Global Well-posedness result for nonlinear unilateral parabolic equations in Musielak spaces

Abstract:

This study investigates the existence and uniqueness of solutions for a class of nonlinear parabolic equations characterized by integrable data within time-dependent Musielak-Orlicz spaces. By employing density arguments and advanced variational techniques, we demonstrate the existence of entropy solutions. The framework developed herein is robust enough to encompass various mathematical models, including those involving double-phase growth, variable exponents, and Orlicz-type growth conditions

Keywords: Uniqueness or solutions, Musielak-Orlicz spaces, Entropy Solutions, Double-phase growth, Variable Exponents

1 Introduction:

The study of Nonlinear Partial Differential Equations (PDEs) holds significant importance in modeling complex natural and physical phenomena [20]. Among these equations, Parabolic Equations receive particular attention, especially those featuring nonlinear coefficients or variable growth. In recent years, there has been increasing interest in equations involving more general functional spaces, such as Musielak-Orlicz spaces, which represent a generalization of Orlicz spaces, variable exponents, and double-phase problems [4] [15] [18]. The pioneering work for studying PDE problems in these spaces was established in earlier works [5] [21]. The importance of these spaces lies in their ability to describe the behavior of materials with non-uniform or variable properties.

When dealing with problems with low data integrability, it becomes necessary to work with Entropy Solutions instead of traditional weak solutions to ensure the existence and stability of the solution [23]. Issues of the existence of weak solutions in anisotropic and reflexive Musielak-Orlicz spaces have been addressed in previous references [12]. Further research has also been conducted on entropy solutions for parabolic equations exhibiting Orlicz growth [25] [26], and entropy solutions for the $p(x)$-Laplace equation [24].

This research focuses on the problem of Global Well-posedness for nonlinear unilateral parabolic equations in Musielak spaces with data that is only time-integrable. The following equivalent problem is considered:

$$\begin{cases} \partial_t u - \text{ div } A(t,x,\nabla u) + g(t,x,\nabla u) = f(t,x) & in\ \Omega_T := (0,T] \times \Omega Q \\ \qquad u(x,0) = u_0(x) & in\ \Omega \\ \qquad u(t,x) = 0 & on \sum = \partial\Omega \times (0,T) \end{cases} \tag{1.1}$$

where $f \in L^1(\Omega_T)$ and $u_0 \in L^1(\Omega)$. We assume the following hypotheses:
let $A(t,x,\xi) : (0,T)\times\Omega\times\ I\!R^N \to\ I\!R^N$ is a Caratheodory function (i.e., measurable in t) and x for fixed ξ and continuous with respect to ξ for fixed (t,x).

$$A(t,x,0) = 0\ for\ almost\ every\ (t,x) \in \Omega_T \tag{1.2}$$

there exist an N-function M (see Definition 2.1) and constants $c_1 \in (0,1)$, such that for all $\xi \in\ I\!R$ we have

$$c_1(M(t,x,|\xi|) + M^*(t,x|A(t,x,\xi)|)) \leq A(t,x,\xi)\xi \tag{1.3}$$

where M^* is the complementary function of M (see Definition 2.2).

$$[A(t,x,\xi) - A(t,x,\eta)].(\xi - \eta) > 0 \; for \;\; all \; \xi, \eta \in \; I\!R^N, \xi \neq \eta \; and \; a.e.(t, x \in \Omega_T) \tag{1.4}$$

let $g(t,x,\xi) : (0,T) \times \Omega \times I\!R^N \to I\!R^N$ is a Caratheodory function (i.e., measurable in t) and x for fixed ξ and continuous with respect to ξ for fixed (t,x)

$$|g(t,x,\xi)| \leq L(t,x) + M(t,x,|\xi|) \tag{1.5}$$

where $L_1(.) \in L^1(\Omega)$

where Ω is a bounded domain with a Lipschitz boundary, and T is a positive number. The primary goal is to prove the global well-posedness result for nonlinear unilateral parabolic equations in Musielak spaces with simple data integrability in time-dependent spaces.

Contribution of the Research:

The main contribution of this research lies in the use of an adensity argument to establish the existence of entropy solutions, covering a variety of problems, including those involving Orlicz growth, variable exponents, and double-phase growth. This is achieved by imposing a condition on the behavior of the function M concerning its integrability, which ensures that smooth functions are norm-dense in the associated Sobolev space [16].

Structure of the Research:

The research is organized as follows: Section 2 presents the definitions and preliminary results related to Musielak-Orlicz spaces. Section 3 provides the proof of the existence of entropy solutions for problem (1.1) using the density argument.

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Motivated by fluids of nonstandard rheology, we focus on the general form of growth conditions for the leading term of the operator , which makes Musielak-Orlicz spaces a suit-able function space for the considered problem . We do not assume any growth condition of doubliog type on the function M.
Instead, we impose a condition that balances the behavior of M with respect to its variable, ensuring that smooth functions are modularly dense in the related Sobolev-type space.
Musielak-Orlicz spaces, which generalize Orlicz, variable exponent and double-phase spaces , There is a large amount of literature on PDE problems in the framework of Musielak-Orlicz spaces [4, 15, 18]. We refer to [12] for the existence of weak solutions in isotropic, separable and reflexive Musielak-Orlicz-Sobolev spaces. The groundwork for studying PDE problems in anisotropic Musielak-Orlicz spaces wes established in the works of [5, 21]. For more recent reslts concering PDEs in Musielak-Orlicz spaces, the readers may refer to monograph[6] and the review paper [7].
As we consider problems with data of low integrability, it is reasonable to work with entropy solutions ,as they require lwss regularity in the data than standard weak solutions , At the same time, Benilan et al. We refer to [23] for this issue in the nonreflexive Orlicz- Dobolev space. Also, Entropy solutions for the $p(x)$-Laplace equations were investigated in [24], with further research on entropy solutions exhibiting Orlicz growth available in [25, 26]. For studies on the existence of entropy solutions in Musielak-Orlicz-Sobolev spaces, see [16] . and for $(p(x), q(x))$ growth in parablic equations in [1].

$$\begin{cases} \partial_t u - \text{ div } A(t,x,\nabla u) + g(t,x,\nabla u) = f(t,x) & in\ \Omega_T \\ u(x,0) = u_0(x) & in\ \Omega \\ u(t,x) = 0 & on \sum \end{cases}$$

where $f \in L^1(\Omega_T)$, $u_0 \in L^1(\Omega)$, and A was assumed to be controlled by a N-function. The authors therein employed a delicate time approximation method to achieve smoothness in the time direction, we do not require separate approximations for time and space variables. We assume that the regularity of the modular function is strong enough to ensure the density of smooth functions in the related Sobolev-type space . In fact , this denity can be guaranteed by the balance condition in the isotropic Musielak-Orlicz space , see Lemma 2.2. Utilizing this density result ,the existence and uniqueness of entropy solutions for equation (1,1). We rely on the density of smooth functions in a relevant function space to study problem (1,1), namely

$$W(\Omega_T) := \left\{u \in W_0^{1,x} L_M(\Omega_T) \cap L^2(\Omega_T), \partial_t u \in W^{-1,x} L_{M^*}(\Omega_T) + L^2(\Omega_T)\right\}$$

where $W_0^{1,x} L_M(\Omega_T)$ and $W^{-1,x} L_{M^*}(\Omega_T)$ are defined in Section 2.
To ensure the density , one may assume the regularity of M. Note that the smooth functions are dense in $W(\Omega_T)$ in the modular topology if the following balance condition holds , see Lemma 2.2.

2 MAIN TOOLS

Defintion 2.1 If there exists a function $\varrho : [0, \infty) \times I\!R^+ \to I\!R^+$ which is non-decreasing with respect to each of the variables such that for $(t, x) \in \Omega_T$ and $(\tau, y) \in \Omega_T$,

$$M(t, x, s) \leq \varrho(|t - \tau| + c|x - y|, s) M(\tau, y, s) \quad with \quad \lim_{\varepsilon \to 0^+} sup\ \varrho(\varepsilon, \varepsilon^- N) < \infty$$

We point out that the Balanced condition is only used to ensure the density of smooth functions in our proof. In addition, throughout the papre , we assume that the N-functions $M(t, x)$satisfies the following Y-condition.
(Y)-CONDITION. A N-function M is said to satisfy the Y-condition on a segment[a,b] of the real line $I\!R$, if either
(Y1) : there exist $q_0 \in I\!R^+$ and $1 \leq i \leq N$such that $x_i \in [a, b] \to M(t, x)$ is

increasing when $x_i \geq q_0$ and decreasing when $x_i \leq q_0$. such that, or

(Y2) : there exist $1 \leq i \leq N$such that for all$s \geq 0$ the partial function $x_i \in [a, b] \to M(t, x)$ is monotone on [a,b]. Here x_i stands for the $i^t h$ component of $x \in \Omega$ In Musielak-Orlicz apaces , the norm Poincare inequality is no longer true in general . We remark that the (Y)-condition is only used as a sufficient condition to obtain the norm Poincare inequality (see Lemma 2.1 below), ehich is crucial in the proof of Lemma 2.4. This condition covers the assumption given by Maeda [4]. to provide the Poincare integral form for variable exponents . See[4],the authors used different approximation methods for spatial and time directions making it unnecessary to assume(Y) -condition to obtain information abut the L_M norm of the solution itself . In our situation, we do not separate space and time approximations . We rely on the density argument of Lemma 2.2, ehich shws that for any test function ϕ, it is necessary that $\phi \in L_M(\Omega)$ and $\nabla\phi \in L_M(\Omega_T)$. However , when making a priori estimates for the solutions to the approximation problem , we can only obtain gradient infomation about the approximate solutions due to the growth condition (1.2) . Thus , we need to impose that N-function M astisfies (Y)-condition. Nevertheless, we provide a more straightforward approach that reveals the intinsic connection netween entropy solutions for such equations . In our future work, we will explore more general conditions that our method functions are dense in the anisotropic Musilak-Orlicz spaces.We believe that our method, with slight modifications , will be applicable to anisotropic Musilak-Orlicz spaces. Before we proceed to define the entropy solution to(1,1) . We first introduce the truncation operator $T_{k(r)}$ as follows:

$$T_k(r) =: \begin{cases} r & \text{if} \quad |r| \leq k \\ k\frac{r}{|r|} & \text{if} \quad |r| > k \end{cases}$$

Its primitive $\beta_k : I\!R \to I\!R^+$ defined by

$$\beta_k(r) = \int_0^r T_k(r)dr = \begin{cases} \frac{r^2}{2} & \text{if} \quad |r| \leq k \\ k|r| - \frac{k^2}{2} & \text{if} \quad |r| > k \end{cases} \tag{2.1}$$

It is obvious that $\beta_k(r) \geq 0$ and $\beta_k(r) \leq k|r|$. The definitions of entropy solutions for problem(1,1)are asfollows.

Defintion 2.2 A function $u \in c([0,T];L^1(\Omega))$ is an entropy solution to problem (1,1) if u satisfies the following two conditions:

(1) u is a measuroble function satisfying $T_k(u) \in W_0^{1,x}L_M(\Omega_T)$ for each $k > 0$ and $A(t,x,\nabla T_K(u)) \in L_{M^*}(\Omega_T)$;

(2) For every $k > 0$ and every $\phi \in C^1(\bar{\Omega_T})$ with $\phi = 0$, the inequality

$$\int_\Omega \beta_k(u-\phi)(T)dx - \int_\Omega \beta_k(u_0 - \phi(0))dx + \int_0^T \langle \phi_t, T_k(u-\phi)\rangle dt$$
$$+ \int_0^T \int_\Omega A(t,x,\nabla u).\nabla T_K(u-\phi)dx\ dt + \int_0^T \int_\Omega g(t,x,\nabla u).T_k(u-\phi)dx\ dt \quad (2.2)$$
$$= \int_0^T \int_\Omega f T_K(u-\phi)dx\ dt$$

holds.

Definition 2.3.A function $M(.,s) : \Omega_T \times I\!R^+ \to I\!R^+$ is called an N-function if $M(.,s)$ is ameasurable function for every $s \geq 0, M(t,x,.)$ is strictly increasing with respect to last variable , and $M(t,x,.)$ is a converx function for almost every $(t,x) \in \Omega_T$ with $M(t,x,0) = 0$ $M(t,x,s) \to +\infty$ as $s \to +\infty$ and

$$\lim_{s\to 0} \frac{M(t,x,s)}{s} = 0 \quad , \quad \lim_{s\to +\infty} \frac{M(t,x,s)}{s} = +\infty$$

Definition 2.4 The complementary function M to an N-function M in the sense of Young defined by

$$M^*(t,x,\xi_1) := \sup_{\xi_2 \geq 0} [\xi_1 \xi_2 - M(t,x,\xi_2)] \quad (2.3)$$

for any $\xi_1 \geq 0$ and a.e.$(t,x) \in \Omega_T$ For an N-function, we define the general Musielak-Orlicz class $L_m(\Omega)$ as the set of all measurable functions $u(t,x) : \Omega_T \to I\!R$ such that

$$\int_0^T \int_\Omega M(t,x,|u(t,x)|)dx\ dt < \infty$$

The Musielak-Orlicz space $L_M(\Omega_T)$(resp. $E_M(\Omega_T)$) is defined as the set of all measurabl function $u : \Omega_T \to I\!R$ such that

$$\int_0^T \int_\Omega M\left(t, x, \frac{|u(t,x)|}{\lambda}\right) dx\ dt < +\infty$$

for some $\lambda > 0$(resp. for all$\lambda > 0$). Equipped with Luxemburg norm

$$||u||_{L_m(\Omega_T)} = inf\left\{\lambda > 0 : \int_0^T \int_\Omega M\left(t, x, \frac{|u(t,x)|}{\lambda}\right) dx\ dt \leq 1\right\}$$

Then $L_M(\Omega_T)$ is a Banach space and $E_M(\Omega_T)$ is its closed subset.

An N-function M is called locally integrable on Ω_T , if for any condtant number $c > 0$ and for any compact Ω'_T of Ω_T , the following holds

$$\int_{\Omega'_T} M(t, x, c) dx\ dt < +\infty$$

We remark here that if an n-function M satisfies the balanced condition (B), thaen the function is naturally locally integrable , see [3,5] . It is shown in [6,Lemma2.1] that the continuous embedding $L_M(\Omega_T) \to L^1(\Omega_T)$ holds if either M^* is locally integrable, or M satisfies $\operatorname{essinf}_{(t,x)\in\Omega_T} M(t, x, 1) \geq c > 0$

Definition 2.5 Suppose that the complementary N-function M^* of M is locally integrable on Ω_T, we define

$$W^{1,x}L_M(\Omega_T) = \{u : \Omega_T \to I\!R : u \in L_M(\Omega_T), |\nabla u| \in\ L_M(\Omega_T)\} \tag{2.4}$$

and

$$W^{1,x}E_M(\Omega_T) = \{u : \Omega_T \to I\!R : u \in E_M(\Omega_T), |\nabla u| \in\ E_M(\Omega_T)\} \tag{2.5}$$

We denote ∇u the vector gradient with respect to the space variable . These spaces are normed by $||u||_{w^{1,x}L_M(\Omega_T)} := ||u||_{L_M(\Omega_T)} + ||\nabla u||_{L_M(\Omega_T)}$and then $W^{1,x}L_M(\Omega_T)$ is a Banach space .

Let $X$and Y be subsets of $L^1(\Omega_T)$ not neceessarily related by duality . We say $f_n \to f$ for $\sigma(X, Y)$ if

$$\int_0^T \int_\Omega f_n g dx\ dt \longrightarrow \int_0^T \int_\Omega f g dx\ dt \qquad when\ \ n \to +\infty$$

for all $g \in Y$. If $X = L_M(\Omega_T)$ and $Y = E_{M^*}(\Omega_T)$ we recovre the weak-* convergence and can also denote $f_n \to^* f$.
We define $W^1L_M(\Omega)$ resp. $W^1L_M(\Omega)$ as the set of all measurable function $u : \Omega \to \mathbb{R}$ such that for all $|\alpha| \leq 1$, the function $|D^\alpha u|$ belong to$L_M(\Omega)$ (resp. $E_M(\Omega)$) , than is

$$\int_\Omega M\left(x, \frac{|D^\aleph u|}{\lambda}\right) dx < +\infty \ for \ \ some \ \lambda > 0 \ (resp. \lambda > 0)$$

Note that if a N-function M is locally integrabl, than the set of $C_0^\infty(\Omega)$-functions is contained in $W^1E_M(\Omega)$. Therefore, the norm closure of $C_0^\infty(\Omega)$ -functions is in $W^1E_M(\Omega)$, denoted by in $W_0^1E_M(\Omega)$ is well defined . Moreover, if the pair of complementary N-functions (M, M^*) are both locally integrable, then the space $W_0^1E_M(\Omega)$. defined as the closur of $C_0^\infty(\Omega)$-functions with respect to the weak-* topology $\sigma(L_M, E_M^*)$, is slso well defined.

Definition 2.6.Suppose that thecomplementary N-function M^* of M is locally integrable on Ω_T . we define

$$W_0^{1,x}L_M(\Omega_T) = \{u : (0,T) \to W_0^1L_M(\Omega_T) : u \in L_M(\Omega_T), |\nabla u| \in \ L_M(\Omega_T)\} \quad (2.6)$$

and

$$W_0^{1,x}E_M(\Omega_T) = \{u : (0,T) \to W_0^1L_M(\Omega_T) : u \in E_M(\Omega_T), |\nabla u| \in \ E_M(\Omega_T)\} \quad (2.7)$$

These spaces are equipped with the norm$||u||_{W^{1,x}L_M(\Omega_T)}$.

Lemma 2.1 (Theorem 1.1,[3]). Assume that the pair of complementary N-functions M and M^* satisfy both Definition(1.1), and M satisfies (Y)-condition. Then there exists a constant C depaending only on Ω_T such that for every $u \in W_0^{1,x}L_M(\Omega_T)$ it holds

$$||u||_{L_M(\Omega_T)} \leq C||\nabla u||_{L_M(\Omega_T)}.$$

From Lemma 2.5 the two norms $||.||_{L_M(\Omega_T)} + ||\nabla.||_{L_M(\Omega_T)}$ and $||\nabla.||_{L_M(\Omega_T)}$ are equivalent on $W_0^{1,x}E_M(\Omega_T)$. In addition , it follows from Defintion 2.2 thet

$$\int_0^T \int_\Omega |u_1 u_2| dx\ dt \leq 2||u_1||_{L_M(\Omega_T)}||u_2||_{L^*_M(\Omega_T)} \tag{2.8}$$

for all $u_1 \in L_M(\Omega_T)$ and $u_2 \in L^*_M(\Omega_T)$ We say that a sequence $\xi_{n n=1}^{\infty}$ converges modularly to ξ in $L_M(\Omega_T)$, if there $\lambda > 0$ such that

$$\int_0^T \int_\Omega M\left(t, x, \frac{|\xi_n - \xi|}{\lambda}\right) dx\ dt \to 0\ as\ n \to +\infty$$

For the notion of this convergence, we write$\xi_n \to^M \xi$

Lemma 2.2 (Theorem 5.2,[16]).Assume that A satisfies the conditions (2.2) − (2.4), and the pair of complementary N-function M and M^* satisfy both balance condition . Then, for every $\phi \in W(\Omega_T)$ there exists a saquence $\phi_\delta \delta \subset C_0^\infty((0,T]; C_0^\infty))$ such that

$$\phi_\delta \to \phi \quad in \quad L^2(\Omega_T),$$

$$\partial_t \phi_\delta \to^M \partial_t \phi \quad in \quad W^{-1,x} L_{M^*}(\Omega_T) + L^2(\Omega_T),$$

$$D^\alpha \phi_\delta \to^M D^\alpha \phi, |\alpha| \leq 1 \quad in \quad L_{M^*}(\Omega_T).$$

where $W^{-1,x} L_{M^*}(\Omega_T)$ is defined as

$$W^{-1,x} L_{M^*}(\Omega_T) := (0,T) \to W^{-1} L_{M^*}(\Omega_T) : u = \tilde{u} - divU, with\tilde{u} \quad in \quad L_{M^*}(\Omega_T)\ and \quad u_n \in L_{M^*}(\Omega_T)$$

The following fact is a consequence of modular topolgy.

Lemma 2.3 (Lemma 2,[2]). Let M be an N-function and $u_n, u \in L_{M^*}(\Omega_T)$. If $u_n \to^M u$ modularly , then $u_n \to u$ in $\sigma(L_M, L^*_M)$

Using aproof strategy analogous to that in [17] , we can establish the following lemma.

Lemma 2.4 Suppose that $f \in C_0^\infty(\Omega)$, and A , g satisfies the conditions (1.2)-(1.5), N-function M is regular enough so that the set of smooth functions is dense in $W(\Omega_T)$ in the modular topology , and M satisfies (Y)-condition. Then there exists at least one dis-tributional solution $u \in W(\Omega_T)$ of problem (1.1) satisfying $u(x,0) = u_0(x)$ for almost every $x \in \Omega$. Furthermore, for all $\tau \in (0,T]$, we have

$$-\int_0^\tau \int_\Omega \partial_t \varphi u dx\ dt + \int_\Omega u\varphi dx|_0^\tau + \int_0^\tau \int_\Omega A(t,x,\nabla u).\nabla\varphi dx\ dt$$
$$+\int_0^\tau \int_\Omega g(t,x,\nabla u).\nabla\varphi dx\ dt = \int_0^\tau \int_\Omega f\varphi dx\ dt$$

for every $\varphi \in W(\Omega_T)$ Then , we will present some preliminary lemmas that will be used later.

Lemma 2.5 ((Lemma 2.3,[19]). Let Ω_T be a measurable with finite Lebesgue measure, and let f_n be a sequence of functions in $L^p(\Omega_T)(p \geq 1)$ such that

$$f_n \to f \quad weakly \quad in \quad L^p(\Omega_T)$$

,

$$f_n \to g \quad a.e. \quad in \quad (\Omega_T)$$

Then $f = g$ a.e. in Ω_T

Lemma 2.6 (Theorem 13.47,[20]) Let $f_n, f \in L^1(\Omega_T)$ such that $f_n \geq 0$ a.e. in $,\Omega_T$, $f_n \to f$ a.e. in $,\Omega_T$ and

$$\int_0^T \int_\Omega f_n dx\ dt \to \int_0^T \int_\Omega f dx\ dt\ as\ \ n \to +\infty$$

Then $f_n \to f$ strongly in $L^1(\Omega_T)$.

Theorem 1.1. Assume that $f \in L^1(\Omega_T)$ $,u_0 \in L^1(\Omega)$ and A , g satisfies the conditions $(1,2) \to (1,5)$, N-function M is regular enough so that the the set of smooth functions is dense in $W(\Omega_T)$ in the modular topology, and M satisfies (Y)-condition. Then there exist an entropy solutions for problem (1.1).

We organize this paper in the following framework. In Section 2, we state some basic results that will be used latar . We will prove the main results in Section 3, In the following statement, C stands for a constant, which may vary even within the same inequality.

3 THE PROOFS THE THEOREM (1.1)

Step 1. We first consider the following approximate problems

$$\begin{cases} \partial_t u_n - \text{ div } A(t,x,\nabla u_n) + g(t,x,\nabla u_n) = f(t,x) \quad \in \quad \Omega_T, \\ u_n = 0 \qquad on \ \ \Sigma \\ u_n(0,x) = u_{0n} \qquad in \quad \Omega \end{cases} \tag{3.1}$$

. where the two sequences of functions $f_n \subset C_0^\infty(\Omega_T)$ and $u_{0n} \subset C_0^\infty(\Omega)$ strongly con-vergent respectively to f in $L^1(\Omega_T)$ and to u_0 in $L^1(\Omega)$ such that

$$\begin{aligned} &||f_n||_{L^1(\Omega_T)} \leq ||f||_{L^1(\Omega_T)}, \\ &||u_{0n}||_{L^1(\Omega)} \leq ||u_0||_{L^1(\Omega)}. \\ &g_n(t,x,\xi) = T_n g(t,x,\xi) \end{aligned} \tag{3.2}$$

It follows from Lemma 2.4 that there exists a distributional solution $u_n \in W(\Omega_T)$ for problem (3.1), such that

$$\begin{aligned} \int_0^\tau \int_\Omega \partial_t u_n \varphi dx \ dt + \int_0^\tau \int_\Omega A(t,x,\nabla u_n).\nabla\varphi dx \ dt + \int_0^\tau \int_\Omega g(t,x,\nabla u).\nabla\varphi dx \ dt \\ = \int_0^\tau \int_\Omega f_n \varphi dx \ dt. \end{aligned} \tag{3.3}$$

for every $\varphi \in W(\Omega_T)$

Taking the test function as $T_k(u_n)\chi(0,\tau)$ with $\tau \in (0,T]$ in (3.3). we have

$$\begin{aligned} \int_\Omega \beta_k(u_n)(\tau)dx - \int_\Omega \beta_k(u_{0n})dx + \int_0^\tau \int_\Omega A(t,x,\nabla T_k(u_n).\nabla T_k(u_n)dx \ dt \\ + \int_0^\tau \int_\Omega g_n(t,x,\nabla T_k(u_n).\nabla T_k(u_n)dx \ dt = \int_0^\tau \int_\Omega f_n T_k(u_n)dx \ dt \end{aligned} \tag{3.4}$$

According to the definition of $\beta_k(r)$,(1.5) and (3.2) we deduce

$$\int_0^\tau \int_\Omega A(t,x,\nabla T_k(u_n).\nabla T_k(u_n)dx\,dt+\int_0^\tau \int_\Omega g(t,x,\nabla T_k(u_n).\nabla T_k(u_n)dx\,dt+\int_\Omega \beta_k(u_n)(\tau)dx$$
$$\leq (||f_n||_{L^1(\Omega_T)}+||u_{0n}||_{L^1(\Omega)})+n \leq K(||f||_{L^1(\Omega_T)}+||u_0||_{L^1(\Omega)})+c_1 \leq c_2 k \quad (3.5)$$

Recalling condition (2.3) ,we have

$$c_1 \int_0^\tau \int_\Omega M(t,x,|\nabla T_k(u_n)|)dx\,dt \leq$$
$$\leq \int_0^\tau \int_\Omega A(t,x,\nabla T_k(u_n).\nabla T_k(u_n)dx\,dt \leq C_2 k \quad (3.6)$$

and

$$c_1 \int_0^\tau \int_\Omega M^*(t,x,|A(t,x,\nabla T_k(u_n)|)dx\,dt \leq C_2 k \quad (3.7)$$

Since $L_M(\Omega_T) \to L^1(\Omega_T)$, we know

$$\int_0^\tau \int_\Omega |\nabla T_k(u_n)|dx\,dt \leq C(k+1) \quad (3.8)$$

that is $T_k(u_n)$ is bounded in $L^1(0,T;W_0^{1,1}(\Omega))$ Choosing $k=1$ in the inequality (3.5) , we find that

$$\int_\Omega \beta_1(u_n(\tau))dx \leq ||f||_{L^1(\Omega_T)}+||u_0||_{L^1(\Omega)}$$

for a.e. $\tau \in (0,T]$. Moreover

$$\int_\Omega |u_n(\tau)|dx \leq meas(\Omega)+||f||_{L^1(\Omega_T)}+||u_0||_{L^1(\Omega)}$$

Therefore, we obtain

$$||u_n||_{L^\infty(0,T;L^1(\Omega))} \leq C \quad (3.9)$$

Step 2. Prove the convergence of $\{u_n\}$ in $C([0,T];L^1(\Omega))$ ans finds its subsequence which is almost everywhere covergec in Ω_T.

From (3.3),we can write the weak form as

$$\int_0^T \langle \partial_t(u_n - u_m), \phi\rangle dt + \int_0^T \int_\Omega [A(t,x,\nabla u_n) - A(t,x,\nabla u_n)].\nabla \phi dx\ dt$$
$$= \int_0^T \int_\Omega (f_n - f_m)\phi dx\ dt \qquad (3.10)$$

for all $m, n \in Z$ and $\phi \in W(\Omega_T)$. It follows from the Fenchel's Young inequality and (3.5) that

$$|A(t,x,\nabla u_n).\nabla u_m| \leq (M(t,x,|\nabla u_n|) + M^*(t,x,|A(t,x)\nabla u_m)|) \in L^1(\Omega_T).$$

Define

$$\alpha_{n,m} := \int_0^T \int_\Omega |f_n - f_m| dx\ dt + \int_\Omega |u_{0n} - u_{0m}| dx \qquad (3.11)$$

Since f_n and u_{0n} are convergent in L^1 , we have

$$\lim_{n,m\to+\infty} \quad \alpha_{n,m} = 0$$

Taking $\omega = T_1(u_n - u_m)\chi_{(0,\tau)}$ with $\tau \leq T$ as a test function (3.10), and discarding the positive term we obtain

$$\int_\Omega \varphi_1(u_n - u_m)(\tau)dx \leq \int_\Omega \varphi_1(u_{0n} - u_{0m})dx + ||f_n - u_m||^1_L(\Omega_T)$$
$$\leq ||u_{0n} - u_{0m}||_{L^1(\Omega_T)} + ||f_n - u_m||_{L^1(\Omega_T)} = \alpha_{n,m}.$$

Therefore, we conclude that

$$\int_{\{|u_n - u_m|<1\}} \frac{|u_n - u_m|^2(\tau)}{2} dx + \int_{\{|u_n - u_m|\geq 1\}} \frac{|u_n - u_m|(\tau)}{2} dx$$
$$\leq \int_\Omega [\varphi_1(u_n - u_m)](\tau)dx \leq \alpha_{n,m}.$$

Moreover,

$$\int_\Omega |u_n - u_m|(\tau)dx = \int_{\{|u_n - u_m|<1\}} |u_n - u_m|(\tau)dx + \int_{\{|u_n - u_m|\geq 1\}} |u_n - u_m|(\tau)dx$$

$$\leq \left(\int_{\{|u_n-u_m|<1\}} |u_n - u_m|^2(\tau)dx\right)^{\frac{1}{2}} \ meas\ (\Omega)^{\frac{1}{2}} + 2\delta_{n,m}$$

$$\leq\ (2meas\ (\Omega))^{\frac{1}{2}}\ \delta_{n,m}^{\frac{1}{2}} + 2\delta_{n,m}$$

Thus, we deduce that

$$||u_n - u_m||_{C([0,T];L^1(\Omega))} \to 0 \quad as \ \ n,\ m \to +\infty,$$

which implies that u_n is a Cauchy sequence in $C([0,T];L^1(\Omega))$. Then u_n converges to $u$in $C([0,T];L^1(\Omega))$. We find an a.e. convergent subsequence (still denoted by u_n) in Ω_T such that

$$u_n \longrightarrow u \ \ a.e. \ \ in \ \ \Omega_T \ \ as\ n \to +\infty \tag{3.12}$$

Step 3. Show that the sequence ∇u_n converges almost everywhere in Ω_T to ∇u (up to asubsequence) . Let us first set $\delta > 0$ and denote

$$H_1 := \{\ (t,x) \in \Omega_T : |\nabla u_n| > h \cup (t, x \in \Omega_T) : |\nabla u_n| > h\},$$

$$H_2 := \{\ (t,x) \in \Omega_T : |u_n - u_m| > 1\}$$

and

$$H_3 := \{(t,x) \in \Omega_T : |\nabla u_n| \leq h, |\nabla u_m| \leq h, |u_n - u_m| \leq 1, |\nabla u_n - \nabla u_m| > \delta\}$$

where h will be chosen later . Next , we shall show that ∇u_n is Cauchy sequence in measure. It is easy to check that

$$\{(t,x) \in \Omega_T : |\nabla u_n - \nabla u_m| > \delta\} \subset H_1 \cup H_2 \cup H_3$$

Firstly, we notice that

$$\{(t,x) \in \Omega_T : |\nabla u_n| \geq h\} \subset \{(t,x) \in \Omega_T : |u_n| \geq k\} \cup \{(t,x) \in \Omega_T : |\nabla T_k(u_n)| \geq h\}$$

for all $k > 0$. Thus, using (3.9) and (3.8), we know there exist constants $C > 0$ such that

$$meas\{(t,x) \in \Omega_T : |\nabla u_n| \geq h\} \leq \frac{C}{k} + \frac{C(k+1)}{h}$$

when h is large approriately. By choosing $k = Ch^{\frac{1}{2}}$, we deduce that

$$meas\{(t,x) \in \Omega_T : |\nabla u_n| \geq h\} \leq Ch^{-\frac{1}{2}}$$

Let $\epsilon > 0$. We may Let $h = h(\epsilon)$ large enough such that

$$meas(H_1) \geq \frac{\epsilon}{3} \quad for \quad all \quad n,m > 0 \tag{3.13}$$

Secondly, by Step 1 we know that $\{u_n\}$ is a Cauchy sequence in measure. Then there exists $N_1(\epsilon) \in \mathbb{N}$ such that

$$meas(H_2) \geq \frac{\epsilon}{3} \quad for \quad all \quad n,m > N_1(\epsilon) \tag{3.14}$$

Finally , from condition (H_3), we know there there exists a real - valued $m(h,\delta) > 0$ such that

$$[A(t,x,\nabla\eta) - A(t,x,\nabla\zeta)].(\nabla\eta - \nabla\zeta) \leq m(h,\delta) > 0$$

for all $\eta, \zeta \in \mathbb{R}^N$ with $|\eta|, |\zeta| \leq h, \delta \leq |\eta - \zeta|$. By taking $T_1(u_n - u_m)$ as a test function in (3.10) and integrate on H_3 , we obtain

$$\begin{aligned} m(h,\delta)meas(H_3) &\leq \int_{H_3} [A(t,x,\nabla u_n) - A(t,x\nabla u_m)].(\nabla u_n - \nabla u_m)dx\ dt \\ &\leq \int_0^T \int_\Omega [A(t,x,\nabla u_n) - A(t,x\nabla u_m)].(\nabla u_n - \nabla u_m)dx\ dt \\ &\leq \int_0^T \int_\Omega |f_n - f_m|dx\ dt + \int_\Omega |u_{0n} - u_{om}|dx = \alpha(n,m) \end{aligned}$$

, which implies that

$$meas(H_3) \leq \frac{\alpha(n,m)}{m(h,\delta)} \leq \frac{\epsilon}{3} \quad for\ all \ \ n,m \geq N_2(\epsilon,\delta) \tag{3.15}$$

combining the estimates (3.13) - (3.15), we obtin

$$measd\{(t,x) \in \Omega_T : |\nabla u_n - u_m| > \delta\} \leq \epsilon \quad for\ all \ \ n,m \geq max\{N_1, N_2\}$$

that is $\{\nabla u_n\}$ is a Cauchy sequence in measure . Therefore , we obtain a subsequence of $\{\nabla u_n\}$ ehich is almost everywere conergent in Ω_T . Moreover , a priori estimate (3.6) and weak lower semi-continuity of a convex functional give that

$$T_k(u_n) \rightharpoonup T_k(u) \quad weakly \ \ in \ \ L^1(0,T;W_0^{1,1}(\Omega)) \tag{3.16}$$

Therefor , we deduce from Lemma 2.5 that

$$\nabla u_n \to \nabla u \quad a.e. \quad in \quad \Omega_T \quad as \ \ n \to +\infty \tag{3.17}$$

Step 4. Prove a decay condition for u_n . In this step, we aim to prove that

$$\lim_{\iota\to+\infty} \lim_{n\to+\infty} \int_{\{\iota\leq|u_n|\leq L+1\}} A(t,x,\nabla u_n).\nabla u_n dx\ dt = 0 \tag{3.18}$$

Define the function $T_{\iota,\alpha}(s) = T_\alpha(s - T_\iota(s))$ as

$$T_{\iota,\alpha}(s) = \begin{cases} s-\iota \quad \text{sign(s)} & \text{if} \quad \iota \leq |s| < \iota+\alpha \\ \alpha \ \ sign(s) & \text{if} \quad \iota+\alpha \leq |s| \\ 0 & \text{if} \quad |s| < \iota \end{cases}$$

Using $T_{\iota\alpha}(u_n) = T_\alpha(u_n - T_\iota(u_n))$ as a test function in (3.3), we find

$$\int_{\{\iota\leq|u_n|\leq L+1\}} A(t,x,\nabla u_n).\nabla u_n dx\ dt \leq \tag{3.19}$$

$$\lim_{\iota\to+\infty} \lim_{n\to+\infty} \int_{\{\iota\leq|u_n|\leq L+1\}} A(t,x,\nabla u_n).\nabla u_n dx\ dt = 0 \tag{3.20}$$

Step 5. Establish the convergence of

$$A(t,x,\nabla T_k(u_n)).\nabla T_k(u))$$

We are going to show that

$$A(t,x,\nabla T_k(u_n))\nabla T_k(u)) \to A(t,x,\nabla T_k(u))\nabla T_k(u) \quad stronglyin \quad L^1(\Omega_T) \tag{3.21}$$

In fact , according to (3.7) and condition (1.2) ,we obtain

$$\int_0^T \int_\Omega M^*(t,x,|A(t,x,\nabla T_k(u_n))|)dx\ dt \leq C$$

Therefore, there exists a subsequence of $A(t,x,\nabla T_k(u_n))$ such that as $n \to +\infty$

$$A(t,x,\nabla T_k(u_n)) \rightharpoonup A_k \quad weakly - *in\ L_{M^*}(\Omega_T) \tag{3.22}$$

Recalling the fact that $\nabla u_n \to \nabla u$ a .e . in Ω_T and $A(t,x,\nabla T_k(u_n))$ is continuous with respect to $\nabla T_k(u_n)$, we deduced that

$$A_k = A(t,x,\nabla T_k(u))$$

To prove (3.21), we first show

$$\lim_{n\to+\infty} \int_0^T \int_\Omega A(t,x,\nabla T_k(u_n)).\nabla T_k(u_n)dx\ dt$$
$$= \int_0^T \int_\Omega A(t,x,\nabla T_k(u)).\nabla T_k(u)dx\ dt$$

Since $T_k(u) \in W(\Omega_T)$, it follows from Lemma 2.2 that there exists a sequence $\{T_k(u)\}_\delta \subset C_0^\infty((0,T];C_0^\infty(\Omega))$ such that

$$\{T_k(u)\}_\delta \to^M T_k(u) \quad in\ L_M(\Omega_T)$$

$$\nabla\{T_k(u_n)\}_\delta {\to}^M T_k(u) \quad in \quad L_M(\Omega_T)$$

$$\partial_t\{T_k(u)\}_\delta \to^M \partial_t T_k(u) \qquad in \qquad W^{-1,x}L_{M^*}(\Omega_T) + L^2(\Omega_T)$$

Define $\psi L(r)$ =min$\{\iota+1-|r|)^{+},1\}$. Taking $\psi L(u_n)T_k(u)_{\delta\chi(0,\tau)}$ as a tast function for prtblem (3.1), we , have

$$\int_0^T\int_\Omega \partial_t u_n\psi\iota(u_n)(T_k(u))_\delta dx\ dt$$
$$+\int_0^T\int_\Omega A(t,x,(u_n)).\nabla(\psi\iota(u_n)(T_k(u))_\delta)dx\ dt$$
$$=\int_0^T\int_\Omega f_n\psi\iota(u_n)(T_k(u))_\delta)dx\ dt \tag{3.23}$$

For the first term on the left -hand side of equation (3.20), recalling the fact that $u_n\to u$ a.e in Ω_T as $n\to+\infty$, we deduced from Lebesgue dominated covergence theorem that

$$\lim_{\delta\to 0}\lim_{n\to+\infty}\int_0^T\int_\Omega \partial_t u_n\psi\iota(u_n)(T_k(u))_\delta dx\ dt$$
$$=\lim_{\delta\to 0}\lim_{n\to+\infty}\int_0^T\int_\Omega \partial_t\int_0^{u_n}\psi(r)dr(T_k(u))_\delta dx\ dt$$
$$=\lim_{\delta\to 0}\lim_{n\to+\infty}\Big(\int_0^T\int_\Omega \psi(r)dr(T_k(u))_\delta dx|_0^T-\int_0^T\int_\Omega\int_0^{u_n}\psi(r)dr\partial_t(T_k(u))_\delta dx\ dt\Big)$$
$$=\int_0^T\int_\Omega \partial_t\psi(u)T_k(u)dx\ dt$$

Taking$L\to+\infty$, it is obvious that

$$\lim_{L\to 0}\lim_{n\to+\infty}\lim_{L\to+\infty}\int_0^T\int_\Omega \partial_t\psi(u)(T_k(u))_\delta dx\ dt=\int_0^T\int_\Omega \partial_t uT_k(u)dx\ dt \tag{3.24}$$

For the second term on the left-hand sid of equation (3.23), we know that

$$\int_0^T\int_\Omega A(t,x,\nabla u_n).\nabla(\psi L(u_n)(T_k(u))_\delta)dx\ dt$$
$$=\int_0^T\int_\Omega A(t,x,\nabla u_n).\nabla T_{L+1}(u_n)\psi^{'}(u_n)(T_k(u))_\delta dx\ dt$$
$$+\int_0^T\int_\Omega A(t,x,\nabla u_n).\nabla)(T_k(u))_\delta\psi L(u_n)dx\ dt:=Z_1+Z_2$$

. For Z_1, it follows from (3.20) that

$$\lim_{L\to+\infty}\lim_{\delta\to 0}\lim_{n\to+\infty}\int_0^T\int_\Omega A(t,x,\nabla u_n)\nabla T_{L+1}(u_n)\psi^{'}(u_n)(T_k(u))_\delta dx\ dt$$
$$\leq \lim_{L\to+\infty}\lim_{n\to+\infty} C\int_{\{|u_n|\leq L+1\}} A(t,x,\nabla u_n).\nabla T_{L+1}(u_n)dx\ dt \to 0$$

For Z_2, we deduced from (3.22) that

$$A(t,x,\nabla T_{L+1}(u_n))\psi L(u_n) \to A(t,x,T_{L+1}(\nabla u))\psi L(u)\ \ weaklyin\ \ L^1(\Omega_T)\ as\ n\to+\infty, \tag{3.25}$$

which gives that

$$\lim_{L\to+\infty}\lim_{\delta\to 0}\lim_{n\to+\infty}\int_0^T\int_\Omega A(t,x,\nabla u_n)\nabla(T_k(u)_\delta\psi L(u_n)dx\ dt$$

$$= \lim_{L\to+\infty}\lim_{\delta\to 0}\lim_{n\to+\infty}\int_0^T\int_\Omega A(t,x,\nabla T_{L+1}(u_n))\nabla(T_k(u)_\delta\psi L(u_n)dx\ dt$$
$$+ \lim_{L\to+\infty}\lim_{\delta\to 0}\lim_{n\to+\infty}\int_0^T\int_\Omega A(t,x,\nabla T_{L+1}(u_n))\nabla(T_k(u)_\delta\psi L(u_n)dx\ dt$$
$$= \int_0^T\int_\Omega A(t,x,\nabla u).\nabla T_k(u)dx\ dt$$

The limit as $\delta\to 0$ results from Lemma 2.7. Therefore, we have

$$\lim_{L\to+\infty}\lim_{\delta\to 0}\lim_{n\to+\infty}\int_0^T\int_\Omega A(t,x,\nabla u_n)\nabla(\psi L(u_n)(T_k(u))_\delta dx\ dt$$

$$= \int_0^T\int_\Omega A(t,x,\nabla u)\nabla T_k(u)dx\ dt \tag{3.26}$$

For the term on the right-hand side of equation (3.23), we have

$$\lim_{L\to+\infty}\lim_{\delta\to 0}\lim_{n\to+\infty}\int_0^T\int_\Omega f_n\psi L(u_n)(T_k(u))_\delta dx\ dt = \int_0^T\int_\Omega fT_k(u)dx\ dt \tag{3.27}$$

Thus, it follows from (3.23)- (3.27) that

$$\int_0^T \int_\Omega \partial_t u T_k(u)dx\ dt + \int_0^T \int_\Omega A(t,x,\nabla u).\nabla T_k(u)dx\ dt = \int_0^T \int_\Omega f T_k(u)dx\ dt \tag{3.28}$$

In addition, testing the approximate problem (3.1) by $T_k(u_n)_\chi(0,\tau)$, we have

$$\lim_{n\to+\infty} \int_\Omega \varphi_k(u_n)(T)dx - \int_\Omega \varphi_k(u_{0n})dx + \int_0^T \int_\Omega A(t,x,\nabla T_k(u_n)).\nabla T_k(u_n)dx\ dt$$

$$= \lim_{n\to+\infty} \int_0^T \int_\Omega f_n T_k(u_n)dx\ dt = \int_0^T \int_\Omega f T_k(u)dx\ dt \tag{3.29}$$

Combining with (3.28) and (3.29) ,we obtain

$$\lim_{n\to+\infty} \int_\Omega A(t,x,\nabla T_k(u_n)).\nabla T_k(u_n)dx\ dt$$

$$= \int_0^T \int_\Omega A(t,x,\nabla T_k(u)).\nabla T_k(u)dx\ dt \tag{3.30}$$

Thus , Lemma 2.6 gives that

$$A(t,x,\nabla T_k(u_n)).\nabla T_k(u_n)$$

$$\to A(t,x,\nabla T_k(u)).\nabla T_k(u)$$

Step 6. convergence equi-integrability of the nonlinearities

We shall now prove that $g_n(t,x,\nabla u_n) \to g(t,x,\nabla u)$ strongly in $L^1(Q)$ by using Vitali's theorem. and thanks to (3.17),it suffices to $g_n(t,x,\nabla u_n)$ is uniformly equi-integrable in Q.

Let $E \in Q$ be a measurable subset of Q. We have for any $m > 0$

$$\int_E \Big| g_n(t,x,\nabla u_n)\Big| dxdt = \int_{E\cap\{|u_n|\le\eta\}} \Big|g_n(t,x,\nabla u_n)\Big|dxdt$$

$$+\int_{E\cap\{|u_n|>\eta\}}\left|g_n(t,x,\nabla u_n)dxdt\right|\ \ dxdt$$

On the one hand

$$\int_{E\cap\{|u_n|>\eta\}}\left|g_n(t,x,\nabla u_n)\right|dxdt$$

$$\leq\frac{1}{\eta}\int_Q\left|g_n(t,x,\nabla u_n)\right|u_n dxdt\leq\frac{D}{\eta}$$

Therefore , there exists $\eta=\eta(\varepsilon)$ large enough such that

$$\int_{E\cap\{|u_n|>\eta\}}\left|g_n(t,x,\nabla u_n)\right|dxdt\leq\frac{\varepsilon}{2}\forall n$$

On the other hand

$$\int_{E\cap\{|u_n|\leq\eta\}}|g_n(t,x,\nabla u_n)|dxdt$$

$$\leq\int_E\left|g_n(t,x,T_m(u_n),\nabla T_\eta(u_n))\right|dxdt$$

$$\leq b(\eta)\int_E[d_2(t,x)+\varphi(x,|\nabla T_\eta(u_n)|)]dxdt$$

$$\leq b(\eta)\int_E[d_2(t,x)+\frac{1}{\alpha}d(t,x)|)]dxdt$$

$$+\frac{b(\eta)}{\alpha}\int_E a(T_\eta(u_n),\nabla T_\eta(u_n))\nabla T_\eta(u_n)dxdt$$

By virtue of strong convergence (3.26) and the fact that $d_2(t,x),d(t,x)\in L_1(Q)$, there exists ν suth that

$$|E|<\nu\Rightarrow\int_{E\cap\{|u_n|>\eta\}}\left|g_n(t,x,\nabla u_n)\right|dxdt\leq\frac{\varepsilon}{2}\ \ \forall n$$

Consequently

$$|E|<\nu\Rightarrow\int_E\left|g_n(t,x,\nabla u_n)\right|dxdt\leq\varepsilon\forall n$$

which shows that $g_n(t,x,\nabla u_n)$ is uniformly equi-integrable in Q as required.

Step 7. Now we choose $T_k(u_n-\varphi)$ as a test funtion in (3.1) for $k\in I\!N^+$ and $\varphi\in C^1(\bar{\Omega_T})$ with $\varphi|\Sigma=0$. Set $L=k+||\varphi||_{L^\infty(\Omega_T)}$, we deduce

$$\int_0^T\int_\Omega A(t,x,\nabla u_n).\nabla T_k(u_n-\varphi)dx\ dt+\int_0^T\int_\Omega g_n(t,x,\nabla u_n).\nabla T_k(u_n-\varphi)dx\ dt$$

$$= \int_0^T \int_\Omega A(t,x,\nabla T_L(u_n)).\nabla T_k(T_L(u_n) - \varphi) dx\, dt + \int_0^T \int_\Omega g_n(t,x,\nabla T_L(u_n)).\nabla T_k(T_L(u_n) - \varphi) dx\, dt$$

and

$$\int_0^T \langle \partial_t u_n, T_k(u_n - \varphi)\rangle dt + \int_0^T \int_\Omega A(t,x,\nabla T_L(u_n)).\nabla T_k(T_L(u_n) - \varphi) dx\, dt$$
$$+ \int_0^T \int_\Omega g(t,x,\nabla T_L(u_n)).\nabla T_k(T_L(u_n) - \varphi) dx\, dt = \int_0^T \int_\Omega f_n T_k(u_n - \phi) dx\, dt$$

Since $\partial_t u_n = \partial_t(u_n - \varphi) + \partial_t \varphi$, we have

$$\int_0^T \langle \partial_t u_n, T_k(u_n - \varphi)\rangle dt = \int_\Omega \beta_k(u_n - \varphi)(T) dx - \int_\Omega \beta_k(u_n - \varphi)(0) dx$$
$$+ \int_0^T \langle \partial_t \varphi, T_k(u_n - \varphi)\rangle dt$$

which yields that

$$\int_\Omega \varphi_k(u_n - \varphi)(T) dx - \int_\Omega \beta_k(u_n - \varphi)(0) dx + \int_0^T \langle \varphi_t, T_k(u_n - \varphi)\rangle dt$$
$$+ \int_0^T \int_\Omega A(t,x,\nabla T_L(u_n)).\nabla T_k(T_L(u_n) - \varphi) dx\, dt + \int_0^T \int_\Omega g_n(t,x,\nabla T_L(u_n)).\nabla T_k(T_L(u_n) - \varphi) dx\, dt$$
$$= \int_0^T \int_\Omega f_n T_k(u_n - \varphi) dx\, dt \tag{3.31}$$

Recalling u_n converges to u in $C([0,T]; L^1(\Omega))$, we have $u_n(t)$ in $L^1(\Omega)$ as $n \to +\infty$ for all $t \leq T$. Since β_t is Lipschitz continuous, we get

$$\int_\Omega \varphi_k(u_n - \phi)(T) dx \to \int_\Omega \varphi_k(u - \phi)(T) dx$$

and

$$\int_\Omega \beta_k(u_n - \phi)(T) dx \to \int_\Omega \beta_k(u - \phi)(0) dx$$
$$\int_0^T \int_\Omega g_n(t,x,\nabla T_L(u_n)) \to \int_0^T \int_\Omega g(t,x,\nabla T_L(u))$$

as $n \to +\infty$ The fourth term on the left-hand side of (3.15) can be written as

$$\int_0^T \int_\Omega A(t,x,\nabla T_L(u_n)).\nabla T_k(T_L(u_n) - \phi) dx\, dt + \int_0^T \int_\Omega g(t,x,\nabla T_L(u_n)).\nabla T_k(T_L(u_n) - \phi) dx\, dt$$

$$\int_{|T_L(u_n)-\phi|\leq k} A(t,x,\nabla T_L(u_n)).\nabla T_k(T_L(u_n)dx\,dt - \int_{|T_L(u_n)-\phi|\leq k} A(t,x,\nabla T_L(u_n)).\nabla\phi dx\,dt \tag{3.32}$$

$$\int_{|T_L(u_n)-\phi|\leq k} g(t,x,\nabla T_L(u_n)).\nabla T_k(T_L(u_n)dx\,dt - \int_{|T_L(u_n)-\phi|\leq k} g(t,x,\nabla T_L(u_n)).\nabla\phi dx\,dt$$
$$=: S_1 + S_2$$

where

$$S_1 := \int_{\{|T_L(u_n)-\phi|\leq k\}} A(t,x,\nabla T_L(u_n)).\nabla T_k(u_n)dx\ dt + \int_{|T_L(u_n)-\phi|\leq k} g(t,x,\nabla T_L(u_n)).\nabla T_k(u_n)dx\ dt$$

and

$$S_2 := -\int_{|T_L(u_n)-\phi|\leq k} A(t,x,\nabla T_L(u_n)).\nabla\phi dx\ dt - \int_{|T_L(u_n)-\phi|\leq k} g(t,x,\nabla T_L(u_n)).\nabla\phi dx\ dt$$

Estimate of S_1. We prone that we derive from (3.21)that

$$\begin{aligned}
&\int_{\{|T_L(u_n)-\phi|\leq k\}} A(t,x,\nabla T_L(u)).\nabla T_k(u)dx\ dt \\
&+\int_{\{|T_L(u_n)-\phi|\leq k\}} g(t,x,\nabla T_L(u)).\nabla T_k(u)dx\ dt \\
&= \lim_{n\to+\infty}\int_{\{|T_L(u_n)-\phi|\leq k\}} A(t,x,\nabla T_L(u_n)).\nabla T_k(u_n)dx\ dt \\
&+ \lim_{n\to+\infty}\int_{\{T_L(u_n)-\phi|\leq k\}} g(t,x,\nabla T_L(u_n)).\nabla T_k(u_n)dx\ dt \\
&=: \lim_{n\to+\infty} S_1
\end{aligned} \tag{3.33}$$

Estimate of S_2. For convenience, we define

$$\eta_n := -A(t,x,\nabla T_L(u_n)), E_n := (t,x)\in\Omega_T : |T_L(u_n)-\phi|\leq k$$

and

$$E := (t,x)\in\Omega_T : |T_L(u_n)-\phi|\leq k$$

We can write

$$S_2 = \int_{E_n} \eta_n.\nabla\phi dx\ dt = \int_E \eta_n.\nabla\phi dz + \int_{E_n\ E} \eta_n.\nabla\phi dx\ dt := S_21 + S_22$$

Recalling the fact that for $n \to +\infty$ ty

$$A(t,x,\nabla T_L(u_n)) + g(t,x,\nabla T_L(u_n))$$

$$\to A(t,x,\nabla T_L(u)) + g(t,x,\nabla T_L(u)) \quad weaklyin \quad L^1(\Omega_T), \tag{3.34}$$

we have

$$\lim_{n\to+\infty} S_{21} = -\int_{|T_L(u_n)-\phi|\le k} A(t,x,\nabla T_L(u))\nabla\phi dx\ dt$$

$$-\int_{|T_L(u_n)-\phi|\le k} g_n(t,x,\nabla T_L(u))\nabla\phi dx\ dt$$

Moreover, since $M^*(t,x,\eta)$ satisfies the condition $\lim_{|\eta|\to+\infty} \frac{M^*(t,x,|\eta|)}{|\eta|} = +\infty$. Then for every $\epsilon > 0$, there exists a constant $\Lambda > 0$ such that

$$|\eta| \le \epsilon M^*(t,x,|\eta|) forall |\eta| > \Lambda$$

It follows from (3.6) that

$$|S_{22}| \le ||\nabla\phi||_{L^\infty(\Omega_T)} \int_o^T \int_\Omega |\eta_n|\chi E_n\ E dx\ dt$$

$$= C\left(\int_{|\eta|\le\Lambda} |\eta_n|\chi E_n\ E dx\ dt + \int_{|\eta|\le\Lambda} |\eta_n|\chi E_n\ E dx\ dt\right)$$

$$\le C\left(\Lambda meas(E_n\ E) + \epsilon\int_o^T \int_\Omega M^*(T,X,|\eta_n|) dx\ dt\right)$$

$$\le C\Lambda meas(E_n\ E) + C\epsilon$$

By the arbitrariness of ϵ , we det

$$\lim_{n\to+\infty} |S_{22}| = 0$$

Therefore , we have

$$\lim_{n\to+\infty} S_2 = -\int_{|T_L(u_n)-\phi|\le k} A(t,x,\nabla T_L(u)).\nabla\phi dx\ dt$$

$$- \int_{|T_L(u_n)-\phi|\le k} g_n(t,x,\nabla T_L(u)).\nabla \phi dx\ dt \qquad (3.35)$$

According to (3.16), we obtain

$$\int_o^T \int_\Omega A(t,x,\nabla(u)).\nabla T_k(u-\phi)dx\ dt + \int_o^T \int_\Omega g(t,x,\nabla(u)).\nabla T_k(u-\phi)dx\ dt$$

$$= \int_o^T \int_\Omega A(t,x,\nabla T_L(u)).\nabla T_k(T_L(u)-\phi)dx\ dt + \int_o^T \int_\Omega g(t,x,\nabla T_L(u)).\nabla T_k(T_L(u)-\phi)dx\ dt$$

$$= \int_{|T_L(u_n)-\phi|\le k} A(t,x,\nabla T_L(u)).\nabla T_L(u)dx\ dt + \int_{|T_L(u_n)-\phi|\le k} g(t,x,\nabla T_L(u)).\nabla T_L(u)dx\ dt$$

$$- \int_{|T_L(u_n)-\phi|\le k} A(t,x,\nabla T_L(u)).\nabla \phi dx\ dt - \int_{|T_L(u_n)-\phi|\le k} g(t,x,\nabla T_L(u)).\nabla \phi dx\ dt$$

$$= \lim_{n\to+\infty}(S_1+S_2)$$

Using the strong convergence of f_n,(3.12) and the Lebesgue doninated convergence theorem, we can pass to the limits as $n \to +\infty$ in the other term of (3.36) to conclude

$$\int_\Omega \theta_k(u-\phi)(T)dx - \int_\Omega \theta_k(u_0 - phi(0))dx$$

$$+ \int_o^T \langle \partial_t \phi, T_k(u-\phi)\rangle dt$$

$$+ \int_o^T \int_\Omega A(t,x,\nabla(u)).\nabla T_k(u-\phi)dx\ dt + \int_o^T \int_\Omega g_n(t,x,\nabla(u)).\nabla T_k(u-\phi)dx\ dt$$

$$= \int_o^T \int_\Omega f T_k(u-\phi)dx\ dt \qquad (3.36)$$

for all $k > 0$ and $\phi \in C^1(\bar{\Omega}_T)$ with$\phi|\sigma = 0$. Hence, our solution u satisfies condition (2) Since u_n is the distributional solution of problem (3.1) , then its limit u satisfies condition (1) naturally . This completes the proof.

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