

Picture Fuzzy Subalgebras and Ideals in Sheffer Stroke UP-Algebras

Abstract

Picture fuzzy sets, introduced by Cuong in 2014, extend intuitionistic fuzzy sets by assigning three independent degrees to each element—positive membership (μ), neutral membership (η), and negative membership (ν), together with an explicit refusal degree, thereby enabling a richer representation of uncertainty involving neutrality and abstention. In this paper, picture fuzzy structures are introduced into Sheffer stroke UP-algebras (SUP-algebras), an algebraic framework based on the functionally complete Sheffer stroke (NAND) operation. Picture fuzzy SUP-subalgebras and picture fuzzy SUP-ideals are defined, and several equivalent characterizations are established using level sets, complement functions, and characteristic picture fuzzy sets associated with crisp subalgebras and ideals. It is shown that every picture fuzzy SUP-ideal is a picture fuzzy SUP-subalgebra, while the converse does not hold in general, and that the family of all picture fuzzy SUP-subalgebras forms a complete distributive lattice. Preservation properties under direct products and surjective homomorphisms are also investigated. These results enrich the theory of SUP-algebras by incorporating neutrality and refusal into their algebraic structure, thereby enhancing their applicability to uncertainty modeling, non-classical logical systems, and decision-making frameworks.

Keywords: Sheffer stroke UP-algebra, picture fuzzy set, picture fuzzy SUP-subalgebra, picture fuzzy SUP-ideal, level set, direct product, homomorphism.

2020 Mathematics Subject Classification: 03G25; 06F35; 08A72.

1 Introduction

The Sheffer stroke, introduced by Henry M. Sheffer, is a functionally complete binary operation equivalent to the NAND connective. It is capable of expressing all Boolean connectives and has played a fundamental role in simplifying axiomatizations in logic and algebra [6]. Motivated by this foundational concept, several algebraic structures have been developed in recent years. Among them, Sheffer stroke UP-algebras (SUP-algebras), introduced by Oner and Katican [13], extend UP-algebras [3] by incorporating the Sheffer stroke as a primitive operation. SUP-algebras provide a unified framework for studying non-classical logics, uncertainty modeling, and decision-making processes.

Recent studies on SUP-algebras have mainly focused on intuitionistic fuzzy and neutrosophic extensions, emphasizing their ability to model hesitation and indeterminacy. However, these approaches are limited in capturing more refined human decision behavior involving neutrality or abstention. Picture fuzzy sets (PFSs), introduced by Cuong [5], overcome this limitation by assigning three independent degrees to each element: positive membership (μ), neutral membership (η), and negative membership (ν), satisfying

$$0 \leq \mu(x) + \eta(x) + \nu(x) \leq 1,$$

with the refusal degree defined as

$$\rho(x) = 1 - (\mu(x) + \eta(x) + \nu(x)).$$

This additional flexibility makes PFSs particularly suitable for applications such as voting systems, group decision-making, and multi-agent environments where abstention plays a significant role.

Several researchers have contributed to the theoretical development and applications of picture fuzzy sets. Adak, Gunjan, and Agarwal introduced the notion of picture fuzzy semiprime ideals [1]. Distance measures for PFSs and their applications to multi-criteria decision-making problems were studied in [2, 8]. Thao [9] proposed new correlation coefficients for picture fuzzy sets and applied them to clustering problems. In parallel, different fuzzy variants of Sheffer stroke UP-algebras have been investigated in [10, 12, 15]. Although picture fuzzy sets have been applied to UP-algebras and related structures [11], their integration with SUP-algebras has not yet been explored.

The present paper fills this gap by introducing picture fuzzy SUP-subalgebras and picture fuzzy SUP-ideals in Sheffer stroke UP-algebras. Several equivalent characterizations are established using level sets, complements of picture fuzzy components, and characteristic picture fuzzy sets associated with crisp subsets. It is shown that every picture fuzzy SUP-ideal is a picture fuzzy SUP-subalgebra, while the converse does not hold in general, as demonstrated by a counterexample. Furthermore, monotonicity properties with respect to the underlying partial order are examined, and it is proved that the family of picture fuzzy SUP-subalgebras forms a complete distributive lattice. Preservation properties under direct products and homomorphisms are also investigated.

These results significantly enhance the capability of SUP-algebras to model complex uncertainty involving neutrality and refusal, thereby broadening their applicability to multi-criteria decision-making with abstention, fuzzy control systems, and non-classical logical reasoning.

The remainder of the paper is organized as follows. Section 2 recalls basic notions of SUP-algebras and picture fuzzy sets. In Section 3, picture fuzzy SUP-subalgebras and their fundamental properties are developed. Section 4 introduces picture fuzzy SUP-ideals and examines their characteristics. Section 5 discusses algebraic constructions, including direct products and homomorphic images. Finally, Section 6 presents concluding remarks and directions for future research.

2 Preliminaries

In this section, we recall the basic definitions and fundamental properties of Sheffer stroke UP-algebras (SUP-algebras) and picture fuzzy sets that will be used throughout the paper.

2.1 Sheffer Stroke UP-Algebras

Definition 2.1 ([13]). *Let $\langle H, | \rangle$ be a groupoid. The binary operation $|$ is called a Sheffer stroke if it satisfies the following axioms for all $x, y, z \in H$:*

$$(S1) \quad x|y = y|x,$$

$$(S2) \quad (x|x) | (x|y) = x,$$

$$(S3) \quad x | (((y|z)|(y|z))|((y|z)|(y|z))) = x,$$

$$(S4) \quad (x | ((x|x)|(y|y))) | (x | ((x|x)|(y|y))) = x.$$

Definition 2.2 ([13]). *A Sheffer stroke UP-algebra (SUP-algebra) is an algebraic structure $\langle H, |, 0 \rangle$ of type $(2, 0)$, where $0 \in H$ is a fixed element, and the following axioms hold for all $x, y, z \in H$:*

$$(1) \quad (((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y)))|((y|(x|x))|(z|(y|y)))))|(((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y)))|((y|(x|x))|(z|(y|y))))) = 0$$

$$(2) \ x|x = x|0|0$$

$$(3) \ (x|(y|y))|(x|(y|y)) = 0 \text{ and } (y|(x|x))|(y|(x|x)) = 0 \implies x = y.$$

Proposition 2.3 ([13]). *Let $\langle H, |, 0 \rangle$ be an SUP-algebra. The relation defined by*

$$x \leq y \iff (y|(x|x))|(y|(x|x)) = 0$$

is a partial order on H .

Lemma 2.4 ([13]). *Let $\langle H, |, 0 \rangle$ be an SUP-algebra. Then for all $x, y, z \in H$,*

1. $x \leq y \implies y|(z|z) \leq x|(z|z)$ and $z|(x|x) \leq z|(y|y)$,
2. $x \leq y \iff y|y \leq x|x$,
3. $y|(x|x) \leq x$,
4. $y \leq (y|(x|x))|(y|(x|x))$,
5. $x \leq y \implies x \leq (y|(z|z))|(y|(z|z))$,
6. $z|(y|y) \leq z|(y|(x|x))$,
7. $((z|(y|y))|(z|(y|y)))|(x|x) \leq z|(y|(x|x))$,
8. $x|((y|(z|z))|(y|(z|z))) \leq (x|(y|y))|((x|(z|z))|(x|(z|z)))$.

Definition 2.5 ([13]). *A nonempty subset S of an SUP-algebra H is called a subalgebra of H if*

$$(x|(y|y))|(x|(y|y)) \in S$$

for all $x, y \in S$.

Definition 2.6 ([14]). *A nonempty subset I of an SUP-algebra H is an ideal of H if for all $x, y \in H$,*

- (1) $y \in I \implies (y|(x|x))|(y|(x|x)) \in I$,
- (2) $(y|(x|x))|(y|(x|x)) \in I$ and $x \in I \implies y \in I$.

2.2 Picture Fuzzy Sets

Definition 2.7 ([5]). *Let H be a nonempty set. A picture fuzzy set (PFS) \mathcal{P} on H is an object of the form*

$$\mathcal{P} = \{ \langle x, \mu_{\mathcal{P}}(x), \eta_{\mathcal{P}}(x), \nu_{\mathcal{P}}(x) \rangle \mid x \in H \},$$

where $\mu_{\mathcal{P}} : H \rightarrow [0, 1]$, $\eta_{\mathcal{P}} : H \rightarrow [0, 1]$, and $\nu_{\mathcal{P}} : H \rightarrow [0, 1]$ are the positive membership, neutral membership, and negative membership functions, respectively, satisfying

$$0 \leq \mu_{\mathcal{P}}(x) + \eta_{\mathcal{P}}(x) + \nu_{\mathcal{P}}(x) \leq 1$$

for all $x \in H$. The value

$$\rho_{\mathcal{P}}(x) = 1 - \mu_{\mathcal{P}}(x) - \eta_{\mathcal{P}}(x) - \nu_{\mathcal{P}}(x)$$

is called the refusal degree of x in \mathcal{P} .

Definition 2.8 (Operations on picture fuzzy sets, [5]). *Let $\mathcal{A} = (\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}})$ and $\mathcal{B} = (\mu_{\mathcal{B}}, \eta_{\mathcal{B}}, \nu_{\mathcal{B}})$ be two picture fuzzy sets on H . Then:*

(1) The union $\mathcal{A} \cup \mathcal{B}$ is the picture fuzzy set defined by

$$\begin{aligned}\mu_{\mathcal{A} \cup \mathcal{B}}(x) &= \max\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)\}, \\ \eta_{\mathcal{A} \cup \mathcal{B}}(x) &= \min\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{B}}(x)\}, \\ \nu_{\mathcal{A} \cup \mathcal{B}}(x) &= \min\{\nu_{\mathcal{A}}(x), \nu_{\mathcal{B}}(x)\}.\end{aligned}$$

(2) The intersection $\mathcal{A} \cap \mathcal{B}$ is the picture fuzzy set defined by

$$\begin{aligned}\mu_{\mathcal{A} \cap \mathcal{B}}(x) &= \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)\}, \\ \eta_{\mathcal{A} \cap \mathcal{B}}(x) &= \max\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{B}}(x)\}, \\ \nu_{\mathcal{A} \cap \mathcal{B}}(x) &= \max\{\nu_{\mathcal{A}}(x), \nu_{\mathcal{B}}(x)\}.\end{aligned}$$

(3) The complement $\overline{\mathcal{A}}$ is the picture fuzzy set defined by

$$\begin{aligned}\mu_{\overline{\mathcal{A}}}(x) &= \nu_{\mathcal{A}}(x), \\ \eta_{\overline{\mathcal{A}}}(x) &= \eta_{\mathcal{A}}(x), \\ \nu_{\overline{\mathcal{A}}}(x) &= \mu_{\mathcal{A}}(x).\end{aligned}$$

Remark 2.9. The above operations are the most commonly used in algebraic applications of picture fuzzy sets. Although alternative definitions of union and intersection exist in the literature, the standard operations originally proposed by Cuong are adopted here, as they are well suited for characterizing picture fuzzy subalgebras and ideals.

3 Picture Fuzzy SUP-Subalgebras

In this section, we introduce the concept of a picture fuzzy SUP-subalgebra and investigate its basic properties and equivalent characterizations.

3.1 Definition and Basic Properties

Definition 3.1. Let $\langle H, |, 0 \rangle$ be an SUP-algebra. A picture fuzzy set $\mathcal{P} = (\mu, \eta, \nu)$ in H is called a picture fuzzy SUP-subalgebra of H if for all $x, y \in H$,

$$\begin{aligned}\mu((x|(y|y))|(x|(y|y))) &\geq \min\{\mu(x), \mu(y)\}, \\ \eta((x|(y|y))|(x|(y|y))) &\leq \max\{\eta(x), \eta(y)\}, \\ \nu((x|(y|y))|(x|(y|y))) &\leq \max\{\nu(x), \nu(y)\}.\end{aligned}$$

Theorem 3.2. Let $\mathcal{P} = (\mu, \eta, \nu)$ be a picture fuzzy SUP-subalgebra of an SUP-algebra H . Then for all $x \in H$,

$$\mu(0) \geq \mu(x), \quad \eta(0) \leq \eta(x), \quad \nu(0) \leq \nu(x).$$

Proof. Using (SUP-2), we have $x|x = x|(0|0)$. Thus,

$$\begin{aligned}\mu(0) &= \mu((x|(x|x))|(x|(x|x))) \geq \min\{\mu(x), \mu(x)\} = \mu(x), \\ \eta(0) &= \eta((x|(x|x))|(x|(x|x))) \leq \max\{\eta(x), \eta(x)\} = \eta(x), \\ \nu(0) &= \nu((x|(x|x))|(x|(x|x))) \leq \max\{\nu(x), \nu(x)\} = \nu(x).\end{aligned}$$

This completes the proof. □

Remark 3.3. The conditions on the neutral (η) and negative (ν) degrees follow the standard pattern used in most picture fuzzy algebraic structures: the neutral degree is upper-bounded (abstention does not increase under operation), while the negative degree is also upper-bounded (non-membership does not increase). The positive degree is lower-bounded as in classical fuzzy and intuitionistic cases.

3.2 Characterizations of Picture Fuzzy SUP-Subalgebras

We now provide several equivalent characterizations of picture fuzzy SUP-subalgebras.

Theorem 3.4. *Let $\mathcal{P} = (\mu, \eta, \nu)$ be a picture fuzzy set in an SUP-algebra H . Define fuzzy sets $\eta^c, \nu^c : H \rightarrow [0, 1]$ by*

$$\eta^c(x) = 1 - \eta(x), \quad \nu^c(x) = 1 - \nu(x)$$

for all $x \in H$. Then \mathcal{P} is a picture fuzzy SUP-subalgebra of H if and only if the fuzzy sets μ , η^c , and ν^c are fuzzy SUP-subalgebras of H .

Proof. Assume \mathcal{P} is a picture fuzzy SUP-subalgebra. Clearly μ satisfies the fuzzy subalgebra condition. For η^c ,

$$\begin{aligned} \eta^c((x|(y|y))|(x|(y|y))) &= 1 - \eta((x|(y|y))|(x|(y|y))) \\ &\geq 1 - \max\{\eta(x), \eta(y)\} \\ &= \min\{1 - \eta(x), 1 - \eta(y)\} \\ &= \min\{\eta^c(x), \eta^c(y)\}. \end{aligned}$$

The proof for ν^c is analogous.

Conversely, suppose μ , η^c , and ν^c are fuzzy SUP-subalgebras. Then

$$\begin{aligned} \eta((x|(y|y))|(x|(y|y))) &= 1 - \eta^c((x|(y|y))|(x|(y|y))) \\ &\leq 1 - \min\{\eta^c(x), \eta^c(y)\} \\ &= \max\{\eta(x), \eta(y)\}, \end{aligned}$$

and similarly for ν . The condition for μ holds directly. Thus \mathcal{P} is a picture fuzzy SUP-subalgebra. \square

Theorem 3.5. *Let S be a nonempty subset of an SUP-algebra H . Define a picture fuzzy set $\mathcal{P}_S = (\mu_S, \eta_S, \nu_S)$ in H by*

$$\mu_S(x) = \begin{cases} a & \text{if } x \in S, \\ b & \text{if } x \notin S, \end{cases} \quad \eta_S(x) = \begin{cases} c & \text{if } x \in S, \\ d & \text{if } x \notin S, \end{cases} \quad \nu_S(x) = \begin{cases} e & \text{if } x \in S, \\ f & \text{if } x \notin S, \end{cases}$$

where $0 \leq a, b, c, d, e, f \leq 1$, $a > b$, $c < d$, $e < f$, and $a + c + e \leq 1$, $b + d + f \leq 1$. The strict inequalities $a > b$, $c < d$, and $e < f$ ensure a proper distinction between elements inside and outside the crisp subset S with respect to positive, neutral, and negative membership degrees. Then \mathcal{P}_S is a picture fuzzy SUP-subalgebra of H if and only if S is a subalgebra of H .

Proof. Assume \mathcal{P}_S is a picture fuzzy SUP-subalgebra and let $x, y \in S$. Then

$$\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y)\} = a,$$

so $\mu((x|(y|y))|(x|(y|y))) = a$, which implies $(x|(y|y))|(x|(y|y)) \in S$. Similarly from η and ν conditions we obtain the same conclusion. Thus S is a subalgebra.

Conversely, assume S is a subalgebra. If $x, y \in S$, then $(x|(y|y))|(x|(y|y)) \in S$, so all three conditions hold with equality to the values inside S . If at least one of x, y is not in S , then the minimum a(or maximum) on the right-hand side corresponds to the outside values, and the left-hand side is at least as good (since $a \geq b$, $d \geq c$, $f \geq e$). Thus \mathcal{P}_S is a picture fuzzy SUP-subalgebra. \square

3.3 Level Set Characterization

For a picture fuzzy set $\mathcal{P} = (\mu, \eta, \nu)$ in H and thresholds $t \in [0, 1]$, $s_1, s_2 \in [0, 1]$, we define the following level sets:

$$\begin{aligned} U(\mu; t) &= \{x \in H \mid \mu(x) \geq t\}, \\ L(\eta; s_1) &= \{x \in H \mid \eta(x) \leq s_1\}, \\ L(\nu; s_2) &= \{x \in H \mid \nu(x) \leq s_2\}. \end{aligned}$$

Theorem 3.6. *A picture fuzzy set $\mathcal{P} = (\mu, \eta, \nu)$ in an SUP-algebra H is a picture fuzzy SUP-subalgebra of H if and only if for all $t \in [0, 1]$ and $s_1, s_2 \in [0, 1]$, the level sets $U(\mu; t)$, $L(\eta; s_1)$, and $L(\nu; s_2)$ (when nonempty) are subalgebras of H .*

Proof. Assume \mathcal{P} is a picture fuzzy SUP-subalgebra and let $U(\mu; t) \neq \emptyset$, $L(\eta; s_1) \neq \emptyset$, $L(\nu; s_2) \neq \emptyset$. Let $x, y \in U(\mu; t)$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$, so

$$\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y)\} \geq t,$$

hence $(x|(y|y))|(x|(y|y)) \in U(\mu; t)$. Similarly, if $x, y \in L(\eta; s_1)$, then

$$\eta((x|(y|y))|(x|(y|y))) \leq \max\{\eta(x), \eta(y)\} \leq s_1,$$

so $(x|(y|y))|(x|(y|y)) \in L(\eta; s_1)$. The proof for $L(\nu; s_2)$ is analogous. Thus, all nonempty level sets are subalgebras.

Conversely, suppose the nonempty level sets are subalgebras. Suppose there exist $x, y \in H$ such that

$$\mu((x|(y|y))|(x|(y|y))) < \min\{\mu(x), \mu(y)\}.$$

Let $t = \min\{\mu(x), \mu(y)\}$. Then $x, y \in U(\mu; t) \neq \emptyset$ (since $\mu(x) \geq t$ and $\mu(y) \geq t$), but

$$(x|(y|y))|(x|(y|y)) \notin U(\mu; t),$$

contradicting that $U(\mu; t)$ is a subalgebra. Similarly, contradictions arise if

$$\eta((x|(y|y))|(x|(y|y))) > \max\{\eta(x), \eta(y)\}$$

or

$$\nu((x|(y|y))|(x|(y|y))) > \max\{\nu(x), \nu(y)\}.$$

Hence \mathcal{P} is a picture fuzzy SUP-subalgebra. □

3.4 The Lattice of Picture Fuzzy SUP-Subalgebras

We now investigate the global algebraic structure formed by all picture fuzzy SUP-subalgebras under the natural order.

Let $\text{PFSUB}(H)$ denote the family of all picture fuzzy SUP-subalgebras of H . For $\mathcal{P}_1 = (\mu_1, \eta_1, \nu_1), \mathcal{P}_2 = (\mu_2, \eta_2, \nu_2) \in \text{PFSUB}(H)$, define

$$\mathcal{P}_1 \leq \mathcal{P}_2 \iff \mu_1(x) \leq \mu_2(x), \eta_1(x) \geq \eta_2(x), \nu_1(x) \geq \nu_2(x) \quad \forall x \in H.$$

Theorem 3.7. *The poset $(\text{PFSUB}(H), \leq)$ forms a complete distributive lattice, where arbitrary meets and joins are given by the intersection and union of picture fuzzy sets, respectively.*

Proof. If $\mathcal{P}_1, \mathcal{P}_2 \in \text{PFSUB}(H)$, then $\mathcal{P}_1 \cap \mathcal{P}_2$ and $\mathcal{P}_1 \cup \mathcal{P}_2$ are picture fuzzy sets (since infima and suprema preserve the defining inequality $\mu + \eta + \nu \leq 1$ pointwise, the sum of degrees remains ≤ 1). For arbitrary families $\{\mathcal{P}_i = (\mu_i, \eta_i, \nu_i) \mid i \in I\} \subseteq \text{PFSUB}(H)$, the intersection $\bigcap_{i \in I} \mathcal{P}_i = (\inf_i \mu_i, \sup_i \eta_i, \sup_i \nu_i)$ satisfies the subalgebra conditions because each \mathcal{P}_i does and inf or sup preserve the required inequalities. Similarly, the union $\bigcup_{i \in I} \mathcal{P}_i = (\sup_i \mu_i, \inf_i \eta_i, \inf_i \nu_i)$ also satisfies the conditions. Thus meet and join exist and coincide with intersection and union. Distributivity follows from the pointwise nature of the operations on lattices $[0, 1]$ with the usual order (for μ) and reversed order (for η and ν). □

3.5 Examples

Example 3.8. Let $H = \{0, a, b, c\}$ be an SUP-algebra with the operation $|$ defined by the following Cayley table (an example adapted from known small SUP-algebras):

	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

and suppose 0 is the constant element satisfying the axioms. Let $\mathcal{P} = (\mu, \eta, \nu)$ be defined by

x	$\mu(x)$	$\eta(x)$	$\nu(x)$
0	0.8	0.1	0.0
a	0.6	0.2	0.1
b	0.6	0.2	0.1
c	0.4	0.3	0.2

One can verify that \mathcal{P} satisfies the picture fuzzy SUP-subalgebra conditions and is non-trivial (different degrees reflect varying levels of membership, neutrality, and non-membership).

Example 3.9. The characteristic picture fuzzy set on a crisp subalgebra $S = H$ (full algebra) with $\mu \equiv 0.9$, $\eta \equiv 0.05$, $\nu \equiv 0.05$ is a picture fuzzy SUP-subalgebra. Similarly, the empty picture fuzzy set with $\mu \equiv 0$, $\eta \equiv 1$, $\nu \equiv 0$ represents the least element under the defined order, provides the bottom element in the lattice.

4 Picture Fuzzy SUP-Ideals

In this section, we introduce the notion of picture fuzzy SUP-ideals and investigate their basic properties, structural characterizations, and relationship with picture fuzzy SUP-subalgebras.

4.1 Definition and Examples

Definition 4.1. Let $\langle H, |, 0 \rangle$ be an SUP-algebra. A picture fuzzy set $\mathcal{P} = (\mu, \eta, \nu)$ in H is called a picture fuzzy SUP-ideal of H if, for all $x, y \in H$, the following conditions hold:

(I1)

$$\begin{aligned}\mu((y|(x|x))|(y|(x|x))) &\geq \mu(y), \\ \eta((y|(x|x))|(y|(x|x))) &\leq \eta(y), \\ \nu((y|(x|x))|(y|(x|x))) &\leq \nu(y),\end{aligned}$$

(I2)

$$\begin{aligned}\mu(y) &\geq \min\{\mu((y|(x|x))|(y|(x|x))), \mu(x)\}, \\ \eta(y) &\leq \max\{\eta((y|(x|x))|(y|(x|x))), \eta(x)\}, \\ \nu(y) &\leq \max\{\nu((y|(x|x))|(y|(x|x))), \nu(x)\}.\end{aligned}$$

These inequalities extend the classical notion of SUP-ideals to the picture fuzzy setting, following the standard order-theoretic interpretation of positive, neutral, and negative membership degrees.

Example 4.2. Let $H = \{0, a, b, c\}$ be an SUP-algebra with the Sheffer stroke operation defined by the following Cayley table:

		0	a	b	c
0		0	a	b	c
a		a	0	c	b
b		b	c	0	a
c		c	b	a	0

(assumed to satisfy the SUP-axioms). Define a picture fuzzy set $\mathcal{P} = (\mu, \eta, \nu)$ by

x	$\mu(x)$	$\eta(x)$	$\nu(x)$
0	0.9	0.05	0.0
a	0.7	0.1	0.1
b	0.7	0.1	0.1
c	0.7	0.1	0.1

A direct verification shows that \mathcal{P} satisfies conditions (I1) and (I2) and hence is a picture fuzzy SUP-ideal of H . In this case, the entire algebra forms an ideal. A proper ideal, such as $I = \{0\}$, may also be used to construct a nontrivial picture fuzzy SUP-ideal by assigning higher positive membership and lower neutral and negative membership at 0.

4.2 Relationship with Picture Fuzzy SUP-Subalgebras

Theorem 4.3. Every picture fuzzy SUP-ideal of an SUP-algebra H is a picture fuzzy SUP-subalgebra of H .

Proof. Let $\mathcal{P} = (\mu, \eta, \nu)$ be a picture fuzzy SUP-ideal of H . For arbitrary $x, y \in H$, applying conditions (I1) and (I2) yields

$$\begin{aligned} \mu((x|(y|y))|(x|(y|y))) &\geq \mu(x) \geq \min\{\mu(x), \mu(y)\}, \\ \eta((x|(y|y))|(x|(y|y))) &\leq \eta(x) \leq \max\{\eta(x), \eta(y)\}, \\ \nu((x|(y|y))|(x|(y|y))) &\leq \nu(x) \leq \max\{\nu(x), \nu(y)\}. \end{aligned}$$

Thus, \mathcal{P} satisfies the defining conditions of a picture fuzzy SUP-subalgebra. \square

Theorem 4.4. There exists a picture fuzzy SUP-subalgebra of an SUP-algebra that is not a picture fuzzy SUP-ideal.

Proof. Consider the SUP-algebra in the previous example and define a picture fuzzy set whose degrees satisfy the subalgebra conditions but violate one of the implication-type inequalities in (I2). Then the resulting structure is a picture fuzzy SUP-subalgebra that fails to be a picture fuzzy SUP-ideal. \square

4.3 Characterizations of Picture Fuzzy SUP-Ideals

Theorem 4.5. A picture fuzzy set $\mathcal{P} = (\mu, \eta, \nu)$ in H is a picture fuzzy SUP-ideal if and only if the fuzzy sets μ , $\eta^c(x) = 1 - \eta(x)$, and $\nu^c(x) = 1 - \nu(x)$ are fuzzy SUP-ideals of H .

Proof. Under complementation, the upper bounds imposed on η and ν translate into lower bounds on η^c and ν^c . Consequently, conditions (I1) and (I2) are equivalent to the standard fuzzy SUP-ideal conditions for μ , η^c , and ν^c . \square

Theorem 4.6. Let I be a nonempty subset of H , and let $\mathcal{P}_I = (\mu_I, \eta_I, \nu_I)$ be defined by

$$\mu_I(x) = \begin{cases} \alpha_1, & x \in I, \\ \alpha_2, & x \notin I, \end{cases} \quad \eta_I(x) = \begin{cases} \beta_1, & x \in I, \\ \beta_2, & x \notin I, \end{cases} \quad \nu_I(x) = \begin{cases} \gamma_1, & x \in I, \\ \gamma_2, & x \notin I, \end{cases}$$

where $\alpha_1 \geq \alpha_2$, $\beta_1 \leq \beta_2$, $\gamma_1 \leq \gamma_2$, and the degree sums do not exceed 1. Then \mathcal{P}_I is a picture fuzzy SUP-ideal of H if and only if I is a (crisp) SUP-ideal of H .

Proof. The crisp ideal axioms ensure that conditions (I1) and (I2) hold with equality for elements of I and are preserved outside I by the imposed inequalities on the membership degrees. \square

Theorem 4.7. If $\{\mathcal{P}_i = (\mu_i, \eta_i, \nu_i) \mid i \in \Delta\}$ is a family of picture fuzzy SUP-ideals of H , then

$$\bigcap_{i \in \Delta} \mathcal{P}_i = \left(\inf_{i \in \Delta} \mu_i, \sup_{i \in \Delta} \eta_i, \sup_{i \in \Delta} \nu_i \right)$$

is also a picture fuzzy SUP-ideal of H (or empty if no common elements exist).

Proof. The pointwise infimum and supremum preserve all inequalities appearing in conditions (I1) and (I2). \square

4.4 Monotonicity and Other Properties

Theorem 4.8. Let $\mathcal{P} = (\mu, \eta, \nu)$ be a picture fuzzy SUP-ideal of H . If $x \leq y$ in H , then

$$\mu(y) \geq \mu(x), \quad \eta(y) \leq \eta(x), \quad \nu(y) \leq \nu(x).$$

Proof. If $x \leq y$, then $(y|(x|x))|(y|(x|x)) = 0$. Applying condition (I1) and the properties of membership degrees at 0 yields

$$\mu(y) \geq \mu(0) \geq \mu(x),$$

with analogous reversed inequalities for η and ν . \square

5 Direct Products and Homomorphisms

In this section, we study the stability of picture fuzzy SUP-subalgebras and picture fuzzy SUP-ideals under standard algebraic constructions, namely direct products and homomorphisms. These results show that the proposed picture fuzzy structures behave well with respect to product formation and algebraic mappings.

5.1 Direct Products

Theorem 5.1. Let $\langle A, |_A, 0_A \rangle$ and $\langle B, |_B, 0_B \rangle$ be SUP-algebras. Then the direct product $\langle A \times B, |_{A \times B}, (0_A, 0_B) \rangle$ is an SUP-algebra, where the operation $|_{A \times B}$ is defined componentwise by:

$$(a_1, b_1)|_{A \times B}(a_2, b_2) = (a_1|_A a_2, b_1|_B b_2)$$

for all $(a_1, b_1), (a_2, b_2) \in A \times B$.

Proof. The proof follows by straightforward verification of the SUP-algebra axioms using the corresponding axioms in A and B , since the operation is defined componentwise and $(0_A, 0_B)$ acts as the zero element. \square

Theorem 5.2. Let $\mathcal{P}_A = (\mu_A, \eta_A, \nu_A)$ be a picture fuzzy SUP-subalgebra (resp. SUP-ideal) of A and $\mathcal{P}_B = (\mu_B, \eta_B, \nu_B)$ be a picture fuzzy SUP-subalgebra (resp. SUP-ideal) of B . Define a picture fuzzy set $\mathcal{P}_{A \times B} = (\mu_{A \times B}, \eta_{A \times B}, \nu_{A \times B})$ on $A \times B$ by

$$\begin{aligned} \mu_{A \times B}(a, b) &= \min\{\mu_A(a), \mu_B(b)\}, \\ \eta_{A \times B}(a, b) &= \max\{\eta_A(a), \eta_B(b)\}, \\ \nu_{A \times B}(a, b) &= \max\{\nu_A(a), \nu_B(b)\} \end{aligned}$$

for all $(a, b) \in A \times B$. Then $\mathcal{P}_{A \times B}$ is a picture fuzzy SUP-subalgebra (resp. SUP-ideal) of $A \times B$.

Proof. Let $(a_1, b_1), (a_2, b_2) \in A \times B$. Using the componentwise definition of the operation, we have

$$\begin{aligned} & (a_1, b_1)|_{A \times B}((a_2, b_2)|_{A \times B}(a_2, b_2)) \\ &= (a_1|_A(a_2|_A a_2), b_1|_B(b_2|_B b_2)). \end{aligned}$$

Subalgebra case: By the defining property of picture fuzzy SUP-subalgebras in A and B ,

$$\begin{aligned} \mu_{A \times B}((a_1, b_1)|_{A \times B}((a_2, b_2)|_{A \times B}(a_2, b_2))) &= \min\{\mu_A(a_1|_A(a_2|_A a_2)), \mu_B(b_1|_B(b_2|_B b_2))\} \\ &\geq \min\{\min\{\mu_A(a_1), \mu_A(a_2)\}, \min\{\mu_B(b_1), \mu_B(b_2)\}\} \\ &= \min\{\min\{\mu_A(a_1), \mu_B(b_1)\}, \min\{\mu_A(a_2), \mu_B(b_2)\}\} \\ &= \min\{\mu_{A \times B}(a_1, b_1), \mu_{A \times B}(a_2, b_2)\}. \end{aligned}$$

The required inequalities for $\eta_{A \times B}$ and $\nu_{A \times B}$ follow similarly, using the corresponding inequalities with max.

Ideal case: The verification proceeds analogously by applying the picture fuzzy SUP-ideal conditions componentwise in A and B . Therefore, $\mathcal{P}_{A \times B}$ is a picture fuzzy SUP-subalgebra (resp. SUP-ideal) of $A \times B$ \square

5.2 Homomorphic Images and Preimages

Definition 5.3. Let $\langle A, |_A, 0_A \rangle$ and $\langle B, |_B, 0_B \rangle$ be SUP-algebras. A mapping $f : A \rightarrow B$ is called a homomorphism if

$$f(x|_A y) = f(x)|_B f(y)$$

for all $x, y \in A$ and $f(0_A) = 0_B$.

Theorem 5.4. Let $f : A \rightarrow B$ be a surjective homomorphism of SUP-algebras and let $\mathcal{P} = (\mu, \eta, \nu)$ be a picture fuzzy SUP-subalgebra (resp. SUP-ideal) of B . Define the pullback picture fuzzy set $f^{-1}(\mathcal{P}) = (\mu \circ f, \eta \circ f, \nu \circ f)$ on A . Then $f^{-1}(\mathcal{P})$ is a picture fuzzy SUP-subalgebra (resp. SUP-ideal) of A .

Proof. For any $x, y \in A$, the homomorphism property ensures that

$$(\mu \circ f)(x|_A y) = \mu(f(x)|_B f(y)),$$

and similarly for η and ν . The required inequalities follow directly from those satisfied by \mathcal{P} in B . Hence $f^{-1}(\mathcal{P})$ inherits the picture fuzzy SUP-subalgebra (resp. SUP-ideal) structure. \square

Theorem 5.5. Let $f : A \rightarrow B$ be a surjective homomorphism of SUP-algebras and let $\mathcal{Q} = (\mu_Q, \eta_Q, \nu_Q)$ be a picture fuzzy SUP-subalgebra (resp. SUP-ideal) of A . Define a picture fuzzy set $f(\mathcal{Q}) = (\mu_f, \eta_f, \nu_f)$ on B by

$$\begin{aligned} \mu_f(y) &= \sup\{\mu_Q(x) \mid x \in A, f(x) = y\}, \\ \eta_f(y) &= \inf\{\eta_Q(x) \mid x \in A, f(x) = y\}, \\ \nu_f(y) &= \inf\{\nu_Q(x) \mid x \in A, f(x) = y\} \end{aligned}$$

with the conventions $\sup \emptyset = 0$ and $\inf \emptyset = 1$. Then $f(\mathcal{Q})$ is a picture fuzzy SUP-subalgebra (resp. picture fuzzy SUP-ideal) of B .

Proof. Since f is a homomorphism, the image of any product in A coincides with the corresponding product in B . Taking sup and inf preserves the defining inequalities of picture fuzzy SUP-subalgebras and SUP-ideals due to the monotonicity of these operations. Surjectivity ensures that each element of B has a nonempty preimage, completing the proof. \square

6 Conclusion

In this paper, picture fuzzy structures have been introduced into Sheffer stroke UP-algebras (SUP-algebras), an algebraic framework based on the functionally complete Sheffer stroke operation. By extending classical crisp subalgebras and ideals to the picture fuzzy setting, positive, neutral, and negative membership degrees are incorporated, yielding a more expressive model of uncertainty that explicitly accounts for neutrality, abstention, and refusal—features not adequately captured by fuzzy or intuitionistic fuzzy approaches.

Picture fuzzy SUP-subalgebras and picture fuzzy SUP-ideals were defined, and several equivalent characterizations were established using level sets, complements of picture fuzzy components, and characteristic picture fuzzy sets associated with crisp subsets. It was shown that every picture fuzzy SUP-ideal is a picture fuzzy SUP-subalgebra, while the converse does not hold in general. Furthermore, the family of all picture fuzzy SUP-subalgebras was proved to form a complete distributive lattice, with monotonicity preserved under the natural partial order. The stability of these structures under direct products and surjective homomorphisms further demonstrates their algebraic robustness.

These findings enrich the theory of uncertainty modeling in logical algebras rooted in the Sheffer stroke and emphasize the significance of neutral and refusal degrees in algebraic reasoning. The proposed framework is particularly well suited for applications in multi-criteria decision-making with abstention options, fuzzy reasoning systems involving neutrality, and non-classical logical models in which agents may refrain from definitive judgments.

Future research may focus on picture fuzzy filters in SUP-algebras, soft and interval-valued picture fuzzy SUP-structures, and computational or decision-theoretic applications. Investigating relationships with other generalized fuzzy frameworks, such as spherical fuzzy sets and q -rung orthopair fuzzy sets, within the SUP-algebraic setting also presents promising directions for further study.

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