

CONNECTED DOM- k -FORCING SETS IN GRAPHS

ABSTRACT. A vertex subset D_{kf} in a graph G is referred to as a dom- k -forcing set if it satisfies the properties of both a k -forcing set and a dominating set. The dom- k -forcing number of a graph G , denoted as $F_{dk}(G)$, represents the smallest possible size of a dom- k -forcing set. A connected dom- k -forcing set of a graph G , is a dom- k -forcing set of G that induces a sub graph of G which is connected. The connected dom- k -forcing number of G , $F_{cdk}(G)$, is the minimum size of a connected dom- k -forcing set. In this paper, we introduce the study of connected dom- k -forcing sets in graphs. We derived formulas for finding precise values of connected dom- k -forcing number of tree, grid graph, prism graph, gear graph, path, cycle, etc. Additionally, we characterize connected dom- k -forcing sets in certain special graphs, deriving formulas for connected dom- k -forcing number based on these characterizations.

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Key Words: Connected domination number, k -forcing number, dom- k -forcing number, connected dom- k -forcing number.

1. INTRODUCTION

The domination number of a graph G , denoted by $\gamma(G)$, is the minimum number of vertices in a dominating set. A dominating set is a subset S of the graph's vertex set such that every vertex not in S is adjacent to at least one vertex in S . In other words, all vertices outside the set are directly connected to vertices within it. The domination number reflects how effectively a subset of vertices can oversee or influence the entire graph [1].

If the subgraph induced by a dominating set is connected, then the set is known as a connected dominating set. Among all such sets, a minimum connected dominating set is one with the smallest possible number of vertices. The size of this set is called the connected domination number, denoted by $\gamma_c(G)$ [2].

A subset Z_k of vertices in a graph is called a k -forcing set if its vertices are initially assigned a color while the remaining vertices start as uncolored. The graph undergoes a color propagation process based on the following rule: a colored vertex with at most k uncolored neighbors forces each of those neighbors to become colored. This process continues until all vertices in the graph are colored. The k -forcing number of a graph, denoted as $Z_k(G)$, represents the smallest possible size of a k -forcing set see[9].

A connected k -forcing set of a graph G , is a k -forcing set of G that induces a sub graph of G which is connected. The connected k -forcing number of G , $Z_{ck}(G)$, is the minimum size of connected k -forcing sets. Any connected k -forcing set of order $Z_{ck}(G)$ is called a minimum connected k -forcing set [4].

A vertex subset D_{kf} in a graph G is referred to as a dom- k -forcing set if it satisfies the properties of both a k -forcing set and a dominating set. The dom- k -forcing number of a graph G , denoted as $F_{dk}(G)$, represents the smallest possible size of a dom- k -forcing set.

The present authors derived the dom-forcing number, connected dom-forcing number, and connected dom-forcing propagation time of some well known graphs in [5], [6], [7]. The bonds for k -forcing propagation time is investigated in [8]. In [9] Amos et al. proposed the definition of k -forcing set. In [3] the authors presence a study on the k -forcing number in the context of splitting graphs. Yair Caro et al. in [24] proved a conjuncture post by Amos et al.in [9]. Upper bounds for k - forcing number is investigated in [12]. Zero forcing number of splitting graphs investigated in [18] and new class of graphs introduced in [14], [13], [19].

This study integrates the concepts of connected domination and connected k -forcing to introduce a new graph-theoretic parameter: the connected dom- k -forcing set.

Definition 1.1. *For any $k \in \mathcal{N}, 1 \leq k \leq \Delta$, a connected dom- k -forcing set of a graph G , is a dom- k -forcing set of G that induces a sub graph of G which is connected. The connected dom- k -forcing number of G , $F_{cdk}(G)$, is the minimum size of a connected dom- k - forcing set.*

For instance, consider the graph G illustrated in Figure 1. The set $D_{fk} = \{v_2, v_5\}$ is a valid 2-forcing set as well as a dominating set, the sub graph induced by D_{fk} is connected. Here the connected dom-2-forcing number of G is 2.

When $k = 1$, the connected dom- k -forcing number is equivalent to the connected dom-forcing number, denoted by $F_{cd}(G)$ See [6]. When $k = 2$, this is known as the connected dom-2-forcing number and is denoted by $F_{cd2}(G)$.

In this paper, we introduce the study of connected dom- k -forcing sets in graphs. We characterize connected dom- k -forcing sets in certain special graphs,

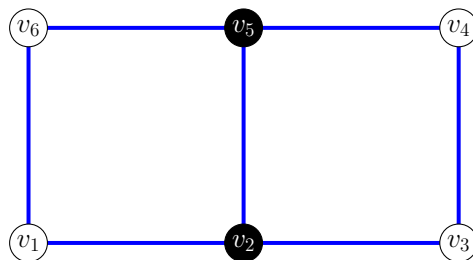


FIGURE 1. The graph $G(V, E)$ with dom-2-forcing set $\{v_2, v_5\}$

deriving formulas for connected dom- k -forcing number based on these characterizations. Additionally, we explore the relationships between this parameter and other well-known graph theory parameters like k -forcing and domination.

2. MOTIVATION

The study of the connected dom- k -forcing number of graphs is at the intersection of domination theory and dynamic graph processes, motivated by both theoretical curiosity and practical applications. In graph theory, domination parameters help to characterise how influence, control or monitoring can be established over a network. k -forcing processes, a generalisation of the zero forcing process, model how information, disease or opinions spread through a system, subject to constraints given by the number k of forced neighbours.

By combining these two concepts—connected domination and k -forcing—we can investigate how an initial set of “influencing” nodes can control the spread in a graph and remain connected throughout the process. This is crucial in situations where continuous communication or cohesion among influencers is necessary, such as in wireless sensor networks, distributed computing or coordinated response systems. Studying the connected dom- k -forcing number not only enriches the theoretical framework of graph parameters but also leads to efficient algorithms and heuristics for real-world network management tasks, including fault detection, resource allocation and containment strategies. Understanding the complexity, bounds and characterisations of this parameter opens up new avenues in combinatorics and applied network science.

3. BOUNDS FOR CONNECTED DOM- k -FORCING NUMBER

In this section, we find some bounds for connected dom- k -forcing number. Also examine the graph having $F_{cdk}(G) = \gamma_c(G)$.

From the definition, it is clear that the union of a connected k -forcing set and a connected dominating set forms a dom- k -forcing set. Consequently, the following relationship holds.

Proposition 3.1. *For a graph G*

- i) $Z_{ck}(G) \leq F_{cdk}(G) \leq Z_{ck}(G) + \gamma_c(G)$*
- ii) $\gamma_c(G) \leq F_{cdk}(G) \leq Z_{ck}(G) + \gamma_c(G)$*

From the definition it is also evident that, for a graph G every connected dom- k -forcing set is also a connected dom- $k + 1$ -forcing set. Hence we have the following result.

Proposition 3.2. *Let G be a graph. Then $F_{cd(k+1)}(G) \leq F_{cdk}(G)$, for all $k \in \mathbb{N}$.*

Also from the definition every connected dom- k -forcing set is also a dom- k -forcing set. Hence we have the following result.

Proposition 3.3. *For any connected graph G ,*

$$F_{dk}(G) \leq F_{cdk}(G).$$

Theorem 3.4. [3] *If G is a connected graph with $\Delta(G) \leq 2$, then $Z_2(G) = 1$.*

Theorem 3.5. *Let G be a connected graph with $\Delta(G) \leq 2$. Then $F_{cd2}(G) = \gamma_c(G)$.*

Proof. Let G be a graph with $\Delta(G) \leq 2$. Then any single vertex in $V(G)$ 2-forces the graph. Let A be any connected dominating set for G . The set A 2-forces the graph since $\Delta(G) \leq 2$. Hence every connected dominating set is also a connected dom-2-forcing set. Therefore, $F_{cd2}(G) = \gamma_c(G)$. \square

There are graphs with $Z_{c2}(G) = 1$ and $F_{d2}(G) > \gamma_c(G)$. Consider the graph given below. The set $Z_2 = \{v_1\}$ is a connected 2- forcing set, but it cannot dominates G . The set $S = \{v_2\}$ is a dominating set for G , but it is not a 2-forcing set. The set $Z_2 \cup S$ forms a 2-forcing as well as a dominating set for G . Hence G is a graph with $Z_{c2}(G) = 1$, $\gamma_c(G) = 1$ and $F_{cd2}(G) = 2$.

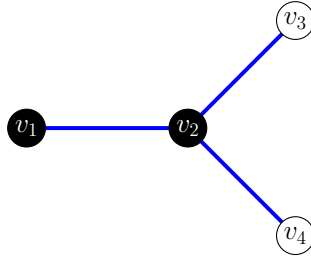


FIGURE 2. A graph G having $Z_{c2}(G) = 1$, $\gamma_c(G) = 1$ and $F_{cd2}(G) = 2$.

From the above example, we can see that the upper bound given in Proposition 3.1 is sharp. We can easily observe the following result.

Proposition 3.6. *If G is a connected graph with minimum degree $\delta \geq k + 1$, then $F_{cdk}(G) > 1$.*

In general, we have the following result

Theorem 3.7. *Let G be a graph with $F_{cdp}(G) = \gamma_c(G)$ for some positive integer p . Then $F_{cdk}(G) = \gamma_c(G)$ for all $k \geq p$.*

4. EXACT VALUES OF CONNECTED DOM- k -FORCING NUMBER

In this section, we find the exact values of connected dom - k -forcing number of several graphs and categorize the graphs where $F_{dk}(G) = F_{cdk}(G)$, $F_{dk}(G) = F_{cdk}(G)$ and $F_{dk}(G) < F_{cdk}(G)$.

4.1. Graphs where $F_{dk}(G) = F_{cdk}(G)$. There are graphs having the same dom- k -forcing number and connected dom- k -forcing number.

The friendship graph F_n is formed by connecting n copies of the complete graph K_3 to a single common vertex see [10].

Proposition 4.1. *Let F_p denote the friendship graph with $p \geq 2$ triangles. Then $F_{d2}(F_p) = p$.*

Now we discuss the case where $k = 2$.

Proposition 4.2. *Let F_p denote the friendship graph with $p \geq 2$ triangles. Then $F_{d2}(F_p) = F_{cd2}(F_p) = p$.*

Proof. Let $v, v_1, v'_1, \dots, v_p, v'_p$ be the vertices of F_p with common vertex v . Consider the subset $A = \{v, v_1, \dots, v_{p-1}\}$ of $V(F_p)$. Then A is a connected dom-2-forcing set, which is minimum and $F_{d2}(F_p) = p$. Hence $F_{d2}(F_p) = F_{cd2}(F_p) = p$. \square

Let $G = K_n$ be a complete graph with n vertices. Then $F_{dk}(K_n) = \max\{n - k, 1\}$, for any positive integer k .

Theorem 4.3. *For the complete graph K_n , $F_{dk}(K_n) = F_{cdk}(K_n) = \max\{n - k, 1\}$, where k is any positive integer.*

Proof. For any complete graph K_n , every dom- k -forcing set is connected. Hence, $F_{dk}(K_n) = F_{cdk}(K_n) = \max\{n - k, 1\}$, for any positive integer k . \square

Let $K_{m,n}$ be a complete bipartite, where $n, m \geq 3$. Then we have $F_{d2}(K_{m,n}) = m + n - 4$. Here, every dom-2-forcing set of $K_{m,n}$ is connected. Hence, we get the following result.

Theorem 4.4. *Let $K_{m,n}$ be a complete bipartite, where $n, m \geq 3$. Then $F_{d2}(K_{m,n}) = F_{cd2}(K_{m,n}) = m + n - 4$.*

The wheel graph, W_n , is a graph obtained by connecting a single vertex to all vertices of a cycle graph C_{n-1} . Also we have $F_{dk}(W_n) = 2$ for $2 \leq k < n - 1$.

Theorem 4.5. *For the wheel graph W_n , $F_{dk}(W_n) = F_{cdk}(W_n) = 2$, where $2 \leq k < n - 1$.*

Proof. Let v_1, v_2, \dots, v_n be the vertices of the wheel graph W_n . Assume that v_1 has degree $n - 1$ and all other vertices have degree 3. Then for $2 \leq k < n - 1$, every minimum dom- k -forcing set contains v_1 and the set connected. Therefore, $F_{dk}(W_n) = F_{cdk}(W_n) = 2$, for $2 \leq k < n - 1$. \square

A helm graph H_m is a graph that is created by attaching a pendant edge to each vertex of an m -wheel graph's cycle, where $m \geq 4$ [11]. We can easily verify that the collection of all vertices in the cycle forms a connected dom-2-forcing set. Hence, the following Theorem.

Theorem 4.6. *Let H_m be the helm graph. Then $F_{d2}(H_m) = F_{cd2}(H_m) = m$.*

The prism graph, also known as the circular ladder graph, is formed by taking the Cartesian product of a cycle C_n and the complete graph K_2 . It is usually represented as $C_n \square K_2$ see [16].

Let G be the prism graph $C_n \square K_2$, where v_1, v_2, \dots, v_n are the vertices of the cycle C_n and v'_1, v'_2, \dots, v'_n be the corresponding vertices in $C_n \square K_2$. Now, consider a graph H obtained by subdividing each edge $v_i v'_i$ (for $i = 1, 2, \dots, n$) exactly once (see [3]).

Theorem 4.7. *Under the above assumptions, let H be the graph obtained from the prism graph by subdividing each edge $v_i v'_i$ (for $i = 1, 2, \dots, n$) exactly once. Then, $F_{d2}(H) = F_{cd2}(H) = n$.*

Proof. Let w_1, w_2, \dots, w_n be the vertices obtained by subdividing each edges $v_i v'_i$ (for $i = 1, 2, \dots, n$) exactly once. Each vertex w_i is adjacent to both v_i and v'_i . Consequently, to dominate w_i , any dominating set of H must contain at least one of the vertices v_i, w_i , or v'_i . Hence $\gamma(H) \geq n$. Also the set $D = \{w_1, w_2, \dots, w_n\}$ dominates H , $\gamma(H) = n$. Here the set D 2-forces the graph H and the subgraph induced by D is connected. Therefore, $F_{d2}(H) = F_{cd2}(H) = n$. \square

Theorem 4.8. *Let G be the graph $C_n \square P_3$, $n > 3$. Then $F_{cd2}(G) = \gamma_c(G) = n$.*

Proof. Let $u_1, \dots, u_n, v_1, \dots, v_n, w_1, \dots, w_n$ be the vertices of the three copies of C_n in $C_n \square P_3$, with $\deg(u_i) = \deg(w_i) = 3$ and $\deg(v_i) = 4$. The set $A = \{v_1, \dots, v_n\}$ is a connected dominating set for G . Here $|A|$ is minimum since if we remove any one of the v_i in A , it affects the domination property. Hence $\gamma_c(G) = n$, the set A 2-forces the entire graph. Therefore, $F_{cd2}(G) = \gamma_c(G) = n$. \square

4.2. Graphs where $F_{dk}(G) < F_{cdk}(G)$. There are graphs with $F_{dk}(G) < F_{cdk}(G)$. From Theorem 3.5 we have the following result.

Theorem 4.9.

- (1) *For a path P_n of order n , $F_{cd2}(P_n) = \gamma_c(P_n) = n - 1$.*
- (2) *For a cycle C_n of order n , $F_{cd2}(C_n) = \gamma_c(C_n) = n - 2$.*

Now we consider the connected dom- k -forcing number of trees.

Definition 4.10. *Let T be a tree. Consider the vertices which are adjacent to the leaf (pendant) vertex and call these vertices as parent leaf vertices.*

For example, consider the tree T illustrated in figure 3. Here, the parent leaf vertices of T are v_3 and v_4 .

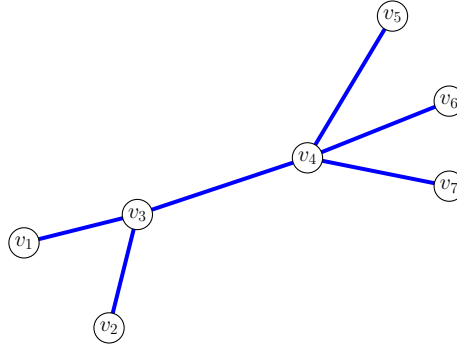


FIGURE 3. The tree T with parent leaf vertices v_3 and v_4 .

Theorem 4.11. *Let T be a tree. The number of leaves adjacent to each parent leaf vertex is at most k if and only if $F_{cdk}(T) = \gamma_c(T) = n - m$, where n is the number of vertices in T and m is the number of leaves in T .*

Proof. First, we assume that the total number of leaves adjacent to each parent leaf vertex of the tree T is at most k . We know that all the vertices of T , except the pendant vertices, form a unique minimum connected dominating set, which k -forces the graph T . Hence $F_{cdk}(T) = \gamma_c(T) = n - m$, where n is the number of vertices in T and m is the number of leaves in T .

Conversely, assume that $F_{cdk}(T) = \gamma_c(T) = n - m$, where n is the number of vertices in T and m is the number of leaves in T . Let H be the subset of the vertex set of T that does not contain the pendant vertices of T . Hence $|H| = n - m$. If possible, assume that there exists a parent leaf vertex, say v , which is adjacent to at least $k + 1$ pendant vertices. All vertices, except the pendant vertices, are in the unique minimum connected dominating set of T . Also from v , we cannot k -force all the pendant vertices adjacent to v . Therefore, the set H cannot be a connected dom- k -forcing set of T , a contradiction. Hence, the total number of leaves adjacent to each parent leaf vertex of the tree T is at most k . \square

We know that the Cartesian product of two graphs G and H , is denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H)$ such that (u, v) is adjacent to (u', v') if and only if

- (1) $u = u'$ and $vv' \in E(H)$, or
- (2) $v = v'$ and $uu' \in E(G)$.

The Cartesian product of two path graphs is known as the grid graph.

Let p and q be positive integers such that $p \leq q$. A $p \times q$ grid graph [15] $G_{p,q} = (V, E)$ is a graph where

$$V(G(p, q)) = \{(i, j) | 1 \leq i \leq p, 1 \leq j \leq q\},$$

$$E(G(p, q)) = \{(i, j), (i, j + 1)\} | 1 \leq i \leq p, 1 \leq j \leq q - 1\} \cup \{(i, j), (i + 1, j)\} | 1 \leq i \leq p - 1, 1 \leq j \leq q\}.$$

The ladder graph L_n is obtained by taking the Cartesian product of path P_n with the complete graph K_2 . It is known that for a ladder graph L_n of order

n , $F_{cd}(L_n) = n = \gamma_c(L_n)$ for $n \geq 4$ (see [6]). Therefore, we have the following Proposition.

Proposition 4.12. *Let L_n be the Ladder graph of order $n \geq 4$. Then $F_{cdk}(L_n) = n$, for all $k \in \mathbb{N}$.*

Let us recall the following result from [15] to prove the succeeding Theorem.

Theorem 4.13. [15] *The connected domination number of $3 \times p$ grid graph $G_{3,p}$ is $\gamma_c(G_{3,p}) = p$.*

Theorem 4.14. *The connected dom-2-forcing number of $3 \times p$ grid graph $G_{3,p}$ is*

$$F_{cd2}(G_{3,p}) = p.$$

Proof. We can see that the set $A = \{(2,1), (2,2), \dots, (2,p)\}$ is a minimum connected domination set and is unique. The set A 2-forces the graph $G_{3,p}$. Hence

$$F_{cd2}(G_{3,p}) = p. \quad \square$$

Theorem 4.15. [21] *The connected domination number of $p \times q$ ($p \geq 4, q \geq 4$) grid graph $G_{p,q}$ is*

$$\gamma_c(G_{p,q}) = \frac{pq - a'_{p,q}}{3} + \bar{r}'_{p,q} + c'_{p,q}.$$

in which

$$a'_{p,q} = p(\text{mod } 3) \cdot q(\text{mod } 3)$$

$$\bar{r}'_{p,q} = \begin{cases} 3 & p(\text{mod } 3) \cdot q(\text{mod } 3) = 4 \\ 2 & p(\text{mod } 3) \cdot q(\text{mod } 3) = 2 \\ 1 & p(\text{mod } 3) \cdot q(\text{mod } 3) = 1 \\ 0 & p(\text{mod } 3) \cdot q(\text{mod } 3) = 0 \end{cases}$$

$$c'_{p,q} = \begin{cases} \min\{\frac{p}{3}, \frac{q}{3}\} & p(\text{mod } 3) = 0 \text{ and } q(\text{mod } 3) = 0 \\ \frac{p}{3} & p(\text{mod } 3) = 0 \text{ and } q(\text{mod } 3) \neq 0 \\ \frac{q}{3} & p(\text{mod } 3) \neq 0 \text{ and } q(\text{mod } 3) = 0 \\ \lfloor \frac{p}{3} \rfloor + \lfloor \frac{q}{3} \rfloor - 1 & p(\text{mod } 3) \neq 0 \text{ and } q(\text{mod } 3) \neq 0 \end{cases}$$

Theorem 4.16. *Let $G_{p,q}$ be a $p \times q$ ($p \geq 4, q \geq 4$) grid graph. Then*

$$F_{cd2}(G_{p,q}) = \gamma_c(G_{p,q}).$$

Proof. Let $G_{p,q}$ be a $p \times q$ ($p \geq 4, q \geq 4$) grid graph. Consider the subsets of the vertex set of $G_{p,q}$

$$A = \{(1,2), (2,2), \dots, (p,2)\},$$

$$B = \{(2,3), (2,4), \dots, (2,q)\}.$$

Also consider the following cases.

Case 1: For any q and $p \pmod{3} = 0$, we consider the vertex set $S = A \cup B \cup C$ in which $C = \{(x,3), \dots, (x,q) | x = 5, 8, \dots, p-1\}$. Theorem 4.15 says that the set S is a minimum connected dominating set with cardinality $\frac{p(q+1)}{3}$.

Case 2: For any p and $q \pmod{3} = 0$, we consider the vertex set $S = A \cup B \cup C'$ in which $C' = \{(3, y), \dots, (p, y) | y = 5, 8, \dots, q - 1\}$. Then the set S is a minimum connected dominating set with cardinality $\frac{(p+1)q}{3}$ (By Theorem 4.15).

Case 3: For $p \pmod{3} = 1$ and $q \pmod{3} = 1$, we consider the vertex set $S = A \cup B_1 \cup C_1 \cup E_1$ for $p = 4$ and $S = A \cup B \cup C_1 \cup D_1 \cup E_1$ for $p > 4$, in which $B_1 = \{(2, 3), (2, 4), \dots, (2, q - 2)\}$, $C_1 = \{(3, y), \dots, (p, y) | y = 5, 8, \dots, q - 2\}$, $D_1 = \{(x, q - 1), (x, q) | x = 5, 8, \dots, p - 5\}$ and $E_1 = \{(p - 1, q - 1), (p - 1, q), (p - 2, q)\}$. Then the set S is a minimum connected dominating set with cardinality $\frac{pq+p+q-3}{3}$ (By Theorem 4.15).

Case 4: For $p \pmod{3} = 1$ and $q \pmod{3} = 2$, we consider the vertex set $S = A \cup B \cup C_2 \cup D_2 \cup E_2$, in which $C_2 = \{(3, y), \dots, (p, y) | y = 5, 8, \dots, q - 3\}$, $D_2 = \{(x, q - 2), (x, q - 1), (x, q) | x = 5, 8, \dots, p - 2\}$ and $E_2 = \{(p, q - 2), (p, q - 1)\}$. Then the set S is a minimum connected dominating set with cardinality $\frac{pq+p+q-2}{3}$ (By Theorem 4.15).

Case 5: For $p \pmod{3} = 2$ and $q \pmod{3} = 1$, we consider the vertex set $S = A \cup B \cup C_3 \cup D_3 \cup E_3$, in which $C_3 = \{(3, y), \dots, (p, y) | y = 5, 8, \dots, q - 2\}$, $D_3 = \{(x, q - 2), (x, q - 1), (x, q) | x = 5, 8, \dots, p - 3\}$ and $E_3 = \{(p, q - 1), (p, q)\}$. Then the set S is a minimum connected dominating set with cardinality $\frac{pq+p+q-2}{3}$ (By Theorem 4.15).

Case 6: For $p \pmod{3} = 2$ and $q \pmod{3} = 2$, we consider the vertex set $S = A \cup B \cup C_4 \cup D_4 \cup E_4$, in which $C_4 = \{(3, y), \dots, (p, y) | y = 5, 8, \dots, q - 3\}$, $D_4 = \{(x, q - 2), (x, q - 1), (x, q) | x = 5, 8, \dots, p - 3\}$ and $E_4 = \{(p, q - 2), (p, q - 1), (p, q)\}$. Then the set S is a minimum connected dominating set with cardinality $\frac{pq+p+q-2}{3}$ (By Theorem 4.15).

In all cases, the set S includes a complete row of vertices, and this alone is enough to 2-force the entire graph. Hence

$$F_{cd2}(G_{p,q}) = \gamma_c(G_{p,q}).$$

□

The above Theorem is illustrated for the grid graph $G_{11,11}$ in Figure 4. The vertices belonging to the sets A, B, C, D , and E are labeled accordingly as A, B, C, D , and E in the figure.

Now consider the prism graph $C_n \square K_2$.

Theorem 4.17. [22] *Let G be a prism graph $C_n \square K_2$, $n > 3$. Then $\gamma_c(G) = n$.*

Theorem 4.18. *Let G be a prism graph $C_n \square K_2$, $n > 3$. Then $F_{cdk}(G) = \gamma_c(G) = n$, for all positive integers $1 \leq k \leq 3$.*

Proof. Let v_1, v_2, \dots, v_n be the vertices of the cycle C_n and let v'_1, v'_2, \dots, v'_n be the corresponding vertices in $C_n \square K_2$. Consider the set $A = \{v_1, v_2, \dots, v_n\}$, which dominates G . The set A is a minimum connected dominating set by the above Theorem. We can see that the set A forces the entire graph G . Hence $F_{cdk}(G) = \gamma_c(G) = n$, for all positive integers $1 \leq k \leq 3$. □

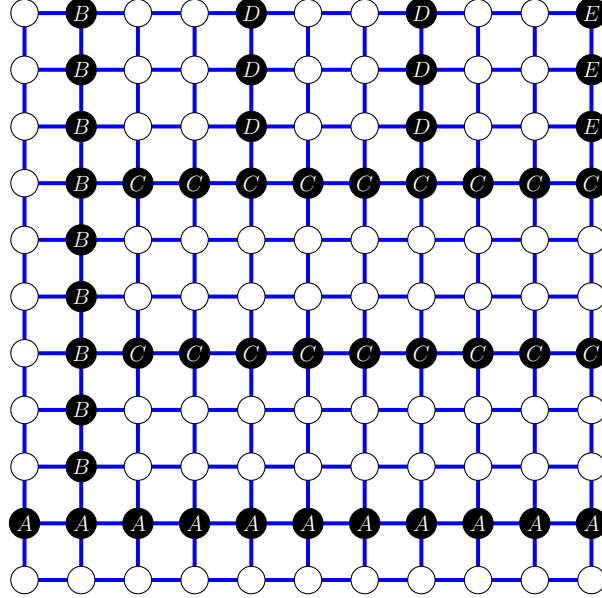


FIGURE 4. The grid graph $G_{11,11}$. The collection of all black colored vertices form a connected dom-2-forcing set as well as a connected dominating set for $G_{11,11}$.

An antiprism graph [23] is a graph having $2n$ vertices on two cycles labeled with $\{v_1, v_2, \dots, v_n\}$ in the inner cycle, $\{v_{n+1}, v_{n+2}, \dots, v_{2n}\}$ in the outer cycle. Joining these two cycles with edges of the form $(v_i v_j : 1 \leq i \leq n, j = n + i)$, $(v_1 v_{2n})$ and $(v_i v_{j-1} : 2 \leq i \leq n, j = n + i)$ such that v_i and $v_{i+1(\text{mod } n)}$ are adjacent. It is denoted as A_n .

Theorem 4.19. [22] *Let A_n be an antiprism graph with $2n$ vertices. Then $\gamma_c(A_n) = n - 1$.*

Theorem 4.20. *Let A_n , $n > 3$, be an antiprism graph. Then $F_{cd2}(A_n) = \gamma_c(A_n) = n - 1$.*

Proof. Let v_1, v_2, \dots, v_n be the vertices of the inner cycle and let $\{v_{n+1}, v_{n+2}, \dots, v_{2n}\}$ be the corresponding vertices in A_n . Consider the set $A = \{v_1, v_2, \dots, v_{n-1}\}$, which dominates A_n . The set A is a minimum connected dominating set by the above Theorem. We can see that the set A 2-forces the entire graph A_n by starting the forcing process from the vertex v_2 . Hence $F_{cd2}(A_n) = \gamma_c(A_n) = n - 1$. \square

In the case of A_8 , the above Theorem is illustrated in Figure 6.

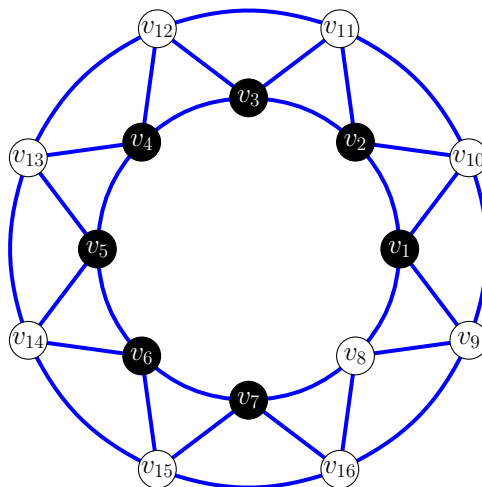


FIGURE 5. The antiprism graph A_8 , the connected dom-2-forcing set $A = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $F_{cd2}(A_8) = 7$.

The n -crossed prism graph [22], R_n , of positive even n vertices is the graph obtained by taking two disjoint cycle graphs C_n labeled with $\{v_1, v_2, \dots, v_n\}$ in the inner cycle, $\{v'_1, v'_2, \dots, v'_n\}$ in the outer cycle. Adding edges (v_k, v'_{2k+1}) and (v_{k+1}, v'_{2k}) , for $k = 1, 3, \dots, (n - 1)$. We can see that R_n is a 3-regular graph.

Theorem 4.21. [22] *Let R_n be the n -crossed prism graph with $2n$ vertices. Then $\gamma_c(R_n) = n$.*

Theorem 4.22. *Let R_n be the n -crossed prism graph. Then $F_{cdk}(R_n) = \gamma_c(R_n) = n$ for all positive integers k .*

Proof. Let v_1, v_2, \dots, v_n be the vertices of the inner cycle and let $\{v'_1, v'_2, \dots, v'_n\}$ be the corresponding vertices of the outer circle. Consider the set $A = \{v_1, v_2, \dots, v_n\}$, which dominates R_n . The set A is a minimum connected dominating set by the above Theorem. We can see that the set A forces the entire graph R_n . Hence $F_{cdk}(R_n) = \gamma_c(R_n) = n$, for all positive integers k . \square

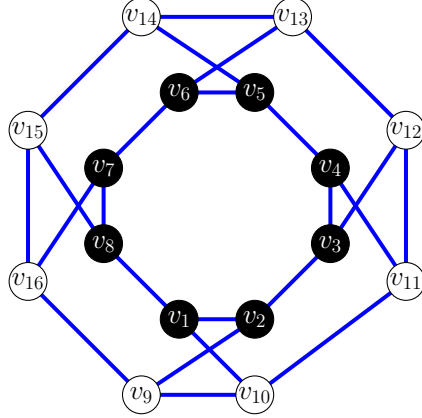


FIGURE 6. The 8-crossed prism graph R_8 , the connected dom-2-forcing set $A = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and $F_{cd2}(A_8) = 8$.

The gear graph is a wheel graph with a vertex added between each pair of adjacent vertices of the outer cycle. The gear graph G_n has $2n + 1$ vertices and $3n$ edges.

Theorem 4.23. [22] For the gear graph G_n , $\gamma_c(G_n) = 1 + \lceil n/2 \rceil$.

Theorem 4.24. For the gear graph G_n ,

$$F_{cdk}(G_n) = \begin{cases} 2 + \lceil n/2 \rceil & \text{for } k = 1 \\ 1 + \lceil n/2 \rceil & \text{for } k > 1 \end{cases}$$

Proof. Let v, v_1, v_2, \dots, v_n be the vertices of the wheel graph with $\deg(v) = n$ and u_1, u_2, \dots, u_n be the vertices added to get the gear graph G_n . Now consider $A = \{v, v_1, v_3, \dots, v_n\}$ if n is odd and $A = \{v, v_1, v_3, \dots, v_{n-1}\}$ if n is even. In both cases, the set A is the minimum connected dominating set by the above Theorem and which is unique up to isomorphism. We can see that the set A 2-forces the graph G_n , since every vertex in the set A has two white neighbors. The set A cannot zero force the graph G_n . Hence, every connected dom-forcing set of G_n must contain strictly more vertices than the connected domination number $\gamma_c(G_n)$. But $A \cup \{u_1\}$ zero forces G_n . Therefore,

$$F_{cdk}(G_n) = \begin{cases} 2 + \lceil n/2 \rceil & \text{for } k = 1 \\ 1 + \lceil n/2 \rceil & \text{for } k > 1 \end{cases}$$

□

The above Theorem is illustrated in Figure 7.

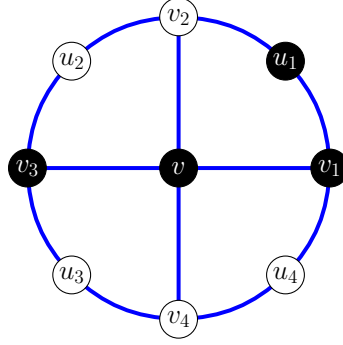


FIGURE 7. The gear graph G_4 , $A = \{v, v_1, u_1, v_3\}$ is a minimum connected dom-forcing set and $B = \{v, v_1, v_3\}$ is a minimum connected dom-2-forcing set. Hence $F_{cd}(G_4) = 4$ and $F_{cd2}(G_4) = 3$.

4.3. Connected dom- k -forcing number of splitting graph of a graph.

The splitting graph of a graph G is the graph $S(G)$ obtained by taking a vertex v' corresponding to each vertex $v \in G$ and join v' to all the vertices of G adjacent to v [17].

Theorem 4.25. [20] *Let G be a connected graph of order n . Then*

$$\gamma_c[S(G)] = \begin{cases} 2 & \text{if } \gamma_c(G) = 1 \\ \gamma_c(G) & \text{if } \gamma_c(G) \geq 2 \end{cases}$$

Theorem 4.26. *Let $S(P_n)$, $n \geq 4$ be the splitting graph of the path P_n . Then $F_{cd2}[S(P_n)] = \gamma_c[S(P_n)] = n - 2$.*

Proof. Let v_1, v_2, \dots, v_n be the vertices of the path graph P_n in $S(P_n)$, and let u_1, u_2, \dots, u_n be the vertices corresponding to v_1, v_2, \dots, v_n which are added to obtain $S(P_n)$. Then the set $A = \{v_2, \dots, v_{n-1}\}$ is a minimum connected dominating set by the Theorem 4.25. We can see that this set 2-forces $S(P_n)$. Hence $F_{cd}[S(P_n)] = n - 2$. \square

Remark: The graph $S(P_2)$, which is isomorphic to P_4 , hence $F_{cd2}[S(P_2)] = \gamma_c[S(P_2)] = 2$. For the graph $S(P_3)$, $A = \{v_1, v_2\}$ is a minimum connected dominating set which 2-forces the entire graph. Hence $F_{cd2}[S(P_3)] = \gamma_c[S(P_3)] = 2$. Therefore, in all cases $F_{cd2}[S(P_n)] = \gamma_c[S(P_n)]$, for all n .

Theorem 4.27. *Let $S(C_n)$, $n \geq 5$ be the splitting graph of the cycle C_n . Then*

$$F_{cd2}[S(C_n)] = \gamma_c[S(C_n)] = n - 2.$$

Proof. Let v_1, v_2, \dots, v_n be the vertices of the cycle graph C_n in $S(C_n)$, and let u_1, u_2, \dots, u_n be the vertices corresponding to v_1, v_2, \dots, v_n which are added to obtain $S(C_n)$. Then $A = \{v_1, \dots, v_{n-2}\}$ is a minimum connected dominating set by the Theorem 4.25. We can see that this set 2-forces $S(C_n)$. Hence $F_{cd}[S(C_n)] = n - 2$. \square

In the case of $S(C_3)$, $A = \{v_1, u_1\}$ is a minimum connected dominating set which 2-forces the entire graph. Hence $F_{cd2}[S(C_3)] = \gamma_c[S(C_3)] = 2$. For $S(C_4)$, $A = \{v_1, v_2\}$ is a minimum connected dominating set and is unique. We can

see that this set is not a 2-forcing set. But $A' = \{v_1, v_2, u_1\}$ be a minimum connected dom-2-forcing set. Hence $F_{cd2}[S(C_3)] = 3$.

Theorem 4.28. For $n \geq 2$, $F_{cd3}[S(L_n)] = \gamma_c[S(L_n)] = n$.

Proof. Let $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ be the vertices of the ladder graph L_n and let $\{u'_1, u'_2, \dots, u'_n, v'_1, v'_2, \dots, v'_n\}$ be the vertices corresponding to $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ which are added to obtain $S(L_n)$. Then $A = \{v_1, \dots, v_n\}$ is a minimum connected dominating set by the Theorem 4.25. We can see that the set A 3-forces the entire graph. Hence $F_{cd3}[S(L_n)] = n$. \square

5. CONCLUSION AND OPEN PROBLEMS

This article deals with the problem of determining the connected dom- k -forcing number of graphs. Section 3 deals with some bounds for the connected dom- k -forcing number. In Section 4, we identified some graphs whose dom- k -forcing number and connected dom- k -forcing number are the same. Also, in this section, we determined some graphs with the property $F_{cdk}(G) = \gamma_c(G)$. In particular, characterizes the trees for which $F_{cdk}(T) = n - m$. Additionally, the connected dom- k -forcing number of the splitting graph of a graph is determined in Section 5. There are few questions that remains open, for example see the following.

- While we have determined $F_{cdk}(G)$ for certain well-known graphs, a comprehensive characterization for a wider variety of graph classes remains an open problem. We extend these results to other families of graphs, such as bipartite graphs, or random graphs, could yield valuable insights.
- Exploring the relationships between $F_{cdk}(G)$ and other graph parameters, such as, chromatic number, independence number, and spectral properties, could uncover deeper connections and will lead to a more unified theory of graph invariants.
- Developing efficient algorithms to compute connected dom-forcing number for a connected graph G .
- Determining the connected dom- k -forcing number of a splitting graph for an arbitrary graph

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