0)(u_{m-1}, v_1)) = 3$$

$$f((u_0, v_0)(u_{m-1}, v_{n-1})) = 4$$

$$f((u_0, v_{n-1}), (u_{m-1}, v_{n-2})) = \begin{cases} 3, & \text{if } n \equiv 0 (mod\ 3) \\ 4, & \text{if } n \equiv 1 (mod\ 3) \\ 5, & \text{if } n \equiv 2 (mod\ 3) \end{cases}$$

$$f((u_0, v_{n-1})(u_{m-1}, v_0)) = \begin{cases} 4, & \text{if } n \equiv 0 (mod\ 3) \\ 0, & \text{if } n \equiv 1 (mod\ 3) \\ 1, & \text{if } n \equiv 2 (mod\ 3) \end{cases}$$

For $1 \leq\ j \leq n-2$,

$$f((u_0, v_j)(u_{m-1} v_{j-1})) = \begin{cases} 5, & \text{if } n \equiv 0 (mod\ 3) \\ 3, & \text{if } n \equiv 1 (mod\ 3) \\ 4, & \text{if } n \equiv 2 (mod\ 3) \end{cases}$$

For $1 \leq\ j \leq n-2$,

$$f((u_0, v_j)(u_{m-1}, v_{j+1})) = \begin{cases} 1, & \text{if } n \equiv 0 (mod\ 3) \\ 4, & \text{if } n \equiv 1 (mod\ 3) \\ 0, & \text{if } n \equiv 2 (mod\ 3) \end{cases}$$

Case 3: $m \equiv 2(mod\ 8), n > 3$

$$f((u_0, v_0)(u_{m-1}, v_1)) = 3$$

$$f((u_0, v_0)(u_{m-1}, v_{n-1})) = 5$$

$$f((u_0, v_j)(u_{m-1}, v_{n-2})) = \begin{cases} 4, & \text{if } n \equiv 0 (mod\ 3) \\ 5, & \text{if } n \equiv 1 (mod\ 3) \\ 3, & \text{if } n \equiv 2 (mod\ 3) \end{cases}$$

$$f((u_0, v_{n-1})(u_{m-1}, v_0)) = \begin{cases} 5, & \text{if } n \equiv 0 (mod\ 3) \\ 3, & \text{if } n \equiv 1 (mod\ 3) \\ 4 & \text{if } n \equiv 2 (mod\ 3) \end{cases}$$

For $1 \leq\ j \leq n-2$,

$$f((u_0, v_j)(u_{m-1}, v_{j-1})) = \begin{cases} 5, & \text{if } n \equiv 0 (mod\ 3) \\ 3, & \text{if } n \equiv 1 (mod\ 3) \\ 4, & \text{if } n \equiv 2 (mod\ 3) \end{cases}$$

For $1 \leq\ j \leq n-2$,

$$f((u_0, v_j)(u_{m-1}, v_{j+1})) = \begin{cases} 6, & \text{if } n \equiv 0 (mod\ 3) \\ 4, & \text{if } n \equiv 1 (mod\ 3) \\ 5, & \text{if } n \equiv 2 (mod\ 3) \end{cases}$$

Case 4: $m \equiv 3(mod\ 8), n > 3$

$$f((u_0, v_0)(u_{m-1}, v_{1)})) = 7$$

$$f((u_0,v_0)(u_{m-1},v_{n-1})) = \begin{cases} 3, & \text{if } n \equiv 0(mod\ 3) \\ 3, & \text{if } n \equiv 1(mod\ 3) \\ 5, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$$f((u_0,v_{n-1})(u_{m-1},v_{n-2})) = \begin{cases} 6, & \text{if } n \equiv 0(mod\ 3) \\ 1, & \text{if } n \equiv 1(mod\ 3) \\ 7, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$$f((u_0,v_{n-1})(u_{m-1},v_0)) = \begin{cases} 0, & \text{if } n \equiv 0(mod\ 3) \\ 2, & \text{if } n \equiv 1(mod\ 3) \\ 6, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq \ j \leq n-2$

$$f((u_0,v_j)(u_{m-1},v_{j-1})) = \begin{cases} 1, & \text{if } n \equiv 0(mod\ 3) \\ 7, & \text{if } n \equiv 1(mod\ 3) \\ 6, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$1 \leq \ j \leq n-2,$

$$f((u_0,v_j)(u_{m-1},v_{j+1})) = \begin{cases} 7, & \text{if } n \equiv 0(mod\ 3) \\ 6, & \text{if } n \equiv 1(mod\ 3) \\ 0, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

Case 5: $m \equiv 4(mod\ 8), n > 3$

$$f((u_0,v_0)(u_{m-1},v_{1)})) = 3$$

$$f((u_0,v_0)(u_{m-1},v_{n-1})) = \begin{cases} 5, & \text{if } n \equiv 0(mod\ 3) \\ 5, & \text{if } n \equiv 1(mod\ 3) \\ 6, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$$f((u_0,v_{n-1})(u_{m-1},v_{n-2})) = \begin{cases} 4, & \text{if } n \equiv 0(mod\ 3) \\ 5, & \text{if } n \equiv 1(mod\ 3) \\ 3, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$$f((u_0,v_{n-1})(u_{m-1},v_0)) = \begin{cases} 5, & \text{if } n \equiv 0(mod\ 3) \\ 4, & \text{if } n \equiv 1(mod\ 3) \\ 2 & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq \ j \leq n-2$

$$f((u_0,v_j)(u_{m-1},v_{j-1})) = \begin{cases} 5, & \text{if } n \equiv 0(mod\ 3) \\ 3, & \text{if } n \equiv 1(mod\ 3) \\ 4, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$1 \leq \ j \leq n-2$

$$f((u_0,v_j)(u_{m-1},v_{j+1})) = \begin{cases} 1, & \text{if } n \equiv 0(mod\ 3) \\ 4, & \text{if } n \equiv 1(mod\ 3) \\ 5, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

Case 6: $m \equiv 5(mod\ 8), n > 3$

$f((u_0,v_0)(u_{m-1},v_1)) = 5$
$f((u_0,v_0)(u_{m-1},v_{n-1})) = 7$

$$f((u_0,v_{n-1})(u_{m-1},v_{n-2})) = \begin{cases} 6, & \text{if } n \equiv 0(mod\ 3) \\ 4, & \text{if } n \equiv 1(mod\ 3) \\ 5, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$$f((u_0,v_{n-1})(u_{m-1},v_0)) = \begin{cases} 5, & \text{if } n \equiv 0(mod\ 3) \\ 0, & \text{if } n \equiv 1(mod\ 3) \\ 6, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq\ j \leq n-2$,

$$f((u_0,v_j)(u_{m-1},v_{j-1})) = \begin{cases} 1, & \text{if } n \equiv 0(mod\ 3) \\ 5, & \text{if } n \equiv 1(mod\ 3) \\ 6, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq\ j \leq n-2$,

$$f((u_0,v_j)(u_{m-1},v_{j+1})) = \begin{cases} 7, & \text{if } n \equiv 0(mod\ 3) \\ 6, & \text{if } n \equiv 1(mod\ 3) \\ 4, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $m \equiv 6(mod\ 8), n > 3$
$f((u_0,v_0)(u_{m-1},v_1)) = 7$
$f((u_0,v_0)(u_{m-1},v_{n-1})) = 3$

$$f((u_0,v_{n-1})(u_{m-1},v_{n-2})) = \begin{cases} 7, & \text{if } n \equiv 0(mod\ 3) \\ 6, & \text{if } n \equiv 1(mod\ 3) \\ 6, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$$f((u_0,v_{n-1})(u_{m-1},v_0)) = \begin{cases} 2, & \text{if } n \equiv 0(mod\ 3) \\ 0, & \text{if } n \equiv 1(mod\ 3) \\ 3, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq\ j \leq n-2$

$$f((u_0,v_j)(u_{m-1},v_{j-1})) = \begin{cases} 1, & \text{if } n \equiv 0(mod\ 3) \\ 0, & \text{if } n \equiv 1(mod\ 3) \\ 7, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq \ j \leq n-2$

$$f((u_0, v_j)(u_{m-1}, v_{j+1})) = \begin{cases} 7, & \text{if } n \equiv 0(mod\ 3) \\ 4, & \text{if } n \equiv 1(mod\ 3) \\ 3, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

Case 7: $m \equiv 7(mod\ 8), n > 3$

$$f((u_0, v_0)(u_{m-1}, v_{1)})) = 3$$

$$f((u_0, v_0)(u_{m-1}, v_{n-1})) = 6$$

$$f((u_0, v_{n-1})(u_{m-1}, v_{n-2})) = \begin{cases} 3, & \text{if } n \equiv 0(mod\ 3) \\ 4, & \text{if } n \equiv 1(mod\ 3) \\ 5, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$f((u_0, v_{n-1})(u_{m-1}, v_0)) = 6$ for all n

For $1 \leq \ j \leq n-2$

$$f((u_0, v_{n-1})(u_{m-1}, v_{j-1})) = \begin{cases} 5, & \text{if } n \equiv 0(mod\ 3) \\ 3, & \text{if } n \equiv 1(mod\ 3) \\ 4, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq \ j \leq n-2$

$$f((u_0, v_j)(u_{m-1}, v_{j+1})) = \begin{cases} 1, & \text{if } n \equiv 0(mod\ 3) \\ 4, & \text{if } n \equiv 1(mod\ 3) \\ 5, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

This coloring pattern and the following example validate that the star chromatic index of $C_m \times C_n$ is 8.

Example 2: Since f has a repeated star edge coloring pattern for all $m \geq 3, n > 3$ it sufficies to check that $C_{10} \times C_{10}$ has a 8-star edge coloring as shown in Figure 5.

$\square$

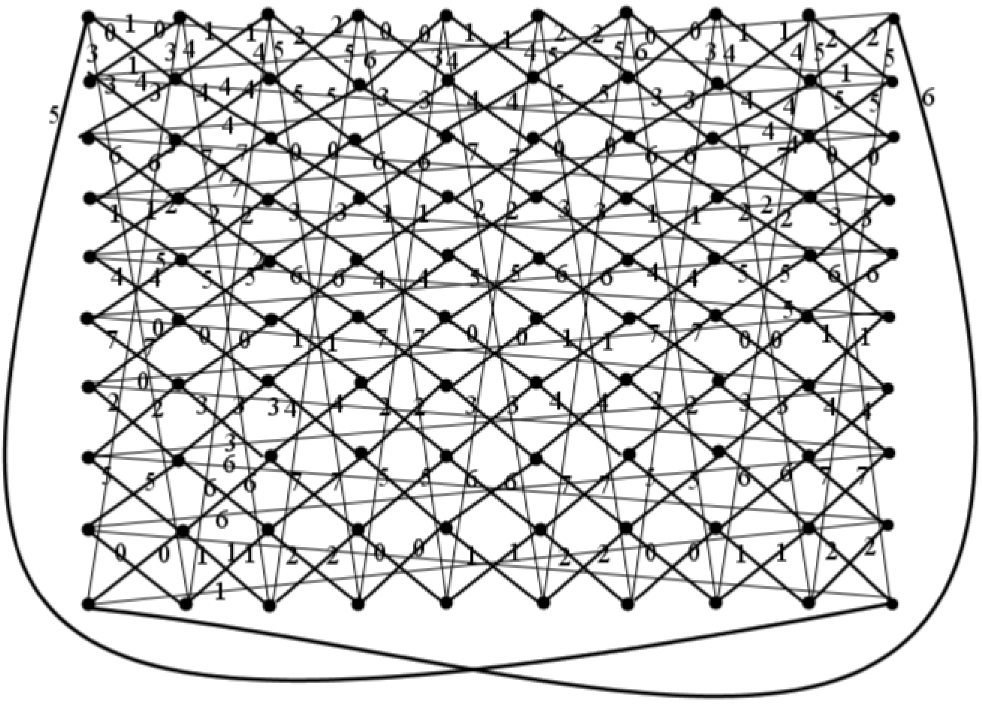

Figure 5: Star edge coloring of $C_{10} \times C_{10}$

6 Conclusion

This research has explored the star chromatic index of $P_m \times C_n$ and $C_m \times C_n$. Further, coloring patterns were given to star edge color any such graph and that can be used as an algorithmic tool to obtain the star edge coloring of large graph structure such as given product graphs, for various values of m and n. This work can be further extended to determine the star chromatic index of other classes of product graphs and to study their algorithmic complexity. Moreover, the proposed coloring patterns may find potential applications in areas such as network design, communication systems, and scheduling problems, where conflict-free edge assignments are essential.

Disclaimer (Artificial intelligence)
Option 1:
Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

References

[1] Bondy J.A., Murty U.S.R, Graph Theory with Applications, *London, Macmillan*, (1976).

[2] Benzegova, L. et al. Star edge coloring of some classes of graphs, *Journal of Graph Theory,* 81(1), (2016), 73 - 82.

[3] Deng, X., et al. Star edge coloring of outerplanar and cubic Graphs: Bounds and Constructions, *Bulletin of Malaysian Mathematical Sciences Society* 48, 93 (2025), 1-12.

[4] Dvorak, Z . et al. Star chromatic index, *Journal of Graph Theory,* 72, (2013), 313 - 326.

[5] Fernando,C.L.R. et al. Star chromatic index of some types of Graphs, *Asian Research Journal of Mathematics*, 21, (2025) 37-43.

[6] Fraser, et al. Overview of Graph Colouring and Some Ramsey-Type Numbers, *Journal of Advances in Mathematics and Computer Science*,5, (2014), 414-428.

[7] Holub, P. et al. Star edge coloring of square grids, *arXiv:* 2005.02864,2020

[8] Lei, Hui. et al. A survey on star edge-coloring of graphs, *arXiv:* 2009.08017V1 [math.CO], 17 Sep 2020.

[9] Kaliraj, K. et al. Star edge coloring of corona product of path and wheel graph families, *Proyeciones Journal of Mathematics*, 37(4), (2018), 593-601.

[10] Kavita, Pradeep. et al. Star chromatic index of direct product of graphs, *International Journal of Pure and Applied Mathematics*, 109(9), (2016), 293-301.

[11] Liu, X.S. et al. An upper bound on the star chromatic index of graphs with $\delta \geq 7$, *Journal of Lanzhou University (Natural sciences)*, 44, (2008), 94-95.

[12] Sivakami R, et al. On Star coloring of Lexicographic product of graphs, *Acta Universitatis Apulensis*, 67, (2021), 49-59.

[13] Omooni,B. et al. Star edge coloring of cartesian product of graphs, *Australian Journal of Combinatorics*, 79, (2021), 15-30.

[14] Xingxing Hu, et al. The star edge coloring of cubic Halin graphs with star chromatic index 5, *arXiv;2511.1314ovl[math.co].*

[15] Wang,Y. et al. Star edge-coloring of graphs with maximum degree four, *Applied Mathematics and Computation*, 333, (2018), 480 - 489.

[16] Yunfeng Tang, et al. Star edge coloring of $K_{2,t}$ -free planar graphs, *AIMS Mathematics*, 8, (2023), 13154–13161.

[17] Hemalatha, P. (2025). STAR CHROMATIC INDEX OF LINE GRAPHS OF THE FAMILY OF LADDER GRAPHS. International Journal of Applied Mathematics, 38(3s), 1253-1265.