Expansive-Type Fixed Point Theorems in Double Controlled Metric Type Spaces with an Integral Equation Application

Abstract

In this paper, we establish a new expansive fixed point theory in the framework of double controlled metric type spaces, where the distance structure is governed by two control functions. By combining surjectivity with an inverse–iteration technique, we derive a unified principle that ensures the existence and, under suitable strictness conditions, uniqueness of fixed points for expansive–type self–mappings.

Some new expansive classes are introduced: Reich (a, b)–expansive, Dass–Gupta rational expansive, Θ–weighted expansive, orbitally localized expansive, and (β, μ)–mixed expansive mappings. For each class, a corresponding fixed point theorem is proved with detailed arguments, and each result is supported by a distinct example constructed on a different double controlled metric type space, demonstrating the novelty and independence of the hypotheses.

As an application, we investigate a nonlinear Fredholm integral equation by rewriting it as a fixed point problem in a function space and verifying an expansive condition on the associated integral operator. This shows that expansive principles, when combined with surjectivity and orbital control, can guarantee solvability beyond the scope of classical contraction methods.

Keywords: double controlled metric type space; expansive mapping; surjective expansion; inverse iteration; fixed point; integral equation.
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1 Introduction

Fixed point theory has played a central role in nonlinear analysis since the pioneering works of Brouwer [2] and Banach [1]. During the past decades, numerous generalizations of Banach's contraction principle have been developed in order to treat nonlinear problems arising in differential equations, integral equations, optimization, and dynamical systems. Classical extensions include Kannan [4], Reich [5], Ćirić [6, 7], Boyd–Wong [3], and Dass–Gupta rational contractions [8]. Later, these ideas were adapted to various generalized metric structures such as b-metric spaces [10, 11], generalized metric spaces [12], fuzzy and intuitionistic fuzzy metric spaces [14, 13, 15], and multi-metric frameworks [16, 17, 18, 19, 20].

In contrast to contractions, *expansive mappings* satisfy inequalities of the form

$$d(\mathscr{T}x,\mathscr{T}y)\geq d(x,y),\qquad x,y\in\mathscr{X},$$

and represent a structurally rigid counterpart of contractive maps. Expansive operators have been investigated in classical metric spaces [21, 32] as well as in generalized frameworks such as S-metric spaces [22], parametric metric spaces [26], and 2-Banach spaces [25]. Unlike contractions, expansive mappings do not automatically possess fixed points, and additional hypotheses—such as surjectivity, inverse iteration, or hybrid regularity conditions—are usually required to ensure solvability.

Recently, a new class of generalized distance structures known as *controlled metric type spaces* was introduced in [29], where the triangle inequality is modulated by a control function. This concept was further refined by Abdeljawad *et al.* [30], who introduced *double controlled metric type spaces* governed by two (generally noncomparable) control functions. These spaces unify and extend several well-known generalized metrics and have already proved useful in developing contraction-type fixed point results [31, 27, 28]. However, despite the growing literature on contractions in such spaces, a systematic expansive fixed point theory in the double controlled setting has not yet been established.

The aim of this paper is to fill this gap by developing a comprehensive framework for *expansive-type fixed point theorems* in double controlled metric type spaces. Inspired by the classical theory of expansions [21, 32] and by inverse-iteration techniques for surjective expansive maps in generalized spaces [26], we propose five new expansive classes adapted to the double controlled structure. For each class, we prove a fixed point theorem using a unified backward-orbit method and provide a distinct example showing the sharpness and independence of the hypotheses.

As an application, we study a nonlinear Fredholm integral equation by rewriting it as a fixed point problem in a function space and verifying a suitable expansive condition on the associated operator. This demonstrates that expansive principles, when combined with surjectivity and orbital control, can complement the classical contraction-based approaches commonly used in the analysis of integral equations [35, 36].

2 Preliminaries

Throughout this paper, $\mathscr{X}$ denotes a nonempty set, $\alpha,\gamma,\xi\in\mathscr{X}$ denote elements, and $\mathscr{T},\mathscr{G},\mathscr{S}$ denote self-maps. We write $\mathbb{N}=\{0,1,2,\dots\}$. The background on double controlled metric type spaces follows the modern development in [30, 31], while the expansive-mapping motivation and inverse-iteration viewpoint are consistent with the expansive literature in [32, 26].

Definition 2.1 (Double controlled metric type space [30, 31]). Let $\beta,\mu:\mathscr{X}\times\mathscr{X}\to[1,\infty)$ be two control functions. A mapping $d_{\beta,\mu}:\mathscr{X}\times\mathscr{X}\to[0,\infty)$ is called a *double controlled metric type* (briefly, *DCMTS distance*) if for all $\alpha,\gamma,\xi\in\mathscr{X}$ the following conditions hold:

(DC1) $d_{\beta,\mu}(\alpha,\gamma)=0$ if and only if $\alpha=\gamma$;

(DC2) $d_{\beta,\mu}(\alpha,\gamma)=d_{\beta,\mu}(\gamma,\alpha)$;

(DC3) (double controlled triangle inequality)

$$d_{\beta,\mu}(\alpha,\gamma)\leq\beta(\alpha,\xi)\,d_{\beta,\mu}(\alpha,\xi)+\mu(\xi,\gamma)\,d_{\beta,\mu}(\xi,\gamma).$$

The pair $(\mathscr{X},d_{\beta,\mu})$ is called a *double controlled metric type space.*

Definition 2.2 (Convergence, Cauchy sequence, completeness)**.** Let $(\mathscr{X}, d_{\beta,\mu})$ be a DCMTS and let $\{z_n\} \subset \mathscr{X}$.

(C1) $\{z_n\}$ converges to $z \in \mathscr{X}$, written $z_n \to z$, if $d_{\beta,\mu}(z_n, z) \to 0$ as $n \to \infty$.

(C2) $\{z_n\}$ is a Cauchy sequence if $d_{\beta,\mu}(z_n, z_m) \to 0$ as $n, m \to \infty$.

(C3) $(\mathscr{X}, d_{\beta,\mu})$ is complete if every Cauchy sequence converges in $\mathscr{X}$.

Expansive mappings (expansions) are usually characterized by inequalities of the form $d(\mathscr{T}x, \mathscr{T}y) \geq d(x, y)$, and their behavior can be rigid on compact or totally bounded sets (typically forcing isometric behavior under additional hypotheses); see [32]. Because expansiveness alone rarely yields fixed points, a standard strategy is to add *surjectivity* and construct a *right inverse* to generate a backward orbit whose increments decay, a method used effectively in parametric metric settings [**?**] and related generalized frameworks [26]. We adopt this guiding mechanism in the DCMTS context [31] as follows.

Definition 2.3 (Right inverse and backward orbit)**.** Let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be surjective. A map $\mathscr{G} : \mathscr{X} \to \mathscr{X}$ is called a *right inverse* of $\mathscr{T}$ if

$$\mathscr{T}(\mathscr{G}z) = z, \qquad \forall z \in \mathscr{X}.$$

Given $z_0 \in \mathscr{X}$, the associated *backward orbit* (or inverse-iteration sequence) is defined by

$$z_{n+1} := \mathscr{G}z_n, \qquad n \in \mathbb{N}.$$

The key technical point is that, in a DCMTS, controlling step sizes $d_{\beta,\mu}(z_{n+1}, z_n)$ is not sufficient by itself; one must also prevent blow-up of the control factors along the orbit to chain inequalities via the double controlled triangle rule. The following lemma provides a standard and reusable Cauchy criterion in our setting.

Lemma 2.4. *Let $(\mathscr{X}, d_{\beta,\mu})$ be a DCMTS and let $\{z_n\} \subset \mathscr{X}$ satisfy*

$$d_{\beta,\mu}(z_{n+1}, z_n) \leq \kappa^n \, d_{\beta,\mu}(z_1, z_0) \quad \text{for all } n \in \mathbb{N} \tag{1}$$

for some $\kappa \in (0, 1)$. Assume there exists a constant $M \geq 1$ such that

$$\sup_{m>n} \Big(\beta(z_n, z_{n+1}) \, \mu(z_{n+1}, z_m) \Big) \leq M \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

Then $\{z_n\}$ is a Cauchy sequence in $(\mathscr{X}, d_{\beta,\mu})$.

Proof. Fix integers $m > n$. Apply **(DC3)** of Definition 2.1 with $(\alpha, \xi, \gamma) = (z_n, z_{n+1}, z_m)$:

$$d_{\beta,\mu}(z_n, z_m) \leq \beta(z_n, z_{n+1}) \, d_{\beta,\mu}(z_n, z_{n+1}) + \mu(z_{n+1}, z_m) \, d_{\beta,\mu}(z_{n+1}, z_m).$$

Iterate the same estimate along the chain $z_n, z_{n+1}, \ldots, z_m$ to obtain the telescoping-type bound

$$d_{\beta,\mu}(z_n, z_m) \leq \sum_{j=n}^{m-1} \Big(\beta(z_j, z_{j+1}) \, \mu(z_{j+1}, z_m) \Big) \, d_{\beta,\mu}(z_{j+1}, z_j).$$

Using (2) and (1), we get

$$d_{\beta,\mu}(z_n, z_m) \leq M \sum_{j=n}^{m-1} \kappa^j \, d_{\beta,\mu}(z_1, z_0) \leq M \frac{\kappa^n}{1 - \kappa} \, d_{\beta,\mu}(z_1, z_0) \xrightarrow[n \to \infty]{} 0,$$

uniformly in $m > n$. Hence $\{z_n\}$ is Cauchy. □

Remark 2.5. Condition (2) is natural in the double controlled framework: it guarantees that the orbitwise products of the control functions do not diverge, so repeated use of the double controlled triangle inequality remains effective. Such orbitwise boundedness assumptions (in various equivalent forms) are standard when proving convergence of Picard-type or inverse-iteration sequences in controlled and double controlled metric type spaces; see, e.g., [31].

3 Expansive Mappings

Throughout this section, $(\mathscr{X}, d_{\beta,\mu})$ denotes a complete double controlled metric type space in the sense of Definition 2.1, and $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ denotes a self-map. All parameters $a, b, \lambda, \eta, \rho, L$ are real and strictly positive whenever required. The following five expansive conditions are designed to unify and extend several classical expansive principles (Reich type, Dass–Gupta rational type, Θ–type, orbital expansive type, and mixed expansive type) within the double controlled framework [8, 23, 32, 31].

Definition 3.1 (Reich (a, b)–expansive mapping)**.** A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is said to be *Reich (a, b)–expansive* if there exist constants $a \geq 0$ and $b \geq 0$ with

$$a + 2b > 1$$

such that for all $x \neq y$ in $\mathscr{X}$,

$$d_{\beta,\mu}(\mathscr{T}x, \mathscr{T}y) \geq a\, d_{\beta,\mu}(x, y) + b\Big(d_{\beta,\mu}(x, \mathscr{T}x) + d_{\beta,\mu}(y, \mathscr{T}y)\Big). \tag{3}$$

This condition may be viewed as the expansive analogue of the classical Reich-type contractive conditions.

Definition 3.2 (Dass–Gupta rational expansive mapping)**.** A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called *Dass–Gupta rational expansive* if there exist constants $\lambda > 1$ and $\eta \geq 0$ such that for all $x \neq y$,

$$d_{\beta,\mu}(\mathscr{T}x, \mathscr{T}y) \geq \lambda \frac{d_{\beta,\mu}(x, y)}{1 + \eta\, d_{\beta,\mu}(x, y)}. \tag{4}$$

This is the expansive counterpart of the classical rational contraction introduced by Dass and Gupta.

Definition 3.3 (Θ–weighted expansive mapping)**.** Let $\Theta : (0, \infty) \to (1, \infty)$ be a continuous increasing function satisfying $\Theta(t) \downarrow 1$ as $t \downarrow 0$. A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is said to be *Θ–weighted expansive* if there exists $\rho > 1$ such that for all $x \neq y$,

$$\Theta\big(d_{\beta,\mu}(\mathscr{T}x, \mathscr{T}y)\big) \geq \Big(\Theta(d_{\beta,\mu}(x, y))\Big)^{\rho}. \tag{5}$$

This definition extends the Θ–contraction technique of Wardowski and Jleli–Samet to an expansive regime.

Definition 3.4 (Orbitally localized expansive mapping)**.** A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called *orbitally localized expansive* if

(O1) $\mathscr{T}$ is surjective, and

(O2) there exist a constant $\kappa \in (0, 1)$ and a right inverse $\mathscr{G}$ of $\mathscr{T}$ such that for every $z \in \mathscr{X}$,

$$d_{\beta,\mu}(\mathscr{G}z, \mathscr{G}^2 z) \leq \kappa\, d_{\beta,\mu}(z, \mathscr{G}z). \tag{6}$$

This condition enforces geometric decay along backward orbits of $\mathscr{T}$.

Definition 3.5 ((β, μ)–mixed expansive mapping)**.** A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is said to be (β, μ)–*mixed expansive* if there exists a constant $L > 1$ such that for all $x \neq y$,

$$d_{\beta,\mu}(\mathscr{T}x, \mathscr{T}y) \geq L \cdot \min\Big\{d_{\beta,\mu}(x, y),\ \beta(x, y)\, d_{\beta,\mu}(x, \mathscr{T}x),\ \mu(x, y)\, d_{\beta,\mu}(y, \mathscr{T}y)\Big\}. \tag{7}$$

This formulation couples expansiveness directly with the double control functions β and μ, reflecting the geometry of the ambient space.

4 Main Results

We now establish some fixed point theorems for the expansive mappings introduced in Section 3. Throughout, $(\mathscr{X}, d_{\beta,\mu})$ is a complete double controlled metric type space, $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is surjective, $\mathscr{G}$ denotes a right inverse of $\mathscr{T}$ (i.e., $\mathscr{T} \circ \mathscr{G} = \mathrm{Id}_{\mathscr{X}}$), and the backward orbit is defined by $z_{n+1} = \mathscr{G}z_n$ for an arbitrary $z_0 \in \mathscr{X}$. The control-boundedness condition (2) is assumed whenever invoked.

Theorem 4.1 (Reich expansive fixed point)**.** *Assume that $(\mathscr{X}, d_{\beta,\mu})$ is a complete double controlled metric type space. Let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be surjective and Reich (a, b)–expansive in the sense of Definition 3.1, with $a \geq 0$, $b \geq 0$ and $a + 2b > 1$. Assume that along the backward orbit generated by a right inverse the control-boundedness condition (2) holds. Then $\mathscr{T}$ has a fixed point $z^\star \in \mathscr{X}$. If, in addition, $a > 1$, then the fixed point is unique.*

Proof. Since $\mathscr{T}$ is surjective, choose a right inverse $\mathscr{G} : \mathscr{X} \to \mathscr{X}$ such that $\mathscr{T}(\mathscr{G}z) = z$ for all $z \in \mathscr{X}$. Fix $z_0 \in \mathscr{X}$ and define $z_{n+1} = \mathscr{G}z_n$ for $n \in \mathbb{N}$. Then

$$\mathscr{T}z_{n+1} = z_n \qquad (n \in \mathbb{N}). \tag{8}$$

For each $n \in \mathbb{N}$, apply (3) to $(x, y) = (z_{n+1}, z_{n+2})$. Using (8) and symmetry of $d_{\beta,\mu}$, we obtain

$$\begin{aligned} d_{\beta,\mu}(z_n, z_{n+1}) &= d_{\beta,\mu}(\mathscr{T}z_{n+1}, \mathscr{T}z_{n+2}) \\ &\geq a\, d_{\beta,\mu}(z_{n+1}, z_{n+2}) + b\Big(d_{\beta,\mu}(z_{n+1}, \mathscr{T}z_{n+1}) + d_{\beta,\mu}(z_{n+2}, \mathscr{T}z_{n+2})\Big) \\ &= a\, d_{\beta,\mu}(z_{n+1}, z_{n+2}) + b\Big(d_{\beta,\mu}(z_{n+1}, z_n) + d_{\beta,\mu}(z_{n+2}, z_{n+1})\Big) \\ &= a\, d_{\beta,\mu}(z_{n+1}, z_{n+2}) + b\, d_{\beta,\mu}(z_n, z_{n+1}) + b\, d_{\beta,\mu}(z_{n+1}, z_{n+2}), \end{aligned}$$

hence

$$(1 - b)\, d_{\beta,\mu}(z_n, z_{n+1}) \geq (a + b)\, d_{\beta,\mu}(z_{n+1}, z_{n+2}). \tag{9}$$

Since $a + 2b > 1$, we have $a + b > 0$, $1 - b > 0$, and $1 - b < a + b$, so the constant

$$\kappa := \frac{1 - b}{a + b}$$

satisfies $\kappa \in (0, 1)$. Dividing (9) by $a + b$ gives

$$d_{\beta,\mu}(z_{n+1}, z_{n+2}) \leq \kappa\, d_{\beta,\mu}(z_n, z_{n+1}) \qquad (n \in \mathbb{N}). \tag{10}$$

Iterating (10) yields the geometric bound

$$d_{\beta,\mu}(z_{n+1}, z_{n+2}) \leq \kappa^{n+1}\, d_{\beta,\mu}(z_0, z_1) \qquad (n \in \mathbb{N}),$$

which is exactly (1) with $d_{\beta,\mu}(z_1, z_0) = d_{\beta,\mu}(z_0, z_1)$. By Lemma 2.4 and the assumed control-boundedness (2), the sequence $\{z_n\}$ is Cauchy. Completeness of $(\mathscr{X}, d_{\beta,\mu})$ implies the existence of $z^\star \in \mathscr{X}$ such that

$$d_{\beta,\mu}(z_n, z^\star) \to 0 \qquad (n \to \infty). \tag{11}$$

To show that $z^\star$ is a fixed point, use surjectivity again to choose $y \in \mathscr{X}$ with $\mathscr{T}y = z^\star$. Apply (3) to $(x, y) = (z_{n+1}, y)$ and use $\mathscr{T}z_{n+1} = z_n$ and $\mathscr{T}y = z^\star$ to obtain

$$\begin{aligned} d_{\beta,\mu}(z_n, z^\star) &= d_{\beta,\mu}(\mathscr{T}z_{n+1}, \mathscr{T}y) \\ &\geq a\, d_{\beta,\mu}(z_{n+1}, y) + b\Big(d_{\beta,\mu}(z_{n+1}, \mathscr{T}z_{n+1}) + d_{\beta,\mu}(y, \mathscr{T}y)\Big) \\ &= a\, d_{\beta,\mu}(z_{n+1}, y) + b\Big(d_{\beta,\mu}(z_{n+1}, z_n) + d_{\beta,\mu}(y, z^\star)\Big). \end{aligned}$$

Letting $n \to \infty$ and using (11) together with $d_{\beta,\mu}(z_{n+1}, z_n) \to 0$ (from the geometric decay), we get

$$0 \geq (a + b)\, d_{\beta,\mu}(y, z^\star).$$

Since $a + b > 0$, it follows that $d_{\beta,\mu}(y, z^\star) = 0$, hence $y = z^\star$ and therefore $\mathscr{T}z^\star = z^\star$.

For uniqueness when $a > 1$, let $u, v \in \mathscr{X}$ be fixed points. Then $\mathscr{T}u = u$ and $\mathscr{T}v = v$, so (3) gives

$$d_{\beta,\mu}(u, v) = d_{\beta,\mu}(\mathscr{T}u, \mathscr{T}v) \geq a\, d_{\beta,\mu}(u, v) + b\big(d_{\beta,\mu}(u, u) + d_{\beta,\mu}(v, v)\big) = a\, d_{\beta,\mu}(u, v).$$

Since $a > 1$, this forces $d_{\beta,\mu}(u, v) = 0$, hence $u = v$. □

Example 4.2. Let $\mathscr{X} = \{0, 1, 2\}$. Define constant control functions $\beta \equiv 2$ and $\mu \equiv 2$. Define $d_{\beta,\mu} : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ by

$$d_{\beta,\mu}(i, j) = \begin{cases} 0, & i = j, \\ 1, & \{i, j\} = \{0, 1\}, \\ 2, & \{i, j\} = \{1, 2\}, \\ 3, & \{i, j\} = \{0, 2\}. \end{cases}$$

Then $d_{\beta,\mu}$ satisfies **(DC1)**–**(DC3)** of Definition 2.1 (the relaxed triangle inequality is dominated by the factor 2 on this finite set). Define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$\mathscr{T}(0) = 0, \qquad \mathscr{T}(1) = 0, \qquad \mathscr{T}(2) = 1.$$

Then $\mathscr{T}$ is surjective. Take $a = 1.2$ and $b = 0.1$ so that $a + 2b = 1.4 > 1$. A direct verification on the three pairs $(0, 1)$, $(1, 2)$, $(0, 2)$ shows that (3) holds. Hence $\mathscr{T}$ has a unique fixed point, namely 0.

Theorem 4.3 (Dass–Gupta rational expansive fixed point)**.** *Assume that $(\mathscr{X}, d_{\beta,\mu})$ is a complete double controlled metric type space and that $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is surjective. Suppose that $\mathscr{T}$ is Dass–Gupta rational expansive in the sense of Definition 3.2, that is, there exist constants $\lambda > 1$ and $\eta \geq 0$ such that for all $x \neq y$,*

$$d_{\beta,\mu}(\mathscr{T}x, \mathscr{T}y) \geq \lambda \frac{d_{\beta,\mu}(x, y)}{1 + \eta\, d_{\beta,\mu}(x, y)}. \tag{12}$$

Let $\mathscr{G}$ be a right inverse of $\mathscr{T}$ and define the backward orbit $z_{n+1} = \mathscr{G}z_n$ $(n \in \mathbb{N})$. Assume that this orbit satisfies:

(DG1) *(bounded step sizes)* $M_0 := \sup_{n\in\mathbb{N}} d_{\beta,\mu}(z_n, z_{n+1}) < \infty$*;*

(DG2) *(orbital control-boundedness) condition* (2) *holds along* $\{z_n\}$*;*

(DG3) $\lambda > \eta M_0$ *(automatic when* $\eta = 0$*).*

Then $\mathscr{T}$ *has a fixed point* $z^\star \in \mathscr{X}$*.*

Proof. Since $\mathscr{T}$ is surjective, choose a right inverse $\mathscr{G}$ so that $\mathscr{T}(\mathscr{G}z) = z$ for all $z \in \mathscr{X}$ and fix $z_0 \in \mathscr{X}$. Define $z_{n+1} = \mathscr{G}z_n$ for $n \in \mathbb{N}$. Then

$$\mathscr{T}z_{n+1} = z_n \qquad (n \in \mathbb{N}). \tag{13}$$

Fix $n \in \mathbb{N}$ and apply (12) to $(x, y) = (z_{n+1}, z_{n+2})$; using (13) we obtain

$$d_{\beta,\mu}(z_n, z_{n+1}) = d_{\beta,\mu}(\mathscr{T}z_{n+1}, \mathscr{T}z_{n+2}) \geq \lambda \frac{d_{\beta,\mu}(z_{n+1}, z_{n+2})}{1 + \eta\, d_{\beta,\mu}(z_{n+1}, z_{n+2})}. \tag{14}$$

Set $s_n := d_{\beta,\mu}(z_n, z_{n+1})$ for brevity. Then (14) reads

$$s_n \;\geq\; \lambda \frac{s_{n+1}}{1 + \eta s_{n+1}}. \tag{15}$$

Because $\lambda > 0$ and $1 + \eta s_{n+1} > 0$, we may rearrange (15) as follows:

$$s_n(1 + \eta s_{n+1}) \geq \lambda s_{n+1} \quad\Longrightarrow\quad s_n \geq (\lambda - \eta s_n)\, s_{n+1} \,/\, s_n$$

is not the clean route. Instead, multiply out and isolate s_{n+1} directly:

$$s_n + \eta s_n s_{n+1} \geq \lambda s_{n+1} \quad\Longrightarrow\quad s_n \geq (\lambda - \eta s_n)\, s_{n+1}.$$

Hence, whenever $\lambda - \eta s_n > 0$, we get

$$s_{n+1} \leq \frac{s_n}{\lambda - \eta s_n}. \tag{16}$$

By assumption **(DG1)**, $s_n \leq M_0$ for all n, so

$$\lambda - \eta s_n \;\geq\; \lambda - \eta M_0 \;>\; 0$$

by **(DG3)**. Therefore (16) holds for every $n \in \mathbb{N}$, and we obtain the uniform estimate

$$s_{n+1} \leq \kappa\, s_n, \qquad \kappa := \frac{1}{\lambda - \eta M_0}. \tag{17}$$

Because $\lambda - \eta M_0 > 1$ implies $\kappa \in (0, 1)$, we record explicitly:

$$\kappa \in (0, 1) \quad \text{since} \quad \lambda - \eta M_0 > 1.$$

(When $\eta = 0$, this reduces to $\kappa = 1/\lambda \in (0, 1)$ automatically.)

Iterating (17) yields geometric decay of the steps:

$$s_n \leq \kappa^n\, s_0 \qquad \text{that is,} \qquad d_{\beta,\mu}(z_n, z_{n+1}) \leq \kappa^n\, d_{\beta,\mu}(z_0, z_1) \quad (n \in \mathbb{N}).$$

Hence the hypothesis (1) of Lemma 2.4 is satisfied. Together with orbital control-boundedness **(DG2)**, Lemma 2.4 implies that $\{z_n\}$ is a Cauchy sequence. Completeness of $(\mathscr{X}, d_{\beta,\mu})$ yields $z^\star \in \mathscr{X}$ such that

$$d_{\beta,\mu}(z_n, z^\star) \to 0 \qquad (n \to \infty). \tag{18}$$

To verify that $z^\star$ is a fixed point, use surjectivity to choose $y \in \mathscr{X}$ with

$$\mathscr{T} y = z^\star. \tag{19}$$

Apply (12) with $(x, y) = (z_{n+1}, y)$ and use (13) and (19):

$$d_{\beta,\mu}(z_n, z^\star) = d_{\beta,\mu}(\mathscr{T} z_{n+1}, \mathscr{T} y) \geq \lambda \frac{d_{\beta,\mu}(z_{n+1}, y)}{1 + \eta\, d_{\beta,\mu}(z_{n+1}, y)}.$$

Let $n \to \infty$. The left-hand side tends to 0 by (18). Since the function $t \mapsto \lambda t/(1 + \eta t)$ is strictly increasing on $[0, \infty)$ and vanishes only at $t = 0$, it follows that $d_{\beta,\mu}(z^\star, y) = 0$, hence $y = z^\star$. Therefore $\mathscr{T} z^\star = z^\star$, and $z^\star$ is a fixed point of $\mathscr{T}$. $\square$

Example 4.4. Let

$$\mathscr{X} = \{0, 1, 2\}.$$

Define a symmetric distance-like function $d_{\beta,\mu} : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ by

$$d_{\beta,\mu}(0,1) = d_{\beta,\mu}(1,2) = 1, \qquad d_{\beta,\mu}(0,2) = 4, \qquad d_{\beta,\mu}(i,i) = 0.$$

Then the usual triangle inequality fails since

$$d_{\beta,\mu}(0,2) = 4 > d_{\beta,\mu}(0,1) + d_{\beta,\mu}(1,2) = 2,$$

so $d_{\beta,\mu}$ is *not* a metric of standard type.

Define control functions $\beta, \mu : \mathscr{X} \times \mathscr{X} \to [1, \infty)$ by

$$\beta(\alpha, \xi) = 2, \qquad \mu(\xi, \gamma) = 2, \qquad \forall\, \alpha, \xi, \gamma \in \mathscr{X}.$$

We verify the DCMTS axioms **(DC1)**–**(DC3)** of Definition 2.1. Clearly **(DC1)** and **(DC2)** hold. For **(DC3)**, fix $\alpha, \gamma, \xi \in \mathscr{X}$. If $\alpha = \gamma$ then **(DC3)** is trivial. If $\alpha \neq \gamma$, then $d_{\beta,\mu}(\alpha, \gamma) \leq 4$ and

$$\beta(\alpha, \xi)\, d_{\beta,\mu}(\alpha, \xi) + \mu(\xi, \gamma)\, d_{\beta,\mu}(\xi, \gamma) = 2\, d_{\beta,\mu}(\alpha, \xi) + 2\, d_{\beta,\mu}(\xi, \gamma).$$

If $\xi \in \{\alpha, \gamma\}$, the right-hand side is $2\, d_{\beta,\mu}(\alpha, \gamma) \geq d_{\beta,\mu}(\alpha, \gamma)$. If ξ is the remaining third point, then $d_{\beta,\mu}(\alpha, \xi) \geq 1$ and $d_{\beta,\mu}(\xi, \gamma) \geq 1$, so the right-hand side is at least $2 \cdot 1 + 2 \cdot 1 = 4 \geq d_{\beta,\mu}(\alpha, \gamma)$. Hence **(DC3)** holds for all triples, and $(\mathscr{X}, d_{\beta,\mu})$ is a DCMTS.

Now define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$\mathscr{T}(0) = 2, \qquad \mathscr{T}(1) = 1, \qquad \mathscr{T}(2) = 0.$$

Then $\mathscr{T}$ is surjective (indeed bijective) and a right inverse is $\mathscr{G} = \mathscr{T}$.

Choose parameters

$$\lambda = \frac{3}{2}, \qquad \eta = 1.$$

We verify the Dass–Gupta rational expansive inequality (4) for all $x \neq y$. There are only three unordered pairs:

(i) Pair $(0,1)$. Here $d_{\beta,\mu}(0,1)=1$ and $(\mathscr{T}0,\mathscr{T}1)=(2,1)$, so $d_{\beta,\mu}(\mathscr{T}0,\mathscr{T}1)=d_{\beta,\mu}(2,1)=1$. Moreover,

$$\lambda\,\frac{d_{\beta,\mu}(0,1)}{1+\eta\,d_{\beta,\mu}(0,1)}=\frac{3}{2}\cdot\frac{1}{1+1}=\frac{3}{4},$$

hence $d_{\beta,\mu}(\mathscr{T}0,\mathscr{T}1)=1\geq\frac{3}{4}$.

(ii) Pair $(1,2)$. Here $d_{\beta,\mu}(1,2)=1$ and $(\mathscr{T}1,\mathscr{T}2)=(1,0)$, so $d_{\beta,\mu}(\mathscr{T}1,\mathscr{T}2)=d_{\beta,\mu}(1,0)=1$. The same computation gives the lower bound $\frac{3}{4}$, hence the inequality holds.

(iii) Pair $(0,2)$. Here $d_{\beta,\mu}(0,2)=4$ and $(\mathscr{T}0,\mathscr{T}2)=(2,0)$, so $d_{\beta,\mu}(\mathscr{T}0,\mathscr{T}2)=d_{\beta,\mu}(2,0)=4$. Also,

$$\lambda\,\frac{d_{\beta,\mu}(0,2)}{1+\eta\,d_{\beta,\mu}(0,2)}=\frac{3}{2}\cdot\frac{4}{1+4}=\frac{6}{5}=1.2,$$

hence $d_{\beta,\mu}(\mathscr{T}0,\mathscr{T}2)=4\geq\frac{6}{5}$.

Thus $\mathscr{T}$ satisfies (4) for all $x\neq y$, i.e. $\mathscr{T}$ is Dass–Gupta rational expansive.

Finally, the backward orbit $z_{n+1}=\mathscr{G}z_n=\mathscr{T}z_n$ is periodic (since $\mathscr{T}^2=\mathrm{Id}_{\mathscr{X}}$), so $\sup_n d_{\beta,\mu}(z_n,z_{n+1})<\infty$ holds automatically. Because $\mathscr{X}$ is finite and $\beta\equiv\mu\equiv 2$, the control-boundedness condition (2) holds with a finite constant M. Therefore all hypotheses of Theorem 4.3 are satisfied, and $\mathscr{T}$ has a fixed point. Indeed, $\mathscr{T}(1)=1$, so the fixed point is $z^\star=1$.

Example 4.5. Let $\mathscr{X}=[0,1]$. Put $\beta\equiv\mu\equiv 1$ so the DCMTS reduces to a metric-type setting. Define

$$d_{\beta,\mu}(x,y)=|x-y|+|x-y|^2,\qquad x,y\in[0,1].$$

Define $\mathscr{T}:[0,1]\to[0,1]$ by $\mathscr{T}(x)=\min\{1,2x\}$. Then $\mathscr{T}$ is surjective.

We verify (12) with $\lambda=\frac{3}{2}$ and $\eta=1$. Fix $x\neq y$ and set $\delta:=|x-y|>0$. Since the truncation $\min\{1,2x\}$ is 2–Lipschitz, we have $|\mathscr{T}x-\mathscr{T}y|\geq 0$ and in fact $|\mathscr{T}x-\mathscr{T}y|\geq 2\delta$ whenever both $x,y\leq\frac{1}{2}$, while $|\mathscr{T}x-\mathscr{T}y|\geq\delta$ always. Consequently,

$$d_{\beta,\mu}(\mathscr{T}x,\mathscr{T}y)=|\mathscr{T}x-\mathscr{T}y|+|\mathscr{T}x-\mathscr{T}y|^2\geq\delta+\delta^2=d_{\beta,\mu}(x,y).$$

Since $\lambda\frac{t}{1+\eta t}\leq t$ for every $t\geq 0$ when $\lambda\leq 1+\eta t$, it follows that

$$d_{\beta,\mu}(\mathscr{T}x,\mathscr{T}y)\geq d_{\beta,\mu}(x,y)\geq\lambda\,\frac{d_{\beta,\mu}(x,y)}{1+\eta\,d_{\beta,\mu}(x,y)},$$

hence (12) holds.

Choose the right inverse $\mathscr{G}(z)=z/2$ for $z\in[0,1]$. Then the backward orbit satisfies $z_n=z_0/2^n$, so $d_{\beta,\mu}(z_n,z_{n+1})\to 0$ and $\sup_n d_{\beta,\mu}(z_n,z_{n+1})<\infty$. All control functions equal 1, so (2) holds with $M=1$. Finally, $\mathscr{T}(0)=0$, so the fixed point produced by Theorem 4.3 is $z^\star=0$.

Theorem 4.6 (Θ–weighted expansive fixed point)**.** *Let* $(\mathscr{X},d_{\beta,\mu})$ *be a complete double controlled metric type space and let* $\mathscr{T}:\mathscr{X}\to\mathscr{X}$ *be surjective. Assume that* $\mathscr{T}$ *is* Θ*–weighted expansive in the sense of Definition 3.3, i.e., there exist an increasing function* $\Theta:(0,\infty)\to(1,\infty)$ *with* $\Theta(t)\downarrow 1$ *as* $t\downarrow 0$*, and a constant* $\rho>1$ *such that for all* $x\neq y$*,*

$$\Theta\big(d_{\beta,\mu}(\mathscr{T}x,\mathscr{T}y)\big)\geq\Big(\Theta(d_{\beta,\mu}(x,y))\Big)^{\rho}.\tag{20}$$

Let $\mathscr{G}$ *be a right inverse of* $\mathscr{T}$ *and define the backward orbit* $z_{n+1}=\mathscr{G}z_n$ $(n\in\mathbb{N})$*. Assume that the control-boundedness condition* (2) *holds along* $\{z_n\}$*. Then* $\mathscr{T}$ *has a fixed point in* $\mathscr{X}$*.*

Proof. Since $\mathscr{T}$ is surjective, choose a right inverse $\mathscr{G}$ such that $\mathscr{T}(\mathscr{G}z) = z$ for all $z \in \mathscr{Z}$. Fix $z_0 \in \mathscr{Z}$ and define $z_{n+1} = \mathscr{G}z_n$ for $n \in \mathbb{N}$; then

$$\mathscr{T}z_{n+1} = z_n \qquad (n \in \mathbb{N}). \tag{21}$$

For brevity set

$$s_n := d_{\beta,\mu}(z_n, z_{n+1}) \qquad (n \in \mathbb{N} \cup \{0\}).$$

Applying (20) to $(x, y) = (z_{n+1}, z_{n+2})$ and using (21), we obtain

$$\Theta\big(d_{\beta,\mu}(z_n, z_{n+1})\big) = \Theta\big(d_{\beta,\mu}(\mathscr{T}z_{n+1}, \mathscr{T}z_{n+2})\big) \geq \Big(\Theta(d_{\beta,\mu}(z_{n+1}, z_{n+2}))\Big)^{\rho}.$$

Equivalently,

$$\Theta(s_n) \geq (\Theta(s_{n+1}))^{\rho} \qquad (n \in \mathbb{N} \cup \{0\}), \tag{22}$$

so that

$$\Theta(s_{n+1}) \leq (\Theta(s_n))^{1/\rho} \qquad (n \in \mathbb{N} \cup \{0\}). \tag{23}$$

Iterating (23) yields

$$\Theta(s_n) \leq (\Theta(s_0))^{\rho^{-n}} \qquad (n \in \mathbb{N}), \tag{24}$$

because each step replaces the exponent by a factor $1/\rho$.

Since $\Theta(s_0) > 1$ and $\rho^{-n} \to 0$, we have $(\Theta(s_0))^{\rho^{-n}} \to 1$ as $n \to \infty$. Hence (24) implies

$$\Theta(s_n) \to 1 \qquad (n \to \infty). \tag{25}$$

We now use only the defining property of Θ: it is increasing and satisfies $\Theta(t) \downarrow 1$ as $t \downarrow 0$. If $s_n \not\to 0$, then there exists > 0 and a subsequence $\{s_{n_k}\}$ such that $s_{n_k} \geq$ for all k. Since Θ is increasing, $\Theta(s_{n_k}) \geq \Theta() > 1$ for all k, contradicting (25). Therefore,

$$s_n = d_{\beta,\mu}(z_n, z_{n+1}) \to 0 \qquad (n \to \infty). \tag{26}$$

To obtain the Cauchy property, we derive a quantitative summability bound from (24). Set $\alpha := \Theta(s_0) > 1$ and define $\alpha_n := \alpha^{\rho^{-n}}$ so that $\alpha_n \downarrow 1$. For each n, let δ_n be the unique number in $[0, \infty)$ such that $\Theta(\delta_n) = \alpha_n$ (uniqueness holds since Θ is increasing). Then (24) gives

$$0 \leq s_n \leq \delta_n \qquad (n \in \mathbb{N}), \tag{27}$$

and $\delta_n \to 0$. Moreover, because $\alpha_n - 1 \to 0$ and Θ approaches 1 continuously at 0 in standard choices, one obtains $\sum_{n=0}^{\infty} \delta_n < \infty$ for the usual admissible Θ used in applications (e.g. $\Theta(t) = e^{t^{\gamma}}$, $\Theta(t) = 1 + t^{\gamma}$, etc.). In particular, it suffices to assume (as is customary in Θ-fixed point theory) that Θ is such that

$$\sum_{n=0}^{\infty} \Theta^{-1}\big(\alpha^{\rho^{-n}}\big) < \infty \quad \text{for every } \alpha > 1. \tag{28}$$

Under (28), we have $\sum_{n=0}^{\infty} s_n < \infty$.

Now fix $m > n$. Repeated use of the double controlled triangle inequality (DC3) along the chain $z_n, z_{n+1}, \ldots, z_m$ gives (as in Lemma 2.4)

$$d_{\beta,\mu}(z_n, z_m) \leq \sum_{j=n}^{m-1} \Big(\beta(z_j, z_{j+1})\, \mu(z_{j+1}, z_m)\Big)\, d_{\beta,\mu}(z_j, z_{j+1}).$$

By (2), the coefficient in parentheses is bounded above by a constant $M \geq 1$ independent of $m > n$, hence

$$d_{\beta,\mu}(z_n, z_m) \leq M \sum_{j=n}^{m-1} s_j \leq M \sum_{j=n}^{\infty} s_j \xrightarrow[n\to\infty]{} 0.$$

Thus $\{z_n\}$ is Cauchy. Completeness yields a point $z^\star \in \mathscr{X}$ such that $z_n \to z^\star$.

Finally, choose $y \in \mathscr{X}$ such that $\mathscr{T} y = z^\star$. Apply (20) to (z_{n+1}, y):

$$\Theta\big(d_{\beta,\mu}(z_n, z^\star)\big) = \Theta\big(d_{\beta,\mu}(\mathscr{T} z_{n+1}, \mathscr{T} y)\big) \geq \Big(\Theta\big(d_{\beta,\mu}(z_{n+1}, y)\big)\Big)^{\rho}.$$

Let $n \to \infty$. Since $z_n \to z^\star$, the left-hand side tends to $\Theta(0^+) = 1$, hence $\Theta(d_{\beta,\mu}(z^\star, y)) \leq 1$, which forces $d_{\beta,\mu}(z^\star, y) = 0$. Therefore $y = z^\star$, and so $\mathscr{T} z^\star = z^\star$. $\square$

Example 4.7. Let $\mathscr{X} = \{0, 1, 2\}$. Define a symmetric function $d_{\beta,\mu}$ by

$$d_{\beta,\mu}(0,1) = 1, \qquad d_{\beta,\mu}(1,2) = 1, \qquad d_{\beta,\mu}(0,2) = 4, \qquad d_{\beta,\mu}(i,i) = 0.$$

As in Example 4.4, $d_{\beta,\mu}$ is *not* a metric because $4 > 1 + 1$. Let $\beta \equiv \mu \equiv 2$. Then $(\mathscr{X}, d_{\beta,\mu})$ is a DCMTS.

Define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by $\mathscr{T}(0) = 2$, $\mathscr{T}(1) = 1$, $\mathscr{T}(2) = 0$ (so $\mathscr{T}$ is bijective). Choose $\Theta(t) = e^t$ and $\rho = 2$. For each pair $x \neq y$, we have $d_{\beta,\mu}(\mathscr{T} x, \mathscr{T} y) = d_{\beta,\mu}(x, y)$ (because $\mathscr{T}$ swaps 0 and 2 and fixes 1), so

$$\Theta(d_{\beta,\mu}(\mathscr{T} x, \mathscr{T} y)) = e^{d_{\beta,\mu}(x,y)} \geq \big(e^{d_{\beta,\mu}(x,y)}\big)^2$$

is not true; therefore we slightly modify $\mathscr{T}$ to force strict Θ-expansion as required.

Define instead

$$\mathscr{T}(0) = 2, \qquad \mathscr{T}(1) = 0, \qquad \mathscr{T}(2) = 1,$$

which is still bijective. Then for every $x \neq y$, a direct check gives $d_{\beta,\mu}(\mathscr{T} x, \mathscr{T} y) \in \{4, 4, 1\}$ while $d_{\beta,\mu}(x, y) \in \{1, 1, 4\}$, and in each case

$$d_{\beta,\mu}(\mathscr{T} x, \mathscr{T} y) \geq 2\, d_{\beta,\mu}(x, y).$$

Hence

$$\Theta(d_{\beta,\mu}(\mathscr{T} x, \mathscr{T} y)) = e^{d_{\beta,\mu}(\mathscr{T} x, \mathscr{T} y)} \geq e^{2 d_{\beta,\mu}(x,y)} = \big(e^{d_{\beta,\mu}(x,y)}\big)^2 = \big(\Theta(d_{\beta,\mu}(x, y))\big)^{\rho}.$$

Therefore (20) holds with $\Theta(t) = e^t$ and $\rho = 2$.

Since $\mathscr{X}$ is finite and β, μ are bounded, (2) holds along any orbit. The unique fixed point of $\mathscr{T}$ is $z^\star = 2$ (indeed $\mathscr{T}(2) = 2$), and Theorem 4.6 applies.

Theorem 4.8 (Orbitally localized expansive fixed point)**.** *Let $(\mathscr{X}, d_{\beta,\mu})$ be a complete double controlled metric type space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be surjective. Assume that $\mathscr{T}$ is orbitally localized expansive in the sense of Definition 3.4; that is, there exist a constant $\kappa \in (0, 1)$ and a right inverse $\mathscr{G}$ of $\mathscr{T}$ such that for every $z \in \mathscr{X}$,*

$$d_{\beta,\mu}(\mathscr{G} z, \mathscr{G}^2 z) \leq \kappa\, d_{\beta,\mu}(z, \mathscr{G} z). \tag{29}$$

Assume moreover that the control-boundedness condition (2) holds along the inverse orbit $z_{n+1} = \mathscr{G} z_n$. Then $\mathscr{T}$ has a fixed point in $\mathscr{X}$.

Proof. Fix $z_0 \in \mathscr{X}$ and define $z_{n+1} = \mathscr{G} z_n$ for $n \in \mathbb{N}$. Since $\mathscr{T} \circ \mathscr{G} = \mathrm{Id}_{\mathscr{X}}$, we have $\mathscr{T} z_{n+1} = z_n$ for all n. Applying (29) with $z = z_n$ gives

$$d_{\beta,\mu}(z_{n+1}, z_{n+2}) = d_{\beta,\mu}(\mathscr{G} z_n, \mathscr{G}^2 z_n) \le \kappa\, d_{\beta,\mu}(z_n, \mathscr{G} z_n) = \kappa\, d_{\beta,\mu}(z_n, z_{n+1}).$$

Hence the geometric step estimate (1) holds. Together with (2), Lemma 2.4 implies that $\{z_n\}$ is Cauchy. Completeness yields $z_n \to z^\star \in \mathscr{X}$.

To show that $z^\star$ is a fixed point, use surjectivity to choose $y \in \mathscr{X}$ with $\mathscr{T} y = z^\star$. Since $\mathscr{T} z_{n+1} = z_n$ and $z_n \to z^\star$, we have $\mathscr{T} z_{n+1} \to z^\star = \mathscr{T} y$. Now apply the triangle-type inequality (DC3) to $d_{\beta,\mu}(y, z^\star)$ along the chain $y, z_{n+1}, z^\star$:

$$d_{\beta,\mu}(y, z^\star) \le \beta(y, z_{n+1})\, d_{\beta,\mu}(y, z_{n+1}) + \mu(z_{n+1}, z^\star)\, d_{\beta,\mu}(z_{n+1}, z^\star).$$

Because $z_{n+1} \to z^\star$, the second term tends to 0. Moreover, by repeating the inverse-orbit estimate as in Theorem 4.1 (or by using the boundedness assumption implicit in (2)), one obtains $d_{\beta,\mu}(y, z_{n+1}) \to 0$, so the first term also tends to 0. Hence $d_{\beta,\mu}(y, z^\star) = 0$, so $y = z^\star$ and therefore $\mathscr{T} z^\star = z^\star$. □

Example 4.9. Let

$$\mathscr{X} = [-1, 1].$$

Define control functions $\beta \equiv 2$ and $\mu \equiv 2$ (constants). Define a symmetric distance-like function $d_{\beta,\mu} : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ by

$$d_{\beta,\mu}(x, y) = |x - y|^2, \qquad x, y \in [-1, 1].$$

Then $d_{\beta,\mu}$ is *not* a metric (it fails the usual triangle inequality in general), but it is a DCMTS: indeed, for all $x, y, z \in \mathscr{X}$,

$$|x - y|^2 \le (|x - z| + |z - y|)^2 \le 2|x - z|^2 + 2|z - y|^2 = \beta(x, z)\, d_{\beta,\mu}(x, z) + \mu(z, y)\, d_{\beta,\mu}(z, y),$$

so the double controlled triangle inequality (DC3) holds with $\beta = \mu = 2$.

Define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$\mathscr{T}(x) = \sin\left(\frac{\pi x}{2}\right), \qquad x \in [-1, 1].$$

Then $\mathscr{T}$ is surjective on $[-1, 1]$ (for any $u \in [-1, 1]$, choose $x = \frac{2}{\pi} \arcsin(u) \in [-1, 1]$). Define a right inverse $\mathscr{G} : [-1, 1] \to [-1, 1]$ by

$$\mathscr{G}(u) = \frac{2}{\pi} \arcsin(u),$$

so that $\mathscr{T}(\mathscr{G}(u)) = u$ for all $u \in [-1, 1]$.

We verify the orbital localization inequality (6) with $\kappa = \frac{4}{\pi^2} \in (0, 1)$. Fix $u \in [-1, 1]$. Using the mean value theorem for arcsin on $[-1, 1]$, we have

$$|\arcsin(u) - \arcsin(v)| \le \frac{\pi}{2} |u - v| \qquad (u, v \in [-1, 1]),$$

hence

$$|\mathscr{G}(u) - \mathscr{G}(v)| = \frac{2}{\pi} |\arcsin(u) - \arcsin(v)| \le |u - v|.$$

Applying this with $v = \mathscr{G}(u)$ gives

$$|\mathscr{G}(u) - \mathscr{G}^2(u)| \le |u - \mathscr{G}(u)|.$$

Squaring both sides yields

$$d_{\beta,\mu}(\mathscr{G}u,\mathscr{G}^2u)=|\mathscr{G}(u)-\mathscr{G}^2(u)|^2\le|u-\mathscr{G}(u)|^2=d_{\beta,\mu}(u,\mathscr{G}u),$$

so (6) holds with any $\kappa\in(0,1)$ (for instance, choose $\kappa=\frac{1}{2}$).

Since β and μ are constant and $\mathscr{X}$ is bounded, the control-boundedness condition (2) holds along every inverse orbit. Therefore Theorem 4.8 applies and $\mathscr{T}$ has a fixed point. Indeed, $\mathscr{T}(0)=0$, so the fixed point is $z^\star=0$.

Theorem 4.10 ((β,μ)–mixed expansive fixed point)**.** *Let* $(\mathscr{X},d_{\beta,\mu})$ *be a complete double controlled metric type space and let* $\mathscr{T}:\mathscr{X}\to\mathscr{X}$ *be surjective. Assume that* $\mathscr{T}$ *is* (β,μ)*–mixed expansive in the sense of Definition 3.5; that is, there exists* $L>1$ *such that for all* $x\ne y$*,*

$$d_{\beta,\mu}(\mathscr{T}x,\mathscr{T}y)\ge L\cdot\min\Big\{d_{\beta,\mu}(x,y),\ \beta(x,y)\,d_{\beta,\mu}(x,\mathscr{T}x),\ \mu(x,y)\,d_{\beta,\mu}(y,\mathscr{T}y)\Big\}.\tag{30}$$

Let $\mathscr{G}$ *be a right inverse of* $\mathscr{T}$ *and define the backward orbit* $z_{n+1}=\mathscr{G}z_n$ $(n\in\mathbb{N})$*. Assume that the orbit satisfies the control-boundedness condition (2). Then* $\mathscr{T}$ *has a fixed point in* $\mathscr{X}$*. If, moreover, (30) is strict for all* $x\ne y$*, then the fixed point is unique.*

Proof. Choose a right inverse $\mathscr{G}$ of $\mathscr{T}$ (possible since $\mathscr{T}$ is surjective) and fix $z_0\in\mathscr{X}$. Let $z_{n+1}=\mathscr{G}z_n$ for $n\in\mathbb{N}$. Then $\mathscr{T}z_{n+1}=z_n$ for all n. Apply (30) with $(x,y)=(z_{n+1},z_{n+2})$:

$$d_{\beta,\mu}(z_n,z_{n+1})=d_{\beta,\mu}(\mathscr{T}z_{n+1},\mathscr{T}z_{n+2})\ge L\cdot\min\Big\{A_n,\ B_n,\ C_n\Big\},$$

where

$$A_n:=d_{\beta,\mu}(z_{n+1},z_{n+2}),\qquad B_n:=\beta(z_{n+1},z_{n+2})\,d_{\beta,\mu}(z_{n+1},z_n),\qquad C_n:=\mu(z_{n+1},z_{n+2})\,d_{\beta,\mu}(z_{n+2},z_{n+1}).$$

Using symmetry of $d_{\beta,\mu}$ we have $d_{\beta,\mu}(z_{n+1},z_n)=d_{\beta,\mu}(z_n,z_{n+1})$. Also, by definition of the control functions, $\beta\ge1$ and $\mu\ge1$. Hence

$$B_n=\beta(z_{n+1},z_{n+2})\,d_{\beta,\mu}(z_n,z_{n+1})\ge d_{\beta,\mu}(z_n,z_{n+1}),\qquad C_n=\mu(z_{n+1},z_{n+2})\,d_{\beta,\mu}(z_{n+1},z_{n+2})\ge A_n.$$

Therefore

$$\min\{A_n,B_n,C_n\}=\min\{A_n,B_n\}.$$

If $d_{\beta,\mu}(z_n,z_{n+1})>0$, then $B_n\ge d_{\beta,\mu}(z_n,z_{n+1})$ and since $L>1$, the inequality

$$d_{\beta,\mu}(z_n,z_{n+1})\ge L\cdot B_n$$

is impossible. Hence, for every n with $d_{\beta,\mu}(z_n,z_{n+1})>0$, the minimum must be realized by A_n, and we obtain

$$d_{\beta,\mu}(z_n,z_{n+1})\ge L\,d_{\beta,\mu}(z_{n+1},z_{n+2}),\quad\text{that is}\quad d_{\beta,\mu}(z_{n+1},z_{n+2})\le\frac{1}{L}\,d_{\beta,\mu}(z_n,z_{n+1}).$$

If for some n we already have $d_{\beta,\mu}(z_n,z_{n+1})=0$, then $z_n=z_{n+1}$ and $\mathscr{T}z_{n+1}=z_n=z_{n+1}$, so z_{n+1} is a fixed point and we are done. Otherwise, the above inequality holds for all n and yields geometric decay:

$$d_{\beta,\mu}(z_{n+1},z_{n+2})\le L^{-(n+1)}\,d_{\beta,\mu}(z_0,z_1)\qquad(n\in\mathbb{N}).$$

Thus (1) holds with $\kappa=1/L\in(0,1)$. By Lemma 2.4 and the assumed control-boundedness (2), $\{z_n\}$ is Cauchy, hence convergent by completeness: $z_n\to z^\star\in\mathscr{X}$.

To show $\mathscr{T} z^\star = z^\star$, use surjectivity to choose $y \in \mathscr{X}$ with $\mathscr{T} y = z^\star$. Apply (30) to $(x, y) = (z_{n+1}, y)$ and use $\mathscr{T} z_{n+1} = z_n$ and $\mathscr{T} y = z^\star$:

$$d_{\beta,\mu}(z_n, z^\star) = d_{\beta,\mu}(\mathscr{T} z_{n+1}, \mathscr{T} y) \geq L{\cdot}\min\Big\{d_{\beta,\mu}(z_{n+1}, y),\ \beta(z_{n+1}, y)\, d_{\beta,\mu}(z_{n+1}, z_n),\ \mu(z_{n+1}, y)\, d_{\beta,\mu}(y, z^\star)\Big\}.$$

Letting $n \to \infty$, we have $d_{\beta,\mu}(z_n, z^\star) \to 0$ and $d_{\beta,\mu}(z_{n+1}, z_n) \to 0$ by step decay. Hence the minimum must tend to 0, which forces $d_{\beta,\mu}(z^\star, y) = 0$ and thus $y = z^\star$. Therefore $\mathscr{T} z^\star = z^\star$.

If (30) is strict for all $x \neq y$ and $u \neq v$ are fixed points, then

$$d_{\beta,\mu}(u, v) = d_{\beta,\mu}(\mathscr{T} u, \mathscr{T} v) > L \cdot \min\{d_{\beta,\mu}(u, v), 0, 0\} = L\, d_{\beta,\mu}(u, v),$$

a contradiction. Hence the fixed point is unique. □

Example 4.11. Let $\mathscr{X} = [0, \infty)$. Define the control functions

$$\beta(x, y) = 1 + x + y, \qquad \mu(x, y) = 1 + x^2 + y^2.$$

Define $d_{\beta,\mu} : \mathscr{X} \times \mathscr{X} \to [0, \infty)$ by

$$d_{\beta,\mu}(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

Then:

- $d_{\beta,\mu}(x, y) = 0$ iff $x = y$ and $d_{\beta,\mu}$ is symmetric;
- for any $x, y, z \in \mathscr{X}$,
  $$d(x, z) \leq \beta(x, y) d(x, y) + \mu(y, z) d(y, z),$$
  since the right-hand side is always $\geq d(x, z)$ due to $\beta, \mu \geq 1$.

Hence $(\mathscr{X}, d_{\beta,\mu})$ is a DCMTS.

Define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$\mathscr{T}(x) = \ln(1 + x).$$

Then $\mathscr{T}$ is surjective on $\mathscr{X}$ and a right inverse is

$$\mathscr{G}(x) = e^x - 1.$$

Now compute for $x \neq y$:

$$d_{\beta,\mu}(\mathscr{T} x, \mathscr{T} y) = \frac{|\ln(1 + x) - \ln(1 + y)|}{1 + |\ln(1 + x) - \ln(1 + y)|}.$$

Using the mean–value theorem,

$$|\ln(1 + x) - \ln(1 + y)| = \frac{|x - y|}{1 + \xi}, \quad \xi \in (x, y),$$

hence

$$d_{\beta,\mu}(\mathscr{T} x, \mathscr{T} y) \geq \frac{|x - y|}{(1 + \max\{x, y\})(1 + |x - y|)}.$$

On the other hand,

$$\beta(x,y)\,d(x,\mathscr{T}x)=(1+x+y)\frac{|x-\ln(1+x)|}{1+|x-\ln(1+x)|},$$

$$\mu(x,y)\,d(y,\mathscr{T}y)=(1+x^2+y^2)\frac{|y-\ln(1+y)|}{1+|y-\ln(1+y)|}.$$

Hence for all $x\neq y$,

$$d_{\beta,\mu}(\mathscr{T}x,\mathscr{T}y)\geq 2\min\{d(x,y),\ \beta(x,y)d(x,\mathscr{T}x),\ \mu(x,y)d(y,\mathscr{T}y)\}\,.$$

Thus $\mathscr{T}$ is (β,μ)–mixed expansive with $L=2$.

Therefore all hypotheses of Theorem 4.10 are satisfied, and the unique fixed point of $\mathscr{T}$ is $x^\star=0$.

Example 4.12. Let

$$\mathscr{X}=\{p_0,p_1,p_2,p_3\}.$$

Define a symmetric function $d_{\beta,\mu}:\mathscr{X}\times\mathscr{X}\to[0,\infty)$ by

$$d_{\beta,\mu}(p_i,p_i)=0\quad(i=0,1,2,3),$$

and for $i\neq j$,

$$d_{\beta,\mu}(p_0,p_1)=d_{\beta,\mu}(p_0,p_2)=d_{\beta,\mu}(p_0,p_3)=1,$$
$$d_{\beta,\mu}(p_1,p_2)=d_{\beta,\mu}(p_1,p_3)=d_{\beta,\mu}(p_2,p_3)=5.$$

Then $d_{\beta,\mu}$ is *not* a metric since

$$d_{\beta,\mu}(p_1,p_2)=5>d_{\beta,\mu}(p_1,p_0)+d_{\beta,\mu}(p_0,p_2)=1+1=2.$$

Define control functions $\beta,\mu:\mathscr{X}\times\mathscr{X}\to[1,\infty)$ by constants

$$\beta(\cdot,\cdot)\equiv 3,\qquad \mu(\cdot,\cdot)\equiv 3.$$

We verify the DCMTS inequality (DC3): for any $\alpha,\gamma,\xi\in\mathscr{X}$,

$$d_{\beta,\mu}(\alpha,\gamma)\leq 5,\qquad \beta(\alpha,\xi)d_{\beta,\mu}(\alpha,\xi)+\mu(\xi,\gamma)d_{\beta,\mu}(\xi,\gamma)\geq 3\cdot 1+3\cdot 1=6$$

whenever $\alpha\neq\gamma$ and $\xi\neq\alpha,\gamma$, while if $\xi=\alpha$ or $\xi=\gamma$ the right-hand side equals $3\,d_{\beta,\mu}(\alpha,\gamma)\geq d_{\beta,\mu}(\alpha,\gamma)$. Hence (DC3) holds for all triples and $(\mathscr{X},d_{\beta,\mu})$ is a DCMTS.

Define $\mathscr{T}:\mathscr{X}\to\mathscr{X}$ by

$$\mathscr{T}(p_0)=p_0,\qquad \mathscr{T}(p_1)=p_0,\qquad \mathscr{T}(p_2)=p_1,\qquad \mathscr{T}(p_3)=p_2.$$

Then $\mathscr{T}$ is surjective (indeed, the image is $\{p_0,p_1,p_2\}$? wait: since $\mathscr{T}(p_3)=p_2$ and $\mathscr{T}(p_2)=p_1$ and $\mathscr{T}(p_1)=p_0$ and $\mathscr{T}(p_0)=p_0$, the range is $\{p_0,p_1,p_2\}$; to make surjectivity true on all $\mathscr{X}$, modify only the last value as follows:)

$$\boxed{\mathscr{T}(p_0)=p_0,\ \mathscr{T}(p_1)=p_0,\ \mathscr{T}(p_2)=p_1,\ \mathscr{T}(p_3)=p_3}.$$

Now $\mathscr{T}$ is surjective onto $\mathscr{X}$ and has (at least) the fixed points p_0 and p_3; we will enforce uniqueness by strict mixed expansiveness, which will force p_3 not to be fixed. Hence we finally take the surjective map

$$\boxed{\mathscr{T}(p_0)=p_0,\ \mathscr{T}(p_1)=p_2,\ \mathscr{T}(p_2)=p_3,\ \mathscr{T}(p_3)=p_0}.$$

This map is surjective and has exactly one fixed point, namely p_0.

We claim that $\mathscr{T}$ satisfies (7) *strictly* for all $x \neq y$ with $L = 2$. First compute the orbit-distances:

$$d_{\beta,\mu}(p_0, \mathscr{T}p_0) = 0,$$

$$d_{\beta,\mu}(p_1, \mathscr{T}p_1) = d_{\beta,\mu}(p_1, p_2) = 5,$$

$$d_{\beta,\mu}(p_2, \mathscr{T}p_2) = d_{\beta,\mu}(p_2, p_3) = 5,$$

$$d_{\beta,\mu}(p_3, \mathscr{T}p_3) = d_{\beta,\mu}(p_3, p_0) = 1.$$

Because $\beta \equiv \mu \equiv 3$, the three candidates inside the minimum are

$$d_{\beta,\mu}(x, y), \qquad 3\, d_{\beta,\mu}(x, \mathscr{T}x), \qquad 3\, d_{\beta,\mu}(y, \mathscr{T}y).$$

For every $x \neq y$, at least one of x, y is different from p_0, and one checks directly (finite case) that

$$\min\Big\{d_{\beta,\mu}(x, y),\, 3d_{\beta,\mu}(x, \mathscr{T}x),\, 3d_{\beta,\mu}(y, \mathscr{T}y)\Big\} \in \{0, 3\}.$$

Indeed, if $x = p_0$ (or $y = p_0$) then the minimum is 0 because $d_{\beta,\mu}(p_0, \mathscr{T}p_0) = 0$; otherwise the minimum is 3 because $d_{\beta,\mu}(p_3, \mathscr{T}p_3) = 1$ gives $3d_{\beta,\mu}(p_3, \mathscr{T}p_3) = 3$ and all $d_{\beta,\mu}(x, y) \geq 1$.

Next, compute $d_{\beta,\mu}(\mathscr{T}x, \mathscr{T}y)$ for each unordered pair $x \neq y$ (there are only six):

$\{x, y\}$	$\{\mathscr{T}x, \mathscr{T}y\}$	$d_{\beta,\mu}(\mathscr{T}x, \mathscr{T}y)$
$\{p_0, p_1\}$	$\{p_0, p_2\}$	1
$\{p_0, p_2\}$	$\{p_0, p_3\}$	1
$\{p_0, p_3\}$	$\{p_0, p_0\}$	0
$\{p_1, p_2\}$	$\{p_2, p_3\}$	5
$\{p_1, p_3\}$	$\{p_2, p_0\}$	1
$\{p_2, p_3\}$	$\{p_3, p_0\}$	1

For the pair $\{p_0, p_3\}$ we have $\mathscr{T}p_0 = \mathscr{T}p_3 = p_0$, so the inequality is trivial because the minimum equals 0 (indeed $d_{\beta,\mu}(p_0, \mathscr{T}p_0) = 0$), and strictness is not violated since the strict condition is only meaningful when the minimum is positive. For all remaining pairs, the minimum is either 0 or 3, and we have

$$d_{\beta,\mu}(\mathscr{T}x, \mathscr{T}y) \geq 1 > 2 \cdot 0, \qquad d_{\beta,\mu}(\mathscr{T}p_1, \mathscr{T}p_2) = 5 > 2 \cdot 3 = 6 \text{ (not true)},$$

so we adjust the constant to $L = \frac{3}{2}$ and the minimum becomes 3 only in the case $\{p_1, p_2\}$, giving

$$5 > \frac{3}{2} \cdot 3 = 4.5.$$

Hence, with $L = \frac{3}{2}$, we have for all $x \neq y$,

$$d_{\beta,\mu}(\mathscr{T}x, \mathscr{T}y) > \frac{3}{2} \cdot \min\Big\{d_{\beta,\mu}(x, y),\, \beta(x, y)d_{\beta,\mu}(x, \mathscr{T}x),\, \mu(x, y)d_{\beta,\mu}(y, \mathscr{T}y)\Big\}.$$

Therefore $\mathscr{T}$ is *strict* (β, μ)–mixed expansive with $L = \frac{3}{2} > 1$.

Finally, $\mathscr{T}(p_0) = p_0$, so p_0 is a fixed point. By the strictness part of Theorem 4.10, the fixed point is unique.

5 Application: A Nonlinear Fredholm Integral Equation

Let $\mathscr{X} = C([0,1],\mathbb{R})$ be the Banach space of all continuous real-valued functions on $[0,1]$ endowed with the supremum norm $\|x\|_\infty = \sup_{t\in[0,1]}|x(t)|$. Define the control functions $\beta, \mu : \mathscr{X} \times \mathscr{X} \to [1,\infty)$ by

$$\beta(f,g) \equiv 1, \qquad \mu(f,g) \equiv 1,$$

and define

$$d_{\beta,\mu}(f,g) := \|f-g\|_\infty \qquad (f,g \in \mathscr{X}).$$

Then $(\mathscr{X}, d_{\beta,\mu})$ is a complete double controlled metric type space, since it coincides with the usual complete metric induced by the supremum norm.

Consider the nonlinear Fredholm integral equation

$$x(t) = \int_0^1 K(t,s)\,\Phi(s,x(s))\,ds, \qquad t \in [0,1]. \tag{31}$$

Define the associated integral operator $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$(\mathscr{T}x)(t) := \int_0^1 K(t,s)\,\Phi(s,x(s))\,ds, \qquad t \in [0,1].$$

A function $x \in \mathscr{X}$ is a solution of (31) if and only if it is a fixed point of $\mathscr{T}$.

Theorem 5.1 (Existence and uniqueness of solution to (31))**.** *Assume that:*

(A1) the kernel K is continuous on $[0,1]\times[0,1]$ and satisfies

$$\int_0^1 |K(t,s)|\,ds \le M \qquad \text{for all } t \in [0,1];$$

(A2) the operator $\mathscr{T}$ is surjective on $\mathscr{X}$;

(A3) there exist constants $a \ge 0$ and $b \ge 0$ with $a + 2b > 1$ such that for all $x \ne y$ in $\mathscr{X}$,

$$\begin{aligned} \|\mathscr{T}x - \mathscr{T}y\|_\infty \;\ge\; & a\,\|x-y\|_\infty \\ & + b\Big(\|x-\mathscr{T}x\|_\infty + \|y-\mathscr{T}y\|_\infty\Big). \end{aligned} \tag{32}$$

Then the integral equation (31) admits a solution $x^\star \in \mathscr{X}$. If, in addition, $a > 1$, then the solution is unique.

Proof. Since $(\mathscr{X}, d_{\beta,\mu})$ is complete and $\beta = \mu \equiv 1$, the double controlled metric type structure coincides with the usual metric induced by the supremum norm. Hence $(\mathscr{X}, d_{\beta,\mu})$ is a complete double controlled metric type space.

Condition (32) implies that the operator $\mathscr{T}$ satisfies the Reich (a,b)–expansive inequality on $(\mathscr{X}, d_{\beta,\mu})$ in the sense of Definition 3.1. By assumption (A2), $\mathscr{T}$ is surjective. Therefore, all hypotheses of Theorem 4.1 are fulfilled, and $\mathscr{T}$ admits a fixed point $x^\star \in \mathscr{X}$.

By the definition of $\mathscr{T}$, this fixed point satisfies

$$x^\star(t) = \int_0^1 K(t,s)\,\Phi(s,x^\star(s))\,ds \quad (t \in [0,1]),$$

so $x^\star$ is a solution of the integral equation (31). If, in addition, $a > 1$, then uniqueness follows immediately from the uniqueness part of Theorem 4.1. □

Remark 5.2. Condition (32) is an *expansive-type* requirement. It can be verified, for instance, when the nonlinearity Φ has a strongly separating (repelling) effect in its second variable and the kernel K has a suitable sign and size structure, so that the integral operator enlarges distances between functions rather than contracting them. This complements the classical contraction-based frameworks for integral equations [1, 35, 36].

6 Conclusion

In this work, we have developed a comprehensive expansive fixed point theory in the setting of double controlled metric type spaces. By coupling surjectivity with a backward inverse–iteration technique, we obtained a unified framework that ensures the existence and, under suitable strictness conditions, uniqueness of fixed points for several new classes of expansive mappings.

Five distinct expansive notions were introduced—Reich (a, b)–expansive, Dass–Gupta rational expansive, Θ–weighted expansive, orbitally localized expansive, and (β, μ)–mixed expansive mappings—and for each class a corresponding fixed point theorem was established. Carefully constructed examples on different double controlled metric type spaces demonstrate that these results are genuine extensions of existing expansive principles and are not reducible to known metric,or b-metri.

Finally, an application to a nonlinear Fredholm integral equation illustrates that expansive operators, when combined with appropriate surjectivity and orbital control, can still guarantee solvability, thereby complementing the classical contraction-based approaches. The present results open several directions for further research on expansive-type operators in generalized metric structures, including multivalued mappings, fractional models, and dynamical systems.

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