Dual Generalized Pandita Numbers

Abstract. In this paper, we introduce the dual generalized Pandita numbers defined over the bidimensional Clifford algebra of hyperbolic numbers. As special cases, we examine the dual Pandita and dual Pandita Lucas numbers. We derive Binet formulas, construct generating functions, and establish summation identities for these sequences. Furthermore, we present matrix representations associated with the proposed number sequences.

2010 Mathematics Subject Classification. 11B37, 11B39, 11B83.

Keywords. Pandita numbers, Pandita-Lucas numbers, dual numbers, dual Pandita numbers.

1. Introduction

Dual numbers, first introduced by W.K. Clifford in 1873, represent a fascinating mathematical construct with a wide range of applications. They play a pivotal role in screw theory, the modeling of planar joints, and iterative techniques for displacement analysis in spatial mechanisms. Additionally, dual numbers are instrumental in the inertial force analysis of spatial systems and continue to find relevance in various branches of kinematics and robotics.Here are some general information about the applications of dual numbers.

- Engineering and Physics:

  Used in electrical engineering and control systems.

  Applied in wave analysis and signal processing.

  Utilized in mechanical engineering for vibration analysis, among other applications.

- Mathematics and Geometry:

  Alongside complex numbers, dual numbers contribute to the extension of mathematical structures.

  Employed in geometry to represent various transformations.

- Computer Science:

  Found in graphics and image processing.

  Used in robotics and control systems for modeling and analysis.
- Finance and Economics:

  Applied in risk analysis and financial engineering.

  Utilized in option pricing and portfolio management.
- Optimization Problems:

  Used for finding solutions in optimization problems.

  Acts as a tool in linear programming and decision-making models.
- Quantum Mechanics:

  Employed in quantum computers and quantum mechanics for mathematical representation.

Next, we give some information raleted to hypercomplex number system and then we give some properities about dual number. As discussed in [7], the hypercomplex numbers systems are extensions of real numbers. Some examples of hypercomplex number systems ,which is commutative , are complex numbers, hyperbolic numbers and dual numbers.

- Complex numbers are formed by extending the real number system with the imaginary unit, denoted as ”i”, which satisfies the equation $i^2 = -1$. Complex numbers is defined as follows,

$$C = \{z = a + ib : a, b \in R, i^2 = -1\}.$$

- As discussed in [13], hyperbolic numbers extend the real number system with the hyperbolic unit j, where $j^2 = 1$. Hyperbolic numbers is defined as follows,

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\}.$$

- As discussed in[5], dual numbers extend the real number system by introducing a new element ε, where $\varepsilon^2 = 0$. Dual numbers is defined as follows,

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

Let us now revisit the definition of generalized Pandita numbers.

A generalized Pandita sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$W_n = 2W_{n-1} - W_{n-2} + W_{n-3} - W_{n-4} \tag{1.1}$$

with the initial values W_0, W_1, W_2, W_3 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-1)} - W_{-(n-2)} + W_{-(n-3)} - W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

The initial values of the generalized Pandita numbers for both positive and negative subscripts are presented in Table 1.

Table 1. A few generalized Pandita numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$W_0 - W_1 + 2W_2 - W_3$
2	W_2	$W_1 + W_2 - W_3$
3	W_3	$W_0 + W_1 - W_2$
4	$W_1 - W_0 - W_2 + 2W_3$	$2W_0 - 2W_1 + 2W_2 - W_3$
5	$W_1 - 2W_0 - W_2 + 3W_3$	$3W_2 - 2W_3$
6	$W_1 - 3W_0 - 2W_2 + 5W_3$	$3W_1 - 2W_2$
7	$2W_1 - 5W_0 - 4W_2 + 8W_3$	$3W_0 - 2W_1$
8	$3W_1 - 8W_0 - 6W_2 + 12W_3$	$W_0 - 3W_1 + 6W_2 - 3W_3$
9	$4W_1 - 12W_0 - 9W_2 + 18W_3$	$5W_1 - 2W_0 - W_2 - W_3$
10	$6W_1 - 18W_0 - 14W_2 + 27W_3$	$3W_0 + W_1 - 5W_2 + 2W_3$
11	$9W_1 - 27W_0 - 21W_2 + 40W_3$	$4W_0 - 8W_1 + 8W_2 - 3W_3$
12	$13W_1 - 40W_0 - 31W_2 + 59W_3$	$4W_1 - 4W_0 + 5W_2 - 4W_3$
13	$19W_1 - 59W_0 - 46W_2 + 87W_3$	$9W_1 - 12W_2 + 4W_3$

If we set $W_0 = 0, W_1 = 1, W_2 = 2, W_3 = 3$ then $\{W_n\}$ is the well-known Pandita sequence and if we set $W_0 = 4, W_1 = 2, W_2 = 2, W_3 = 5$ then $\{W_n\}$ is the well-known Pandita -Lucas sequence. In other words, Pandita sequence $\{P_n\}_{n\geq 0}$ and Pandita -Lucas sequence $\{S_n\}_{n\geq 0}$ are defined by the second-order recurrence relations

$$P_n = 2P_{n-1} - P_{n-2} + P_{n-3} - P_{n-4}, \qquad P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 3, \qquad n \geq 4, \tag{1.2}$$

and

$$S_n = 2S_{n-1} - S_{n-2} + S_{n-3} - S_{n-4}, \qquad S_0 = 4, S_1 = 2, S_2 = 2, S_3 = 5, \qquad n \geq 4. \tag{1.3}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{S_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = P_{-(n-1)} - P_{-(n-2)} + 2P_{-(n-3)} - P_{-(n-4)}$$

and

$$S_{-n} = S_{-(n-1)} - S_{-(n-2)} + 2S_{-(n-3)} - S_{-(n-4)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.2) and (1.3) hold for all integer n.

We can list some important properties of generalized Pandita numbers that are needed.

- Binet formula of generalized Pandita sequence can be calculated using its characteristic equation which is given as

$$x^4-2x^3+x^2-x+1=(x^3-x^2-1)(x-1)=0$$

The roots of characteristic equation are

$$\begin{aligned}
\alpha &= \frac{1}{3}+\left(\frac{29}{54}+\sqrt{\frac{31}{108}}\right)^{1/3}+\left(\frac{29}{54}-\sqrt{\frac{31}{108}}\right)^{1/3}, \\
\beta &= \frac{1}{3}+\omega\left(\frac{29}{54}+\sqrt{\frac{31}{108}}\right)^{1/3}+\omega^2\left(\frac{29}{54}-\sqrt{\frac{31}{108}}\right)^{1/3}, \\
\gamma &= \frac{1}{3}+\omega^2\left(\frac{29}{54}+\sqrt{\frac{31}{108}}\right)^{1/3}+\omega\left(\frac{29}{54}-\sqrt{\frac{31}{108}}\right)^{1/3}, \\
\delta &= 1,
\end{aligned}$$

where

$$\omega=\frac{-1+i\sqrt{3}}{2}=\exp(2\pi i/3).$$

Using these roots and the recurrence relation, Binet formula can be given as

$$\begin{aligned}
W_n &= \frac{z_1\alpha^n}{3\alpha-2}+\frac{z_2\beta^n}{3\beta-2}+\frac{z_3\gamma^n}{3\gamma-2}+z_4 \\
&= A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4,
\end{aligned}$$

where z_1, z_2 and z_3 are given below

$$\begin{aligned}
z_1 &= (\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+1)W_1-W_0), \\
z_2 &= (\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+1)W_1-W_0), \\
z_3 &= (\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+1)W_1-W_0), \\
z_4 &= -W_3+W_2+W_0.
\end{aligned}$$

and

$$\begin{aligned}
A_1 &= \frac{z_1}{3\alpha-2}, \qquad (1.4) \\
A_2 &= \frac{z_2}{3\beta-2}, \\
A_3 &= \frac{z_3}{3\gamma-2}, \\
A_4 &= z_4.
\end{aligned}$$

Binet formula of Pandita and Pandita Lucas sequences are

$$P_n = \frac{\alpha^{n+3}}{3\alpha-2} + \frac{\beta^{n+3}}{3\beta-2} + \frac{\gamma^{n+3}}{3\gamma-2} - 1,$$

and

$$S_n = \alpha^n + \beta^n + \gamma^n + 1,$$

respectively.

The generating function for generalized Pandita numbers is

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 2W_0)x + (W_2 - 2W_1 + W_0)x^2 + (W_3 - 2W_2 + W_1 - W_0)x^3}{1 - 2x + x^2 - x^3 + x^4}.$$

For more details about generalized Pandita numbers, see [17].

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n.

Lemma 1. *[18, Lemma 1.4].Suppose that $f_{W_n}(x) = \sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is the exponential generating function of the generalized Pandita sequence $\{W_n\}$.*

Then $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is given by

$$\begin{aligned}
\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \frac{(\alpha W_3 - \alpha(2-\alpha)W_2 + (-\alpha^2+\alpha+1)W_1 - W_0)}{3\alpha-2} e^{\alpha x} \\
&\quad + \frac{(\beta W_3 - \beta(2-\beta)W_2 + (-\beta^2+\beta+1)W_1 - W_0)}{3\beta-2} e^{\beta x} \\
&\quad + \frac{(\gamma W_3 - \gamma(2-\gamma)W_2 + (-\gamma^2+\gamma+1)W_1 - W_0)}{3\gamma-2} e^{\gamma x} \\
&\quad + (-W_3 + W_2 + W_0)e^x.
\end{aligned}$$

The previous Lemma 1 gives the following results as particular examples.

Corollary 2. *Exponential generating function of Pandita and Pandita-Lucas numbers*

a): $\sum\limits_{n=0}^{\infty} P_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\frac{\alpha^{n+3}}{3\alpha-2} + \frac{\beta^{n+3}}{3\beta-2} + \frac{\gamma^{n+3}}{3\gamma-2} - 1)\frac{x^n}{n!} = \frac{\alpha^3 e^{\alpha x}}{3\alpha-2} + \frac{\beta^3 e^{\beta x}}{3\beta-2} + \frac{\gamma^3 e^{\gamma x}}{3\gamma-2} - e^x.$

b): $\sum\limits_{n=0}^{\infty} S_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\alpha^n + \beta^n + \gamma^n + 1)\frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^{\gamma x} + e^x.$

Next, we provide an overview of selected publications in the literature that pertain to dual numbers.

- Göcen,Dikmen Kaya and Soykan [9] presented the dual generalized Fibonacci matrices as

$$DWn = \begin{pmatrix} W_{n+1} + \varepsilon W_{n+2} & W_n + \varepsilon W_{n+1} \\ W_n + \varepsilon W_{n+1} & W_{n-1} + \varepsilon W_n \end{pmatrix} = \begin{pmatrix} W_{n+1} + \varepsilon(Wn + 1 + Wn) & W_n + \varepsilon W_{n+1} \\ W_n + \varepsilon W_{n+1} & W_{n+1} - Wn + \varepsilon W_n \end{pmatrix}$$

with initial conditions $DW_0 = \begin{pmatrix} W_1 + \varepsilon(W_0 + W_1) & W_0 + \varepsilon W_1 \\ W_0 + \varepsilon W_1 & W_1 - W_0 + \varepsilon W_0 \end{pmatrix}$,

$$DW_1 = \begin{pmatrix} W_0 + W_1 + \varepsilon(W_0 + 2W_1) & W_1 + \varepsilon(W_0 + W_1) \\ W_1 + \varepsilon(W_0 + W_1) & W_0 + \varepsilon W_1 \end{pmatrix} \text{ as } \varepsilon^2 = 0$$

.

- Halici [11] studied Dual Fibonacci Octonions as

$$p=\sum_{s=0}^{7}F_{n+s}e_s$$

where Fibonacci given by $F_n=F_{n-1}+F_{n-2}$, $F_0=0$, $F_1=1$.

- Aydın [6] studied Dual Jacobsthal Quaternions as

$$QJ_{k;n}=J_{k;n}+i_1J_{k;n+1}+i_2J_{k;n+2}+i_3J_{k;n+3}$$

where $J_n=J_{n-1}+2J_{n-2}$, $J_0=0$, $J_1=1$.

- Nurkan ,Guven, [4] studied Dual Fibonacci Quaternions as

$$\widetilde{Q}n=(F_n+F_{n+1})+i(F_{n+1}+F_{n+2})+j(F_{n+2}+F_{n+3})+k(F_{n+3}+F_{n+4})$$

where Fibonacci given by $F_n=F_{n-1}+F_{n-2}$, $F_0=0$, $F_1=1$.

- Gürses, Şentürk, Yüce [10] studied dual-generalized complex Fibonacci and Lucas numbers, respectively, as

$$\begin{aligned}\widetilde{\mathcal{F}}_n &= F_n+jF_{n+1}+\varepsilon F_{n+2}+j\varepsilon F_{n+3},\\ \widetilde{\mathcal{L}}_n &= L_n+jL_{n+1}+\varepsilon L_{n+2}+j\varepsilon L_{n+3},\end{aligned}$$

where Fibonacci and Lucas numbers, respectively, given by $F_n=F_{n-1}+F_{n-2}$, $F_0=0$, $F_1=1$, $L_n=L_{n-1}+L_{n-2}$, $L_0=2$, $L_1=1$.

- Yılmaz and Soykan , [12] studied dual generalized Guglielmo numbers given by

$$\widetilde{W}_n=W_n+\varepsilon W_{n+1}$$

where generalized Guglielmo numbers are $W_n=3W_{n-1}-3W_{n-2}+W_{n-3}$ with the initial conditation $W_0,W_1,W_2 \quad (n\geq 2)$.

- Ayrılma and Soykan , [3] introduced On Dual Edouard Numbers are

$$DE_n=7DE_{n-1}-7DE_{n-2}+DE_{n-3}$$

  where generalized Edouard numbers are $E_n=7E_{n-1}-7E_{n-2}+E_{n-3}$ with the initial conditation $E_0=0,E_1=1,E_2=7$

Following this, we provide details on dual hyperbolic sequences as they are presented in literature.

- Demirci and Soykan, [2] studied hyperbolic generalized Adrien numbers given by

$$HA_n = 3HA_{n-1} - HA_{n-2} - HA_{n-4}$$

where generalized Adrien numbers are $A_n = 3A_{n-1} - A_{n-2} + A_{n-4}$ with the initial conditation $A_0 = 0, A_1 = 1, A_2 = 3, A_3 = 8,\ n \geq 4$.

- Kalca and Soykan , [8] studied dual hyperbolic generalized Pandita numbers given by

$$\hat{P}_n = 2\hat{P}_{n-1} - \hat{P}_{n-2} + \hat{P}_{n-3} - \hat{P}_{n-4}$$

where generalized Pandita numbers are $P_n = 2P_{n-1} - P_{n-2} + P_{n-3} - P_{n-4}$ with the initial conditation $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 3,\ n \geq 4.$

In this paper, we define the dual generalized Pandita numbers in the next section and give some properties of them.

2. Dual Generalized Pandita Numbers and their Generating Functions and Binet's Formulas

In this section, we introduce the dual generalized Pandita numbers and derive their corresponding generating functions and Binet formulas. We now define the dual generalized Pandita numbers over the algebra $\mathbb{H}_{\mathbb{D}}$of dual dual numbers.The nth dual generalized Pandita number is

$$DW_n = W_n + \varepsilon W_{n+1}. \tag{2.1}$$

The sequence $\{DW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$DW_{-n} = W_{-n} + \varepsilon W_{-n+1},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.2) holds for all integer n.

Note that

$$\begin{aligned} DW_0 &= W_0 + \varepsilon W_1, \\ DW_1 &= W_1 + \varepsilon W_2, \\ DW_2 &= W_2 + \varepsilon W_3, \\ DW_3 &= W_3 + \varepsilon W_4 = W_3 + \varepsilon(W_1 - W_0 - W_2 + 2W_3). \end{aligned}$$

It can be easily shown that

$$DW_n = 2DW_{n-1} - DW_{n-2} + DW_{n-3} - DW_{n-4} \tag{2.2}$$

and

$$DW_{-n} = DW_{-(n-1)} - DW_{-(n-2)} + 2DW_{-(n-3)} - DW_{-(n-4)}$$

The initial values of the dual generalized Pandita numbers for both positive and negative subscripts are listed in Table 2.

A few dual generalized Pandita numbers

n	DW_n	DW_{-n}
0	DW_0	DW_0
1	DW_1	$DW_0 - DW_1 + 2DW_2 - DW_3$
2	DW_2	$DW_1 + DW_2 - DW_3$
3	DW_3	$DW_0 + DW_1 - DW_2$
4	$DW_1 - DW_0 - DW_2 + 2DW_3$	$2DW_0 - 2DW_1 + 2DW_2 - DW_3$
5	$DW_1 - 2DW_0 - DW_2 + 3DW_3$	$3DW_2 - 2DW_3$
6	$DW_1 - 3DW_0 - 2DW_2 + 5DW_3$	$3DW_1 - 2DW_2$
7	$2DW_1 - 5DW_0 - 4DW_2 + 8DW_3$	$3DW_0 - 2DW_1$
8	$3DW_1 - 8DW_0 - 6DW_2 + 12DW_3$	$DW_0 - 3DW_1 + 6DW_2 - 3DW_3$
9	$4DW_1 - 12DW_0 - 9DW_2 + 18DW_3$	$5DW_1 - 2DW_0 - DW_2 - DW_3$
10	$6DW_1 - 18DW_0 - 14DW_2 + 27DW_3$	$3DW_0 + DW_1 - 5DW_2 + 2DW_3$
11	$9DW_1 - 27DW_0 - 21DW_2 + 40DW_3$	$4DW_0 - 8DW_1 + 8DW_2 - 3DW_3$
12	$13DW_1 - 40DW_0 - 31DW_2 + 59DW_3$	$4DW_1 - 4DW_0 + 5DW_2 - 4DW_3$
13	$19DW_1 - 59DW_0 - 46DW_2 + 87DW_3$	$9DW_1 - 12DW_2 + 4DW_3$

As special cases, the nth dual Pandita numbers and the nth dual Pandita Lucas numbers are given as

$$DP_n = P_n + \varepsilon P_{n+1} \tag{2.3}$$

and

$$DS_n = S_n + \varepsilon S_{n+1} \tag{2.4}$$

respectively. The sequences $\{DP_n\}_{n\geq 0}$ and $\{DS_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$DP_{-n} = P_{-(n-1)} - P_{-(n-2)} + 2P_{-(n-3)} - P_{-(n-4)}$$

and

$$DS_{-n} = S_{-(n-1)} - S_{-(n-2)} + 2S_{-(n-3)} - S_{-(n-4)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.3) and (2.4) holds for all integer n.

For dual Pandita numbers (taking $W_n = P_n,\ \ P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 3,$) we get

$$\begin{aligned} DP_0 &= \varepsilon, \\ DP_1 &= 2\varepsilon + 1, \\ DP_2 &= 3\varepsilon + 2 \end{aligned}$$

and for dual Pandita Lucas numbers (taking $W_n = S_n,\ \ S_0 = 4, S_1 = 2, S_2 = 2, S_3 = 5,$) we get

$$\begin{aligned} DS_0 &= 2\varepsilon + 4, \\ DS_1 &= 2\varepsilon + 2 \\ DS_2 &= 5\varepsilon + 2. \end{aligned}$$

Selected values of the dual Pandita numbers and dual Pandita Lucas numbers for both positive and negative subscripts are presented in Table 3 and Table 4, respectively.

Table 3. Dual Pandita numbers

n	DP_n	DP_{-n}
0	ε	ε
1	$2\varepsilon + 1$	0
2	$3\varepsilon + 2$	0
3	$5\varepsilon + 3$	-1
4	$8\varepsilon + 5$	$-\varepsilon - 1$
5	$12\varepsilon + 8$	$-\varepsilon$

Table 4. Dual Pandita- Lucas numbers

n	DS_n	DS_{-n}
0	$2\varepsilon + 4$	$2\varepsilon + 4$
1	$2\varepsilon + 2$	$4\varepsilon + 1$
2	$5\varepsilon + 2$	$\varepsilon - 1$
3	$6\varepsilon + 5$	$-\varepsilon + 4$
4	$7\varepsilon + 6$	$4\varepsilon + 3$
5	$11\varepsilon + 7$	$3\varepsilon - 4$

We now present the Binet formula for the dual generalized Pandita numbers, and for the remainder of the paper, we adopt the following notational conventions.

$$\widehat{\alpha} = 1 + \varepsilon\alpha, \tag{2.5}$$

$$\widehat{\beta} = 1 + \varepsilon\beta, \tag{2.6}$$

$$\widehat{\gamma} = 1 + \varepsilon\gamma \tag{2.7}$$

$$\widehat{\delta} = \widehat{1} = 1 + \varepsilon, \tag{2.8}$$

Note that we have the following identities:

$$\begin{aligned}
\widehat{\alpha}^2 &= 1+2\alpha\varepsilon, \\
\widehat{\beta}^2 &= 1+2\varepsilon\beta, \\
\widehat{\alpha}\widehat{\beta} &= 1+(\alpha+\beta)\varepsilon, \\
\widehat{\gamma}^2 &= 1+\gamma^2+2\varepsilon\gamma, \\
\widehat{\delta}^2 &= \widehat{1}^2=2+2\varepsilon, \\
\widehat{\gamma}\widehat{\delta} &= 1+\varepsilon(1+\gamma).
\end{aligned}$$

Theorem 3. *(Binet's Formula) For any integer n, the nth dual generalized Pandita number is*

$$DW_n = A_1\alpha^n\widehat{\alpha}+A_2\beta^n\widehat{\beta}+A_3\gamma^n\widehat{\gamma}+\widehat{1}A_4. \tag{2.9}$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}, \widehat{\delta}$ are given as (2.5)-(2.8)

Proof. Using Binet's formula of the generalized Pandita numbers given below

$$W_n = A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4.$$

where A_1, A_2, A_3, A_4 are given (1.4) we get

$$\begin{aligned}
DW_n &= W_n+\varepsilon W_{n+1}, \\
&= A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4+\varepsilon(A_1\alpha^{n+1}+A_2\beta^{n+1}+A_3\gamma^{n+1}+A_4) \\
&= A_1\alpha^n(1+\varepsilon\alpha)+A_2\beta^n(1+\varepsilon\beta)+A_3\gamma^n(1+\varepsilon\gamma)+A_4(1+\varepsilon) \\
&= A_1\alpha^n\widehat{\alpha}+A_2\beta^n\widehat{\beta}+A_3\gamma^n\widehat{\gamma}+\widehat{1}A_4.
\end{aligned}$$

This proves (2.9).

As special cases, for any integer n, the Binet's Formula of nth dual Pandita number is

$$DP_n = \frac{\alpha^{n+3}\widehat{\alpha}}{3\alpha-2}+\frac{\beta^{n+3}\widehat{\beta}}{3\beta-2}+\frac{\gamma^{n+3}\widehat{\gamma}}{3\gamma-2}-\widehat{1} \tag{2.10}$$

and the Binet's Formula of nth dual Pandita Lucas number is

$$DS_n = \widehat{\alpha}\alpha^n+\widehat{\beta}\beta^n+\widehat{\gamma}\gamma^n+\widehat{1}, \tag{2.11}$$

Next, we present generating function.

Theorem 4. *The generating function for the dual generalized Pandita numbers is*

$$f_{DW_n}(x)=\sum_{n=0}^{\infty}DW_nx^n=\frac{DW_0+(DW_1-2DW_0)x+(DW_2-2DW_1+DW_0)x^2+(DW_3-2DW_2+DW_1-DW_0)x^3}{1-2x+x^2-x^3+x^4}.$$

Proof. Using the definition of dual Pandita numbers, and substracting $xf(x)$, $x^2f(x)$ and $x^3f(x)$ from $f(x)$ we obtain $(1-2x+x^2-x^3+x^4)f_{DW_n}(x)$

$$\begin{aligned}
& (1-2x+x^2-x^3+x^4)f_{DW_n}(x) \\
= & \sum_{n=0}^{\infty} DW_n x^n - 2x\sum_{n=0}^{\infty} DW_n x^n + x^2\sum_{n=0}^{\infty} DW_n x^n - x^3\sum_{n=0}^{\infty} DW_n x^n + x^4\sum_{n=0}^{\infty} DW_n x^n, \\
= & \sum_{n=0}^{\infty} DW_n x^n - 2\sum_{n=0}^{\infty} DW_n x^{n+1} + \sum_{n=0}^{\infty} DW_n x^{n+2} - \sum_{n=0}^{\infty} DW_n x^{n+3} + \sum_{n=0}^{\infty} DW_n x^{n+4}, \\
= & \sum_{n=0}^{\infty} DW_n x^n - 2\sum_{n=1}^{\infty} DW_{(n-1)} x^n + \sum_{n=2}^{\infty} DW_{(n-2)} x^n - \sum_{n=3}^{\infty} DW_{(n-3)} x^n + \sum_{n=4}^{\infty} DW_{(n-4)} x^n, \\
= & (DW_0 + DW_1x + DW_2x^2 + DW_3x^3) - 2(DW_0x + DW_1x^2 + DW_2x^3) + (DW_0x^2 + DW_1x^3) - DW_0x^3 \\
& + \sum_{n=4}^{\infty}(DW_n - 2DW_{n-1} - DW_{n-2} - DW_{n-3} + DW_{n-4})x^n, \\
= & DW_0 + (DW_1 - 2DW_0)x + (DW_2 - 2DW_1 + DW_0)x^2 + (DW_3 - 2DW_2 + DW_1 - DW_0)x^3.
\end{aligned}$$

And rearranging above equation, we get (4). □

The following results are immediate consequences of the preceding Theorem.

Corollary 5. *For all integers n, we have following identities:*

- **a):** $\sum_{n=0}^{\infty} DP_n x^n = \dfrac{x+\varepsilon}{1-2x+x^2-x^3+x^4}$.
- **b):** $\sum_{n=0}^{\infty} DS_n x^n = \dfrac{(-4\varepsilon-1)x^3+(3\varepsilon+2)x^2+(-2\varepsilon-6)x+2\varepsilon+4}{1-2x+x^2-x^3+x^4}$.

Theorem (4) gives the following results as special cases,

$(1-2x+x^2-x^3+x^4)f_{DP_n}(x) = DP_0 + (DP_1 - 2DP_0)x + (DP_2 - 2DP_1 + DP_0)x^2 + (DP_3 - 2DP_2 + DP_1 - DP_0)x^3 = x+\varepsilon$,

$(1-2x+x^2-x^3+x^4)f_{DS_n}(x) = DS_0 + (DS_1 - 2DS_0)x + (DS_2 - 2DS_1 + DS_0)x^2 + (DS_3 - 2DS_2 + DS_1 - DS_0)x^3 = (-4\varepsilon-1)x^3+(3\varepsilon+2)x^2+(-2\varepsilon-6)x+2\varepsilon+4$.

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty} DW_n \frac{x^n}{n!}$ of the sequence DW_n.

Lemma 6. *Suppose that $f_{DW_n}(x) = \sum\limits_{n=0}^{\infty} DW_n \frac{x^n}{n!}$ is the exponential dual generating function of the generalized Pandita sequence $\{DW_n\}$.*

Then $\sum\limits_{n=0}^{\infty} DW_n \frac{x^n}{n!}$ is given by

$$\sum_{n=0}^{\infty} DW_n \frac{x^n}{n!} == A_1e^{\alpha x}\widehat{\alpha} + A_2e^{\beta x}\widehat{\beta} + A_3e^{\gamma x}\widehat{\gamma} + A_4e^{x}\widehat{1}.$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}, \widehat{\delta}$ are given as (2.5)-(2.8)

Proof. Using Binet's formula

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4.$$

where A_1, A_2, A_3, A_4 are given in (1.4) we get

$$\begin{aligned}
\sum_{n=0}^{\infty} DW_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty} W_n \frac{x^n}{n!} + \varepsilon \sum_{n=0}^{\infty} W_{n+1} \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} (A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4) \frac{x^n}{n!} + \varepsilon \sum_{n=0}^{\infty} (A_1\alpha^{n+1} + A_2\beta^{n+1} + A_3\gamma^{n+1} + A_4) \frac{x^n}{n!} \\
&= (A_1 e^{\alpha x} + A_2 e^{\beta x} + A_3 e^{\gamma x} + A_4 e^x) + \varepsilon (A_1 \alpha e^{\alpha x} + A_2 \beta e^{\beta x} + A_3 \gamma e^{\gamma x} + A_4 e^x) \\
&= A_1 e^{\alpha x}(1 + \varepsilon\alpha) + A_2 e^{\beta x}(1 + \varepsilon\beta) + A_3 e^{\gamma x}(1 + \varepsilon\gamma) + A_4 e^x (1 + \varepsilon) \\
&= A_1 e^{\alpha x}\widehat{\alpha} + A_2 e^{\beta x}\widehat{\beta} + A_3 e^{\gamma x}\widehat{\gamma} + A_4 e^x \widehat{1}
\end{aligned}$$

This proves (6). □

The previous Lemma 6 gives the following results as particular examples.

COROLLARY 7. *Exponential dual generating function of Pandita and Pandita-Lucas numbers are*

a): $\sum_{n=0}^{\infty} DP_n \frac{x^n}{n!} = \frac{\alpha^3 e^{\alpha x}\widehat{\alpha}}{3\alpha - 2} + \frac{\beta^3 e^{\beta x}\widehat{\beta}}{3\beta - 2} + \frac{\gamma^3 e^{\gamma x}\widehat{\gamma}}{3\gamma - 2} - e^x \widehat{1}.$

b): $\sum_{n=0}^{\infty} DS_n \frac{x^n}{n!} = e^{\alpha x}\widehat{\alpha} + e^{\beta x}\widehat{\beta} + e^{\gamma x}\widehat{\gamma} + e^x \widehat{1}.$

3. Obtaining Binet Formula From Generating Function

We next find Binet's formula generalized dual Pandita number $\{DW_n\}$ by the use of generating function for DW_n.

THEOREM 8. *Binet's formula of generalized dual Pandita numbers:*

$$DW_n = \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}. \tag{3.1}$$

where

$$\begin{aligned}
q_1 &= DW_0\alpha^3 + (DW_1 - 2DW_0)\,\alpha^2 + (DW_0 - 2DW_1 + DW_2)\,\alpha - DW_0 + DW_1 - 2DW_2 + DW_3, \\
q_2 &= DW_0\beta^3 + (DW_1 - 2DW_0)\,\beta^2 + (DW_0 - 2DW_1 + DW_2)\,\beta - DW_0 + DW_1 - 2DW_2 + DW_3, \\
q_3 &= DW_0\gamma^3 + (DW_1 - 2DW_0)\,\gamma^2 + (DW_0 - 2DW_1 + DW_2)\,\gamma - DW_0 + DW_1 - 2DW_2 + DW_3, \\
q_4 &= DW_0\delta^3 + (DW_1 - 2DW_0)\,\delta^2 + (DW_0 - 2DW_1 + DW_2)\,\delta - DW_0 + DW_1 - 2DW_2 + DW_3.
\end{aligned}$$

Proof. Let

$$h(x) = x^4 - x^3 + x^2 - 2x + 1.$$

Then for some α, β, γ and δ we write

$$h(x) = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x).$$

i.e.,

$$x^4 - x^3 + x^2 - 2x + 1 = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x). \tag{3.2}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$ are the roots of $h(x)$. This gives α, β, γ and δ as the roots of

$$h(\frac{1}{x}) = \frac{1}{x^2} - \frac{2}{x} - \frac{1}{x^3} + \frac{1}{x^4} + 1 = 0.$$

This implies $x^4 - x^3 + x^2 - 2x + 1 = 0$. Now, by it follows that

$$\sum_{n=0}^{\infty} DW_n x^n = \frac{(DW_1 - DW_0 - 2DW_2 + DW_3)\, x^3 + (DW_0 - 2DW_1 + DW_2)\, x^2 + (DW_1 - 2DW_0)\, x + DW_0}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)}.$$

Then we write

$$\frac{(DW_1 - DW_0 - 2DW_2 + DW_3)\, x^3 + (DW_0 - 2DW_1 + DW_2)\, x^2 + (DW_1 - 2DW_0)\, x + DW_0}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)} \tag{3.3}$$

$$= \frac{B_1}{(1 - \alpha x)} + \frac{B_2}{(1 - \beta x)} + \frac{B_3}{(1 - \gamma x)} + \frac{B_4}{(1 - \delta x)}. \tag{3.4}$$

So

$$\begin{aligned} & (DW_1 - DW_0 - 2DW_2 + DW_3)\, x^3 + (DW_0 - 2DW_1 + DW_2)\, x^2 + (DW_1 - 2DW_0)\, x + DW_0 \\ = \; & B_1(1 - \beta x)(1 - \gamma x)(1 - \delta x) + B_2(1 - \alpha x)(1 - \gamma x)(1 - \delta x) \\ & + B_3(1 - \alpha x)(1 - \beta x)(1 - \delta x) + B_3(1 - \alpha x)(1 - \beta x)(1 - \gamma x). \end{aligned}$$

If we consider $x = \frac{1}{\alpha}$, we get $DW_0 + \frac{1}{\alpha^2}(DW_0 - 2DW_1 + DW_2) - \frac{1}{\alpha^3}(DW_0 - DW_1 + 2DW_2 - DW_3) + \frac{1}{\alpha}(DW_1 - 2DW_0) = -B_1\left(\frac{1}{\alpha}\beta - 1\right)\left(\frac{1}{\alpha}\gamma - 1\right)\left(\frac{1}{\alpha}\delta - 1\right).$

This gives

$$\begin{aligned} B_1 &= \alpha^3(DW_0 + \frac{1}{\alpha^2}(DW_0 - 2DW_1 + DW_2) + \frac{1}{\alpha^3}(DW_1 - 5DW_0 - 4DW_2 + DW_3) + \frac{1}{\alpha}(DW_1 - 2DW_0)) \\ &= \frac{DW_0\alpha^3 + (DW_1 - 2DW_0)\,\alpha^2 + (DW_0 - 2DW_1 + DW_2)\,\alpha - DW_0 + DW_1 - 2DW_2 + DW_3}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} B_2 &= \frac{DW_0\beta^3 + (DW_1 - 2DW_0)\,\beta^2 + (DW_0 - 2DW_1 + DW_2)\,\beta - DW_0 + DW_1 - 2DW_2 + DW_3}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)}, \\ B_3 &= \frac{DW_0\gamma^3 + (DW_1 - 2DW_0)\,\gamma^2 + (DW_0 - 2DW_1 + DW_2)\,\gamma - DW_0 + DW_1 - 2DW_2 + DW_3}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)}, \\ B_4 &= \frac{DW_0\delta^3 + (DW_1 - 2DW_0)\,\delta^2 + (DW_0 - 2DW_1 + DW_2)\,\delta - DW_0 + DW_1 - 2DW_2 + DW_3}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}. \end{aligned}$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty} DW_n x^n = B_1(1 - \alpha x)^{-1} + B_2(1 - \beta x)^{-1} + B_3(1 - \gamma x)^{-1} + B_4(1 - \delta x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} DW_n x^n = B_1 \sum_{n=0}^{\infty} \alpha^n x^n + B_2 \sum_{n=0}^{\infty} \beta^n x^n + B_3 \sum_{n=0}^{\infty} \gamma^n x^n + B_4 \sum_{n=0}^{\infty} \delta^n x^n = \sum_{n=0}^{\infty} (B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n) x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$DW_n = B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n.$$

The following identity reveals a connection between the dual Pandita numbers and the Pandita–Lucas numbers.

Corollary 9.

Theorem 10. *For all integers m, n the following identities holds:*

$$DW_{m+n} = P_{m-2}DW_{n+3} + (P_{m-4} - P_{m-3} - P_{m-5})DW_{n+2} + (P_{m-3} - P_{m-4})DW_{n+1} - DW_nP_{m-3}.$$

Proof. First we assume that $m, n \geq 0$.The Theorem (10) can be proved by mathematical induction on m. If $m = 0$ we get

$$DW_n = P_{-2}DW_{n+3} + (P_{-4} - P_{-3} - P_{-5})DW_{n+2} + (P_{-3} - P_{-4})DW_{n+1} - DW_nP_{-3}.$$

which is true since $P_{-2} = 0, P = -1, P_{-4} = -1, P_{-5} = 0$. Assume that the equality holds for $m \leq k$. For $m = k + 1$, we get

$$\begin{aligned} DW_{k+1+n} &= 2DW_{n+k} - DW_{n+k-1} + DW_{n+k-2} - DW_{n+k-3}, \\ & 2(P_{m-2}DW_{n+3} + (P_{m-4} - P_{m-3} - P_{m-5})DW_{n+2} + (P_{m-3} - P_{m-4})DW_{n+1} - DW_nP_{m-3}) \\ & -(P_{m-3}DW_{n+3} + (P_{m-5} - P_{m-4} - P_{m-6})DW_{n+2} + (P_{m-4} - P_{m-5})DW_{n+1} - DW_nP_{m-4}) \\ & +(P_{m-4}DW_{n+3} + (P_{m-6} - P_{m-5} - P_{m-7})DW_{n+2} + (P_{m-5} - P_{m-6})DW_{n+1} - DW_nP_{m-5}) \\ & -(P_{m-5}DW_{n+3} + (P_{m-7} - P_{m-6} - P_{m-8})DW_{n+2} + (P_{m-6} - P_{m-7})DW_{n+1} - DW_nP_{m-6}). \end{aligned}$$

Consequently, by mathematical induction on m, this proves Theorem 10.

The other cases of m, n can be proved smilarly for all integers m, n. □

Taking $DW_n = DP_n$ or $DW_n = DS_n$ in above Theorem, respectively, we get:

Corollary 11.

$$\begin{aligned} DP_{m+n} &= P_{m-2}DP_{n+3} + (P_{m-4} - P_{m-3} - P_{m-5})DP_{n+2} + (P_{m-3} - P_{m-4})DP_{n+1} - DP_nP_{m-3}, \\ DS_{m+n} &= P_{m-2}DS_{n+3} + (P_{m-4} - P_{m-3} - P_{m-5})DS_{n+2} + (P_{m-3} - P_{m-4})DS_{n+1} - DS_nP_{m-3}. \end{aligned}$$

4. SIMSON'S FORMULA

In this section, we present Simpson's formula for the dual generalized Pandita numbers . This is a special case of [16, Theorem 4.1].

Theorem 12. *(Simpson's formula for dual generalized Pandita numbers) For all integers n we have,*

$$\begin{vmatrix} DW_{n+3} & DW_{n+2} & DW_{n+1} & DW_n \\ DW_{n+2} & DW_{n+1} & DW_n & DW_{n-1} \\ DW_{n+1} & DW_n & DW_{n-1} & DW_{n-2} \\ DW_n & DW_{n-1} & DW_{n-2} & DW_{n-3} \end{vmatrix} = \begin{vmatrix} DW_3 & DW_2 & DW_1 & DW_0 \\ DW_2 & DW_1 & DW_0 & DW_{-1} \\ DW_1 & DW_0 & DW_{-1} & DW_{-2} \\ DW_0 & DW_{-1} & DW_{-2} & DW_{-3} \end{vmatrix}$$

$= (DW_0 + DW_2 - DW_3)(-DW_3^3 + 3DW_2^3 - DW_1^3 + DW_0^3 + (5DW_2 - 2DW_1)DW_3^2 + (4DW_0 - 5DW_1 - 8DW_3)DW_2^2 + (4DW_0 + 4DW_2 - 5DW_3)DW_1^2$

$+(DW_2 - 3DW_1 - DW_3)DW_0^2 + 9DW_1DW_2DW_3 - 3DW_0DW_2DW_3 + 5DW_0DW_1DW_3 - 7DW_0DW_1DW_2)$

.

Proof. Using Theorem 3 it can be proved by using induction use [16, Theorem 4.1]. □

From the Theorem 12, we get the following Corollary.

Corollary 13. *For all integers n, the Simson's formulas of dual Pandita numbers and dual Pandita Lucas numbers are given as,*

a):

$$\begin{aligned} n &= 0 \\ \begin{vmatrix} DP_{n+3} & DP_{n+2} & DP_{n+1} & DP_n \\ DP_{n+2} & DP_{n+1} & DP_n & DP_{n-1} \\ DP_{n+1} & DP_n & DP_{n-1} & DP_{n-2} \\ DP_n & DP_{n-1} & DP_{n-2} & DP_{n-3} \end{vmatrix} &= \begin{vmatrix} DP_3 & DP_2 & DP_1 & DP_0 \\ DP_2 & DP_1 & DP_0 & DP_{-1} \\ DP_1 & DP_0 & DP_{-1} & DP_{-2} \\ DP_0 & DP_{-1} & DP_{-2} & DP_{-3} \end{vmatrix} \\ &= \begin{vmatrix} 5\varepsilon+3 & 3\varepsilon+2 & 2\varepsilon+1 & \varepsilon \\ 3\varepsilon+2 & 2\varepsilon+1 & \varepsilon & 0 \\ 2\varepsilon+1 & \varepsilon & 0 & 0 \\ \varepsilon & 0 & 0 & -1 \end{vmatrix} = \varepsilon^4 + \varepsilon^3 + \varepsilon^2 + 2\varepsilon + 1 \\ &= 2\varepsilon + 1 \end{aligned}$$

b):

$$\begin{aligned} n &= 0 \\ \begin{vmatrix} DS_{n+3} & DS_{n+2} & DS_{n+1} & DS_{n} \\ DS_{n+2} & DS_{n+1} & DS_{n} & DS_{n-1} \\ DS_{n+1} & DS_{n} & DS_{n-1} & DS_{n-2} \\ DS_{n} & DS_{n-1} & DS_{n-2} & DS_{n-3} \end{vmatrix} &= \begin{vmatrix} DS_{3} & DS_{2} & DS_{1} & DS_{0} \\ DS_{2} & DS_{1} & DS_{0} & DS_{-1} \\ DS_{1} & DS_{0} & DS_{-1} & DS_{-2} \\ DS_{0} & DS_{-1} & DS_{-2} & DS_{-3} \end{vmatrix} \\ &= \begin{vmatrix} 6\varepsilon+5 & 5\varepsilon+2 & 2\varepsilon+2 & 2\varepsilon+4 \\ 5\varepsilon+2 & 2\varepsilon+2 & 2\varepsilon+4 & 4\varepsilon+1 \\ 2\varepsilon+2 & 2\varepsilon+4 & 4\varepsilon+1 & \varepsilon-1 \\ 2\varepsilon+4 & 4\varepsilon+1 & \varepsilon-1 & -\varepsilon+4 \end{vmatrix} \\ &= -31\varepsilon^4-31\varepsilon^3-31\varepsilon^2-62\varepsilon-31 \\ &= -62\varepsilon-31. \end{aligned}$$

respectively.

5. Linear Sums

In this section, we give the summation formulas of the dual generalized Pandita numbers with positive and negative subscripts.

Now, we present the summation formulas of the generalized Pandita numbers.

Theorem 14. *For the generalized Pandita numbers, we have the following formulas:*

(a): $\sum\limits_{k=0}^{n} W_k = -(n+3)W_{n+3}+(n+4)W_{n+2}+(n+4)W_n+3W_3-4W_2-3W_0.$

(b): $\sum\limits_{k=0}^{n} W_{2k} = \frac{1}{3}(-3(n+2)W_{2n+2}+(3n+8)W_{2n+1}+2W_{2n}+(3n+7)W_{2n-1}+7W_3-8W_2-W_1-6W_0).$

(c): $\sum\limits_{k=0}^{n} W_{2k+1} = \frac{1}{3}(-(3n+4)W_{2n+2}+(3n+8)W_{2n+1}+W_{2n}+3(n+2)W_{2n-1}+6W_3-8W_2+W_1-7W_0).$

Proof. For the proof, see Soykan [14, Theorem 3.12]. □

Theorem 15. *For the dual Pandita numbers, we have the following formulas:*

(a): $\sum\limits_{k=0}^{n} DW_k = -(n+3)DW_{n+3}+(n+4)DW_{n+2}+(n+4)DW_n+3DW_3-4DW_2-3DW_0.$

(b): $\sum\limits_{k=0}^{n} DW_{2k} = \frac{1}{3}(-3(n+2)DW_{2n+2}+(3n+8)DW_{2n+1}+2DW_{2n}+(3n+7)DW_{2n-1}+7DW_3-8DW_2-DW_1-6DW_0).$

(c): $\sum\limits_{k=0}^{n} DW_{2k+1} = \frac{1}{3}(-(3n+4)DW_{2n+2}+(3n+8)DW_{2n+1}+DW_{2n}+3(n+2)DW_{2n-1}+6DW_3-8DW_2+DW_1-7DW_0).$

Proof. Use Theorem 14 and the definition of DW_n. □

As a special case of the theorem 15, we present the following Corollary.

Corollary 16. *For $n \geq 0$, dual Pandita numbers have the following properties:*

(a): $\sum\limits_{k=0}^{n} DW_k = -(n+3)DW_{n+3} + (n+4)DW_{n+2} + (n+4)DW_n + 1.$

(b): $\sum\limits_{k=0}^{n} DW_{2k} = \frac{1}{3}(-3(n+2)DW_{2n+2} + (3n+8)DW_{2n+1} + 2DW_{2n} + (3n+7)DW_{2n-1} + 3\varepsilon 4).$

(c): $\sum\limits_{k=0}^{n} DW_{2k+1} = \frac{1}{3}(-(3n+4)DW_{2n+2} + (3n+8)DW_{2n+1} + DW_{2n} + 3(n+2)DW_{2n-1} + \varepsilon + 3).$

Corollary 17. *For $n \geq 0$, dual Pandita Lucas numbers have the following properties.*

(a): $\sum\limits_{k=0}^{n} DS_k = -(n+3)DS_{n+3} + (n+4)DS_{n+2} + (n+4)DS_n - 8\varepsilon - 5.$

(b): $\sum\limits_{k=0}^{n} DS_{2k} = \frac{1}{3}(-3(n+2)DS_{2n+2} + (3n+8)DS_{2n+1} + 2DS_{2n} + (3n+7)DS_{2n-1} + -12\varepsilon - 7).$

(c): $\sum\limits_{k=0}^{n} DS_{2k+1} = \frac{1}{3}(-(3n+4)DS_{2n+2} + (3n+8)DS_{2n+1} + DS_{2n} + 3(n+2)DS_{2n-1} + -16\varepsilon - 12).$

Next, we give the ordinary generating functions of some special cases of dual generalized Pandita numbers.

Theorem 18. *The ordinary generating functions of the sequences DW_{2n}, DW_{2n+1} are given as follows:*

(a): $\sum_{n=0}^{\infty} DW_{2n}x^n = \dfrac{DW_2\left(x^3+3x^2-x\right) + DW_0\left(2x^2+2x-1\right) - DW_1\left(x^2-x^3\right) - DW_3\left(x^3+2x^2\right)}{-x^4-x^3+x^2+2x-1}.$

(b): $\sum_{n=0}^{\infty} DW_{2n+1}x^n = \dfrac{DW_0\left(x^3+2x^2\right) - DW_3\left(x^3+x^2+x\right) - DW_1\left(x^3-2x+1\right) + DW_2\left(2x^3+x^2\right)}{-x^4-x^3+x^2+2x-1}.$

Proof. Similary, the proof can be constructed as in [4, Theorem 4].

From the last Theorem, we have the following Corollary which gives sum formula of dual Pandita numbers (Take $DW_n = DP_n$ whit $DP_0 = \varepsilon$,$DP_1 = 2\varepsilon + 1, DP_2 = 3\varepsilon + 2, DP_3 = 5\varepsilon + 3$)

Corollary 19. *For $n \geq 0$ dual Pandita numbers have the following properties.*

(a): $\sum_{n=0}^{\infty} DP_{2n}x^n = \dfrac{-x^2(1+\varepsilon) - x(2+\varepsilon) - \varepsilon}{1-2x+x^2-x^3+x^4},$

(b): $\sum_{n=0}^{\infty} DW_{2n+1}x^n = \dfrac{-x(1+\varepsilon) - x^2 - 2\varepsilon - 1}{1-2x+x^2-x^3+x^4}.$

6. Matrices related with dual Generalized Pandita Numbers

In this sectiion, using dual Pandita numbers, we give some matrices related to dual Pandita numbers. We define the square matrix A of order 4 as

$$A = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

uch that $detA = 1$. Note that

$$A^n = \begin{pmatrix} P_{n+1} & -P_n + P_{n-1} - P_{n-2} & P_n - P_{n-1} & -P_n \\ P_n & -P_{n-1} + P_{n-2} - P_{n-3} & P_{n-1} - P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-2} + P_{n-3} - P_{n-4} & P_{n-2} - P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-3} + P_{n-4} - P_{n-5} & P_{n-3} - P_{n-4} & -P_{n-3} \end{pmatrix}$$

for the proof see [15].

Then we give the following lemma.

Lemma 20. *For $n \geq 0$ the following identitiy is true:*

$$\begin{pmatrix} DW_{n+3} \\ DW_{n+2} \\ DW_{n+1} \\ DW_n \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}.$$

Proof. The identitiy(20) can be proved by mathematical induction on n. If $n = 0$ we obtain

$$\begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$. Thus the following identitiy is true

$$\begin{pmatrix} DW_{k+3} \\ DW_{k+2} \\ DW_{k+1} \\ DW_k \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}.$$

For $n=k+1$, we get

$$\begin{aligned}
\begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix} &= \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} DW_{k+3} \\ DW_{k+2} \\ DW_{k+1} \\ DW_k \end{pmatrix} \\
&= \begin{pmatrix} DW_{k+4} \\ DW_{k+3} \\ DW_{k+2} \\ DW_{k+1} \end{pmatrix}.
\end{aligned}$$

Consequently, by mathematical induction on n, the proof completed. $\square$

We define

$$N_{DW} = \begin{pmatrix} DW_3 & DW_2 & DW_1 & DW_0 \\ DW_2 & DW_1 & DW_0 & DW_{-1} \\ DW_1 & DW_0 & DW_{-1} & DW_{-2} \\ DW_0 & DW_{-1} & DW_{-2} & DW_{-3} \end{pmatrix}, \tag{6.1}$$

$$E_{DW} = \begin{pmatrix} DW_{n+3} & DW_{n+2} & DW_{n+1} & DW_n \\ DW_{n+2} & DW_{n+1} & DW_n & DW_{n-1} \\ DW_{n+1} & DW_n & DW_{n-1} & DW_{n-2} \\ DW_n & DW_{n-1} & DW_{n-2} & DW_{n-3} \end{pmatrix}. \tag{6.2}$$

Now, we have the following theorem with N_{DW} and E_{DW}

THEOREM 21. *Using N_{DW} and E_{DW} , we get*

$$A^n N_{DW} = E_{DW}.$$

Proof. Note that we get

$$\begin{aligned} A^{n}N_{DW} &= \begin{pmatrix} P_{n+1} & -P_{n}+P_{n-1}-P_{n-2} & P_{n}-P_{n-1} & -P_{n} \\ P_{n} & -P_{n-1}+P_{n-2}-P_{n-3} & P_{n-1}-P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-2}+P_{n-3}-P_{n-4} & P_{n-2}-P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-3}+P_{n-4}-P_{n-5} & P_{n-3}-P_{n-4} & -P_{n-3} \end{pmatrix} \begin{pmatrix} DW_{3} & DW_{2} & DW_{1} & DW_{0} \\ DW_{2} & DW_{1} & DW_{0} & DW_{-1} \\ DW_{1} & DW_{0} & DW_{-1} & DW_{-2} \\ DW_{0} & DW_{-1} & DW_{-2} & DW_{-3} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned}
a_{11} &= DW_{1}\left(P_{n}-P_{n-1}\right)-DW_{2}\left(P_{n}-P_{n-1}+P_{n-2}\right)-DW_{0}P_{n}+W_{3}P_{n+1}=DW_{n+3}, \\
a_{12} &= DW_{0}\left(P_{n}-P_{n-1}\right)-DW_{1}\left(P_{n}-P_{n-1}+P_{n-2}\right)-P_{n}DW_{-1}+DW_{2}P_{n+1}=DW_{n+2}, \\
a_{13} &= DW_{-1}\left(P_{n}-P_{n-1}\right)-DW_{0}\left(P_{n}-P_{n-1}+P_{n-2}\right)-P_{n}DW_{-2}+DW_{1}P_{n+1}=DW_{n+1}, \\
a_{14} &= DW_{-2}\left(P_{n}-P_{n-1}\right)-DW_{-1}\left(P_{n}-P_{n-1}+P_{n-2}\right)-P_{n}DW_{-3}+DW_{0}P_{n+1}=DW_{n}, \\
a_{21} &= DW_{3}P_{n}-DW_{2}\left(P_{n-1}-P_{n-2}+P_{n-3}\right)+DW\left(P_{n-1}-P_{n-2}\right)-DW_{0}P_{n-1}=DW_{n+2}, \\
a_{22} &= DW_{2}P_{n}-DW_{-1}P_{n-1}-DW_{1}\left(P_{n-1}-P_{n-2}+P_{n-3}\right)+DW\left(P_{n-1}-P_{n-2}\right)=DW_{n+1}, \\
a_{23} &= DW_{-1}\left(P_{n-1}-P_{n-2}\right)-DW_{-2}P_{n-1}+DW_{1}P_{n}-DW_{0}\left(P_{n-1}-P_{n-2}+P_{n-3}\right)=DW_{n}, \\
a_{24} &= DW_{-2}\left(P_{n-1}-P_{n-2}\right)-DW_{-3}P_{n-1}+DW_{0}P_{n}-DW_{-1}\left(P_{n-1}-P_{n-2}+P_{n-3}\right)=DW_{n-1}, \\
a_{31} &= DW_{1}\left(P_{n-2}-P_{n-3}\right)-DW_{2}\left(P_{n-2}-P_{n-3}+P_{n-4}\right)-DW_{0}P_{n-2}+DW_{3}P_{n-1}=DW_{n+1}, \\
a_{32} &= DW_{0}\left(P_{n-2}-P_{n-3}\right)-DW_{1}\left(P_{n-2}-P_{n-3}+P_{n-4}\right)-DW_{-1}P_{n-2}+DW_{2}P_{n-1}=DW_{n}, \\
a_{33} &= DW_{-1}\left(P_{n-2}-P_{n-3}\right)-DW_{-2}P_{n-2}-DW_{0}\left(P_{n-2}-P_{n-3}+P_{n-4}\right)+DW_{1}P_{n-1}=DW_{n-1}, \\
a_{34} &= DW_{-2}\left(P_{n-2}-P_{n-3}\right)-DW_{-3}P_{n-2}-DW_{-1}\left(P_{n-2}-P_{n-3}+P_{n-4}\right)+DW_{0}P_{n-1}=DW_{n-2}, \\
a_{41} &= DW_{1}\left(P_{n-3}-P_{n-4}\right)-DW_{2}\left(P_{n-3}-P_{n-4}+P_{n-5}\right)-DW_{0}P_{n-3}+DW_{3}P_{n-2}=DW_{n}, \\
a_{42} &= DW_{0}\left(P_{n-3}-P_{n-4}\right)-DW_{1}\left(P_{n-3}-P_{n-4}+P_{n-5}\right)-DW_{-1}P_{n-3}+DW_{2}P_{n-2}=DW_{n-1}, \\
a_{43} &= DW_{-1}\left(P_{n-3}-P_{n-4}\right)-DW_{-2}P_{n-3}-DW_{0}\left(P_{n-3}-P_{n-4}+P_{n-5}\right)+DW_{1}P_{n-2}=DW_{n-2}, \\
a_{44} &= DW_{-2}\left(P_{n-3}-P_{n-4}\right)-DW_{-3}P_{n-3}-DW_{-1}\left(P_{n-3}-P_{n-4}+P_{n-5}\right)+DW_{0}P_{n-2}=DW_{n-3}.
\end{aligned}$$

Using the theorem (10) the proof is done. □

By taking DW_n =DP_n with DP_0, DP_1, DP_2, DP_3 in (6.1) and (6.2)

DW_n =S_n with DS_0, DS_1, DS_2, DS_3 in (6.1) and (6.2) respectively, we get:

$$
\begin{aligned}
N_{DP} &= \begin{pmatrix} 5j+8\varepsilon+12j\varepsilon+3 & 3j+5\varepsilon+8j\varepsilon+2 & 2j+3\varepsilon+5j\varepsilon+1 & j+2\varepsilon+3j\varepsilon \\ 3j+5\varepsilon+8j\varepsilon+2 & 2j+3\varepsilon+5j\varepsilon+1 & j+2\varepsilon+3j\varepsilon & \varepsilon+2j\varepsilon \\ 2j+3\varepsilon+5j\varepsilon+1 & j+2\varepsilon+3j\varepsilon & \varepsilon+2j\varepsilon & -j\varepsilon \\ j+2\varepsilon+3j\varepsilon & \varepsilon+2j\varepsilon & -j\varepsilon & -1 \end{pmatrix}, \\
E_{DP} &= \begin{pmatrix} DP_{n+3} & DP_{n+2} & DP_{n+1} & DP_n \\ DP_{n+2} & DP_{n+1} & DP_n & DP_{n-1} \\ DP_{n+1} & DP_n & DP_{n-1} & DP_{n-2} \\ DP_n & DP_{n-1} & DP_{n-2} & DP_{n-3} \end{pmatrix}, \\
N_{DS} &= \begin{pmatrix} 6j+7\varepsilon+11j\varepsilon+5 & 5j+6\varepsilon+7j\varepsilon+2 & 2j+5\varepsilon+6j\varepsilon+2 & 2j+2\varepsilon+5j\varepsilon+4 \\ 5j+6\varepsilon+7j\varepsilon+2 & 2j+5\varepsilon+6j\varepsilon+2 & 2j+2\varepsilon+5j\varepsilon+4 & 4j+2\varepsilon+2j\varepsilon+1 \\ 2j+5\varepsilon+6j\varepsilon+2 & 2j+2\varepsilon+5j\varepsilon+4 & -4j+2\varepsilon+2j\varepsilon+1 & j+4\varepsilon+2j\varepsilon-1 \\ 2j+2\varepsilon+5j\varepsilon+4 & -4j+2\varepsilon+2j\varepsilon+1 & j+4\varepsilon+2j\varepsilon-1 & \varepsilon-j+4j\varepsilon+4 \end{pmatrix}, \\
E_{DS} &= \begin{pmatrix} DS_{n+3} & DS_{n+2} & DS_{n+1} & DS_n \\ DS_{n+2} & DS_{n+1} & DS_n & DS_{n-1} \\ DS_{n+1} & S_n & DS_{n-1} & DS_{n-2} \\ DS_n & DS_{n-1} & DS_{n-2} & DS_{n-3} \end{pmatrix}.
\end{aligned}
$$

From Theorem (21), we can write the following corollary.

Corollary 22. *The following identities are hold:*

- **a):** $A^n N_{DP} = E_{DP}$.
- **b):** $A^n N_{DS} = E_{DS}$.

7. Conclusions

Recurrence relations define sequences where each term depends on previous ones. These sequences such as Fibonacci, Pell, Jacobsthal, Tribonacci, Padovan, Narayana's Cows, Leonardo, Tetranacci, and Pentanacci arise across fields including engineering, biology, mathematics, and physics. Below, we present their definitions with initial conditions using A_n notation and outline their real-world relevance.

- **Fibonacci Sequence:**

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, \quad F_1 = 1$$

- **Pell Sequence:**

$$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 0, \quad P_1 = 1$$

- **Jacobsthal Sequence:**

$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 0, \quad A_1 = 1$$

- **Tribonacci Sequence:**

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \quad T_0 = 0, \quad T_1 = 1, \quad T_2 = 1$$

- **Padovan Sequence:**

$$P_n = P_{n-2} + P_{n-3}, \quad P_0 = P_1 = P_2 = 1$$

- **Narayana's Cows Sequence:**

$$N_n = N_{n-1} + N_{n-3}, \quad N_0 = N_1 = N_2 = 1$$

- **Leonardo Sequence:**

$$L_n = L_{n-1} + L_{n-2} + 1, \quad L_0 = 1, \quad L_1 = 1$$

- **Tetranacci Sequence:**

$$M_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4}, \quad M_0 = M_1 = M_2 = 0, \quad M_3 = 1$$

- **Pentanacci Sequence:**

$$P_n = P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5}, \quad P_0 = P_1 = P_2 = P_3 = 0, \quad P_4 = 1$$

These sequences demonstrate how mathematical recursions extend into the fabric of our world whether designing structures, analyzing algorithms, modeling nature, or probing the quantum realm. Their recursive beauty continues to inspire both theoretical and practical exploration.

Next, we explore several real-world applications of recurrence relations across disciplines.

- Engineering
  - **Fibonacci:** Models recursive filters in control systems and signal processing.
  - **Padovan and Perrin:** Guide architectural proportions using the plastic number.
  - **Jacobsthal:** Applied in digital circuits for counting and encoding.
- Science
  - **Tribonacci and Tetranacci:** Simulate biological systems with delayed reproduction.
  - **Leonardo:** Reflect branching in plants and trees.
  - **Fibonacci and Narayana's Cows:** Describe phyllotaxis and seed arrangement in botany.
- Mathematics
  - **Recurrence Relations:** Analyze algorithms like mergesort and quicksort.
  - **Pell:** Solve Diophantine equations and approximate square roots with continued fractions.
  - **Jacobsthal and Padovan:** Used in tiling and combinatorics problems.
- Physics

  - **Fibonacci and Tribonacci:** Appear in wave interference and quantum systems.
  - **Pentanacci:** Used in recursive models of particle interactions and fractals.
  - **Padovan:** Linked to equilibrium modeling via the plastic constant.

In this study, we extend the classical framework to fourth-order recurrence systems by introducing the dual Pandita numbers, along with two distinguished subclasses. For these novel sequences, we derive Binet-type formulas, ordinary and exponential generating functions, and generalized Simson-type identities. Our analysis also encompasses closed-form summation formulas, algebraic properties, recurrence behaviors, and matrix-based representations.

Recognizing the theoretical depth and real-world utility of recurrence-based sequences, we first revisit the applications of second-order sequences to establish context. We then position our fourth-order generalizations as a natural progression within this broader mathematical landscape—offering new insights and powerful tools for modeling, analysis, and optimization in both pure and applied settings.

- For a detailed discussion on Gaussian Fibonacci and Gaussian Lucas numbers applied to Pauli Fibonacci and Pauli Lucas quaternions, see [19].
- For an in-depth analysis of Pell numbers in solving three-dimensional difference equation systems, refer to [21].
- For a study on Jacobsthal numbers and their role in special matrices, see [29].
- For a comprehensive analysis of generalized k-order Fibonacci numbers in hybrid quaternions, see [30].
- For the application of Fibonacci and Lucas numbers to Split Complex Bi-Periodic numbers, see [63].
- For the use of generalized bivariate Fibonacci and Lucas polynomials in matrix polynomials, see [64].
- For the application of generalized Fibonacci numbers to binomial sums, see [31].
- For the role of generalized Jacobsthal numbers in hyperbolic numbers, see [32].
- For the application of generalized Fibonacci numbers to dual hyperbolic numbers, see [48].
- For the use of Laplace transform and matrix operations in the characteristic polynomial of Fibonacci numbers, see [49].
- For the application of generalized Fibonacci matrices in cryptographic systems, see [50].
- For higher-order Jacobsthal numbers applied to quaternion structures, see [51].
- For Fibonacci and Lucas identities in Toeplitz-Hessenberg matrices, see [52].
- For Fibonacci numbers in lacunary statistical convergence, see [22].
- For lacunary statistical convergence in intuitionistic fuzzy normed linear spaces, see [58].
- For ideal convergence in intuitionistic fuzzy normed linear spaces, see [60].
- For the applications of k-Fibonacci and $k-$Lucas numbers to spinors, see [57].

- For dual-generalized complex Fibonacci and Lucas numbers in quaternion structures, see [56].
- For special cases of Horadam numbers in Neutrosophic analysis, see [53].
- For hyperbolic Fibonacci numbers applied to quaternions, see [28].
- For Pell numbers in Gaussian hyperbolic number systems, see [54].

In the following, we explore several applications of third-order recurrence sequences across various mathematical and applied contexts.

- For the applications of third order Jacobsthal numbers and Tribonacci numbers to quaternions, see [24] and [23], respectively.
- For the application of Tribonacci numbers to special matrices, see [62].
- For the applications of Padovan numbers and Tribonacci numbers to coding theory, see [44] and [20], respectively.
- For the application of Pell-Padovan numbers to groups, see [27].
- For the application of adjusted Jacobsthal-Padovan numbers to the exact solutions of some difference equations, see [55].
- For the application of Gaussian Tribonacci numbers to various graphs, see [43].
- For the application of third-order Jacobsthal numbers to hyperbolic numbers, see [25].For the application of Narayan numbers to finite groups see [59].
- For the application of generalized third-order Jacobsthal sequence to binomial transform, see [42].
- For the application of generalized Generalized Padovan numbers to Binomial Transform, see [41].
- For the application of generalized Tribonacci numbers to Gaussian numbers, see [40].
- For the application of generalized Tribonacci numbers to Sedenions, see [39].
- For the application of Tribonacci and Tribonacci-Lucas numbers to matrices, see [33].
- For the application of generalized Tribonacci numbers to circulant matrix, see [34].
- For the application of Tribonacci and Tribonacci-Lucas numbers to hybrinomials, see [61].
- For the application of hyperbolic Leonardo and hyperbolic Francois numbers to quaternions, see [26].

In the following lists, we outline several applications of fourth-order recurrence sequences across theoretical and applied domains.

- For the application of Tetranacci and Tetranacci-Lucas numbers to quaternions, see [36].
- For the application of generalized Tetranacci numbers to Gaussian numbers, see [37].
- For the application of Tetranacci and Tetranacci-Lucas numbers to matrices, see [38].
- For the application of generalized Tetranacci numbers to binomial transform, see [35].

We now explore several applications of fifth-order sequences.

- For the application of Pentanacci numbers to matrices, see [47].
- For the application of generalized Pentanacci numbers to quaternions, see [46].

- For the application of generalized Pentanacci numbers to binomial transform, see [45].

References

[1] D.K. Biss, D. Dugger, D.C. Isaksen, Large annihilators in Cayley-Dickson algebras, Communication in Algebra, 36 (2), 632-664, 2008.

[2] Soykan, Y.Demirci ,F.,Dual Generalized Adrien Numbers ,Journal of Advances in Mathematics and Computer Science,40(10),(2025),48-73.

[3] Soykan, Y.Ayrımaz ,E.A Study On Hyperbolic Generalized Edouard Numbers,Asian Journal of Probability and Statistics, 27 (8), 52-79, 2025.

[4] Nurkan SK, Guven İA. Dual Fibonacci quaternions. Advances in Applied Clifford Algebras. 2015;25:403-414.

[5] P. Fjelstad, S.G. Gal, n-dimensional Hyperbolic Complex Numbers, Advances in Applied Clifford Algebras, 8(1) (1998), 47-68, .

[6] Aydın FT. Dual Jacobsthal quaternions. Communications in Advanced Mathematical Sciences. 2020;III(3):130-142.

[7] I. Kantor, A. Solodovnikov, Hypercomplex Numbers, Springer-Verlag, New York (1989).

[8] Kalca, F. Soykan,Y. A Study on Dual Hyperbolic Generalized Pandita Numbers. Archives of Current Research International, 25(8), 412-436,2025.

[9] M.Göcen,C.M.Dikmen,Y.KayaandY.Soykan,A study on dual generalized Fibonacci matrices,Filomat.39(1) (2025), 41–54. https://doi.org/10.2298/FIL2501041G.

[10] Gürses N, Şentürk GY, Salim Yüce S. A comprehensive survey of dual-generalized complex Fibonacci and Lucas numbers. Sigma J Eng Nat Sci.2022;40(1):179-187.

[11] HalıcıS. On dual Fibonacci octonions. Advances in Applied Clifford Algebras. 2015;25:905-914.

[12] Soykan, Y.Yilmaz,B.A Study on Hyperbolic Generalized Guglielmo Numbers,Asian Research Journal of Mathematics,19(12),(2023), 120-147.

[13] G. Sobczyk, The Hyperbolic Number Plane, The College Mathematics Journal, 26(4) (1995), 268-280.

[14] Soykan, Y., Sums and Generating Functions of Special Cases of Generalized Tetranacci Polynomials, International Journal of Advances in Applied Mathematics and Mechanics, 12(4), 34-101, 2025.

[15] Soykan, Y., Properties of Generalized (r,s,t,u)-Numbers, Earthline Journal of Mathematical Sciences, 5(2), 297-327, 2021. https://doi.org/10.34198/ejms.5221.297327.

[16] Soykan, Y., Simson Identity of Generalized m-step Fibonacci Numbers, Int. J. Adv. Appl. Math. and Mech. 7(2), 45-56, 2019.

[17] Soykan Y., Generalized Pandita Numbers, International Journal of Mathematics, Statistics and Operations Research, 3(1), 107-123, 2023.

[18] Kalça, F.Z, Soykan, Y. Gaussian Numbers with Generalized Pandita Numbers Components, Asian Journal of Advanced Research and Reports, 19(7), 32-56, 2025. https://doi.org/10.9734/ajarr/2025/v19i71079.

[19] Azak, A.Z., Pauli Gaussian Fibonacci and Pauli Gaussian Lucas Quaternions. Mathematics, 2022, 10, 4655. https://doi.org/10.3390/math10244655

[20] Basu, M., Das, M., Tribonacci Matrices and a New Coding Theory, Discrete Mathematics, Algorithms and Applications, 6 (1), 1450008, (17 pages), 2014.

[21] Büyük, H., Taşkara, N., On The Solutions of Three-Dimensional Difference Equation Systems Via Pell Numbers, European Journal of Science and Technology, Special Issue 34, 433-440, 2022.

[22] Bilgin, N.G., Fibonacci Lacunary Statistical Convergence of Order γ in IFNLS, International Journal of Advances in Applied Mathematics and Mechanics, 8(4), 28-36, 2021.

[23] Cerda-Morales, G., On a Generalization of Tribonacci Quaternions, Mediterranean Journal of Mathematics 14:239, 1-12, 2017.

[24] Cerda-Morales, G., Identities for Third Order Jacobsthal Quaternions, Advances in Applied Clifford Algebras 27(2), 1043–1053, 2017.

[25] Dikmen, C.M., Altınsoy, M.,On Third Order Hyperbolic Jacobsthal Numbers, Konuralp Journal of Mathematics, 10(1), 118-126, 2022.

[26] Dişkaya, O., Menken, H., Catarino, P.M.M.C., On the Hyperbolic Leonardo and Hyperbolic Francois Quaternions, Journal of New Theory, 42, 74-85, 2023.

[27] Deveci, Ö., Shannon, A.G., Pell–Padovan-Circulant Sequences and Their Applications, Notes on Number Theory and Discrete Mathematics, 23(3), 100–114, 2017.

[28] Daşdemir, A., On Recursive Hyperbolic Fibonacci Quaternions, Communications in Advanced Mathematical Sciences, 4(4), 198-207, 2021. DOI:10.33434/cams.99782

[29] Vasanthi, S., Sivakumar, B., Jacobsthal Matrices and their Properties. Indian Journal of Science and Technology 15(5): 207-215, 2022, https://doi.org/10.17485/IJST/v15i5.1948

[30] Gül, K., Generalized k-Order Fibonacci Hybrid Quaternions, Erzincan University Journal of Science and Technology, 15(2), 670-683, 2022. DOI: 10.18185/erzifbed.1132164

[31] Ulutaş, Y.T., Toy, D., Some Equalities and Binomial Sums about the Generalized Fibonacci Number u_n, Notes on Number Theory and Discrete Mathematics, 28(2), 252–260, 2022. DOI: 10.7546/nntdm.2022.28.2.252-260

[32] Soykan, Y., Taşdemir, E., A Study On Hyperbolic Numbers With Generalized Jacobsthal Numbers Components, International Journal of Nonlinear Analysis and Applications, 13(2), 1965–1981, 2022. http://dx.doi.org/10.22075/ijnaa.2021.22113.2328

[33] Soykan, Y., Matrix Sequences of Tribonacci and Tribonacci-Lucas Numbers, Communications in Mathematics and Applications, 11(2), 281-295, 2020. DOI: 10.26713/cma.v11i2.1102

[34] Soykan, Y. Explicit Euclidean Norm, Eigenvalues, Spectral Norm and Determinant of Circulant Matrix with the Generalized Tribonacci Numbers, Earthline Journal of Mathematical Sciences, 6(1), 131-151, 2021. https://doi.org/10.34198/ejms.6121.131151

[35] Soykan, Y., On Binomial Transform of the Generalized Tetranacci Sequence, International Journal of Advances in Applied Mathematics and Mechanics, 9(2), 8-27, 2021.

[36] Soykan Y, Tetranacci and Tetranacci-Lucas Quaternions, Asian Research Journal of Mathematics, 15(1): 1-24, 2019; Article no.ARJOM.50749.

[37] Soykan, Y., Gaussian Generalized Tetranacci Numbers, Journal of Advances in Mathematics and Computer Science, 31(3): 1-21, Article no.JAMCS.48063, 2019.

[38] Soykan, Y., Matrix Sequences of Tetranacci and Tetranacci-Lucas Numbers, Int. J. Adv. Appl. Math. and Mech. 7(2), 57-69, 2019, (ISSN: 2347-2529).

[39] Soykan, Y., Okumuş, İ., Taşdemir, E., On Generalized Tribonacci Sedenions, Sarajevo Journal of Mathematics, 16(1), 103-122, 2020. ISSN 2233-1964, DOI: 10.5644/SJM.16.01.08

[40] Soykan, Y., Taşdemir, E., Okumuş, İ., Göcen, M., Gaussian Generalized Tribonacci Numbers, Journal of Progressive Research in Mathematics(JPRM), 14 (2), 2373-2387, 2018.

[41] Soykan, Y., Taşdemir, E., Okumuş, İ., A Study on Binomial Transform of the Generalized Padovan Sequence, Journal of Science and Arts, 22(1), 63-90, 2022. https://doi.org/10.46939/J.Sci.Arts-22.1-a06

[42] Soykan, Y., Taşdemir, E., Göcen, M., Binomial Transform of the Generalized Third-Order Jacobsthal Sequence, Asian-European Journal of Mathematics, 15(12), 2022. https://doi.org/10.1142/S1793557122502242.

[43] Sunitha, K., Sheriba. M., Gaussian Tribonacci R-Graceful Labeling of Some Tree Related Graphs, Ratio Mathematica, 44, 188-196, 2022.

[44] Shtayat, J., Al-Kateeb, A., An Encoding-Decoding algorithm based on Padovan numbers, arXiv:1907.02007, 2019.

[45] Soykan, Y., Binomial Transform of the Generalized Pentanacci Sequence, Asian Research Journal of Current Science, 3(1), 209-231, 2021.

[46] Soykan, Y., Özmen, N., Göcen, M., On Generalized Pentanacci Quaternions, Tbilisi Mathematical Journal, 13(4), 169-181, 2020

[47] Sivakumar, B., James, V., A Notes on Matrix Sequence of Pentanacci Numbers and Pentanacci Cubes, Communications in Mathematics and Applications, 13(2), 603–611, 2022. DOI: 10.26713/cma.v13i2.1725

[48] Soykan, Y., On Dual Hyperbolic Generalized Fibonacci Numbers, Indian J Pure Appl Math, 2021. https://doi.org/10.1007/s13226-021-00128-2

[49] Deveci, Ö., Shannon, A.G., On Recurrence Results From Matrix Transforms, Notes on Number Theory and Discrete Mathematics, 28(4), 589–592, 2022. DOI: 10.7546/nntdm.2022.28.4.589-5924

[50] Prasad, K., Mahato, H., Cryptography Using Generalized Fibonacci Matrices with Affine-Hill Cipher, Journal of Discrete Mathematical Sciences & Cryptography, 25(8-A), 2341–2352, 2022. DOI : 10.1080/09720529.2020.1838744

[51] Özkan, E., Uysal, M., On Quaternions with Higher Order Jacobsthal Numbers Components, Gazi University Journal of Science, 36(1), 336-347, 2023. DOI: 10.35378/gujs. 1002454

[52] Goy, T.,Shattuck, M., Fibonacci and Lucas Identities from Toeplitz-Hessenberg Matrices, Appl. Appl. Math, 14(2), 699–715, 2019.

[53] Gökbaş¸ H., Topal, S., Smarandache, F., Neutrosophic Number Sequences: An Introductory Study, International Journal of Neutrosophic Science (IJNS), 20(01), 27-48, 2023. https://doi.org/10.54216/IJNS.200103

[54] Gökbaş, H. Gaussian-Hybrid Numbers Obtained From Pell and Pell-Lucas Sequences. Journal of Advances in Mathematics and Computer Science, 37(11), 1–9, 2022. https://doi.org/10.9734/jamcs/2022/v37i111719

[55] Göcen, M., The Exact Solutions of Some Difference Equations Associated with Adjusted Jacobsthal-Padovan Numbers, Kırklareli University Journal of Engineering and Science 8(1), 1-14, 2022. DOI: 10.34186/klujes.1078836

[56] Şentürk, G.Y., Gürses, N., Yüce, S., Construction of Dual-Generalized Complex Fibonacci and Lucas Quaternions, Carpathian Math. Publ. 2022, 14 (2), 406-418, 2022. doi:10.15330/cmp.14.2.406-418

[57] Kumari, M., Prasad, K., Frontczak, R., On the k-Fibonacci and $k-$Lucas Spinors, Notes on Number Theory and Discrete Mathematics, 29(2), 322-335, 2023. DOI: 10.7546/nntdm.2023.29.2.322-335

[58] Kişi, Ö., Tuzcuoglu, I., Fibonacci Lacunary Statistical Convergence in Intuitionistic Fuzzy Normed Linear Spaces, Journal of Progressive Research in Mathematics 16(3), 3001-3007, 2020.

[59] Kuloğlu, B., Özkan, E., Shannon, A.G., The Narayana Sequence in Finite Groups, Fibonacci Quarterly. 60(5), 212–221, 2022.

[60] Kişi, Ö., Debnathb, P., Fibonacci Ideal Convergence on Intuitionistic Fuzzy Normed Linear Spaces, Fuzzy Information and Engineering, 1-13, 2022. https://doi.org/10.1080/16168658.2022.2160226

[61] Taşyurdu, Y., Polat, Y.E., Tribonacci and Tribonacci-Lucas Hybrinomials, Journal of Mathematics Research; 13(5), 2021.

[62] Yilmaz, N., Taskara, N., Tribonacci and Tribonacci-Lucas Numbers via the Determinants of Special Matrices, Applied Mathematical Sciences, 8(39), 1947-1955, 2014.

[63] Yılmaz, N., Split Complex Bi-Periodic Fibonacci and Lucas Numbers, Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat. 71(1), 153–164, 2022. DOI:10.31801/cfsuasmas.704435

[64] Yılmaz, N., The Generalized Bivariate Fibonacci and Lucas Matrix Polynomials, Mathematica Montisnigri, Vol LIII, 33-44, 2022. DOI: 10.20948/mathmontis-2022-53-5