Automorphism of Zero Divisor Graphs of Nilradicals of Commutative Finite Local Rings

Original Research Article

Abstract

The zero-divisor graph associated with a commutative ring encodes deep algebraic information in a combinatorial framework. In this paper, we investigate the automorphism groups of zero-divisor graphs arising from the nonzero nilradical of finite local rings of the form $\mathbb{Z}_{p^k}$. By exploiting the natural p-adic valuation on nilpotent elements, we obtain a canonical stratification of the vertex set into valuation levels. This structure allows for a precise description of graph automorphisms as products of symmetric groups acting on valuation classes. The results provide a complete characterization of graph symmetries in this local setting and establish a foundational case for the broader theory of automorphisms of zero-divisor graphs over finite rings.

Keywords: zero-divisor graph; nilradical; automorphism group; local ring; p-adic valuation.

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1 Introduction

Zero-divisor graphs have become an important interface between commutative algebra and graph theory since the seminal work of Anderson and Livingston [2]. By encoding algebraic interactions among zero divisors as adjacency relations, these graphs allow algebraic properties of rings to be investigated through graph-theoretic invariants such as connectivity, diameter, girth, clique number, and chromatic number. A substantial body of literature has since developed around these themes, particularly for finite commutative rings and Artinian local rings [5, 6]. Much of the existing research has focused on the structural properties of $\Gamma(R)$ and its variants, including Beck's total graph [5]. Results by [2, 1, 4], and others establish sharp bounds on diameter and girth, characterize completeness and multipartite structure, and relate these graph invariants to ideal-theoretic properties such as nilpotency, chain conditions on prime ideals, and the behavior of the Jacobson radical . In particular, for local rings, the structure of the nilradical plays a decisive role in determining the shape of the associated zero-divisor graph [8]. This role of the nilradical has also been emphasized in the work of [3], where annihilator-based partitions are shown to strongly influence graph structure. In contrast, comparatively little attention has been paid to the automorphism groups of zero-divisor graphs, especially those arising from nilradicals. While it is well understood that $\Gamma(R)$ often reflects ring-theoretic symmetries, explicit descriptions of $\mathrm{Aut}(\Gamma(R))$ are known only for restricted classes of rings [6]. Existing studies on graph automorphisms have largely focused on reduced rings, Gorenstein rings, or special constructions such as idealizations and direct products [9]. Even in the case of finite local rings, where $\Gamma(R)$ is finite and structurally constrained, the automorphism groups have not been systematically described in terms of intrinsic algebraic data, apart from specific families such as Galois rings [7]. This gap is particularly evident for principal ideal local rings such as $\mathbb{Z}_{p^k}$. Although the zero-divisor graphs of these rings are well understood with respect to diameter, girth, and completeness [2, 8], the precise manner in which p-adic valuation and nilpotency stratify the vertex set—and thereby restrict graph automorphisms—has not been fully articulated. More broadly, the extent to which valuation-theoretic data determines $\mathrm{Aut}(\Gamma)$ for nilradicals remains unclear, for local rings cases. The purpose of this paper is to address this gap by providing an analysis of the automorphism groups of zero-divisor graphs arising from the nonzero nilradical of finite local rings of the form $\mathbb{Z}_{p^k}$. We show that the p-adic valuation induces a canonical stratification of the vertex set into invariant layers, each consisting of vertices with identical graph-theoretic neighborhoods. This stratification forces the automorphism group to decompose as a direct product of symmetric groups acting independently on valuation classes. Beyond yielding explicit formulas for $\mathrm{Aut}(\Gamma)$, this approach clarifies the structural reason behind the observed symmetry: in local rings, valuation levels act as rigid combinatorial invariants. The results presented here form a foundational case that informs and motivates subsequent investigations of semilocal and mixed-exponent rings, where similar valuation-based mechanisms govern graph automorphisms but interact across multiple maximal ideals.

2 Algebraic Setting and Nilpotent Structure

Throughout this section, we consider the principal ideal local ring $\mathbb{Z}_{p^k}$, where p is a prime and $k \geq 2$. As a finite commutative local ring with a unique maximal ideal (p), the structure of $\mathbb{Z}_{p^k}$ is governed entirely by p-adic valuation. In particular, the behavior of nilpotent elements is completely determined by divisibility by p. Understanding the precise form of nilpotent elements is essential for the study of zero-divisor graphs of nilradicals. Since adjacency in these graphs is ultimately controlled by annihilation conditions modulo p^k, the p-adic valuation of each vertex serves as a fundamental invariant. The following theorem characterizes the nilpotent elements of $\mathbb{Z}_{p^k}$ explicitly and establishes a canonical representation that will later be used to stratify the vertex set of the associated zero-divisor graph into valuation-based layers.

Theorem 2.1. *Let $R = \mathbb{Z}_{p^k}$, where p is a prime and $k \geq 2$. Then an element $x \in R$ is nilpotent if and only if $p \mid x$. Moreover, every nonzero nilpotent element admits a unique representation $x = p^r a, \quad 1 \leq r \leq k-1, \quad \gcd(a,p) = 1.$*

Proof. Since $\mathbb{Z}_{p^k}$ is a finite commutative ring, its nilradical coincides with the intersection of all prime ideals. Because $\mathbb{Z}_{p^k}$ is local with unique maximal ideal (p), this intersection is precisely (p).

Equivalently, the nilradical of $\mathbb{Z}_{p^k}$ is generated by p, and an element is nilpotent if and only if it belongs to the ideal (p). Let $x \in \mathbb{Z}_{p^k}$. If $p \nmid x$, then x is a unit in R, and hence no power of x can be zero. Therefore, x is not nilpotent. Conversely, if $p \mid x$, then $x \in (p)$, and since $(p)^k = (0)$ in $\mathbb{Z}_{p^k}$, it follows that some power of x must be zero. This proves that x is nilpotent if and only if $p \mid x$. Now suppose x is a nonzero nilpotent element. By the fundamental theorem of arithmetic, there exists a unique integer $r \geq 1$ such that $x = p^r a$, where a is an integer satisfying $\gcd(a, p) = 1$. Since $x \not\equiv 0 \pmod{p^k}$, we must have $r \leq k - 1$. This yields the stated representation. To verify nilpotency explicitly, observe that $x^m = p^{rm} a^m \equiv 0 \pmod{p^k}$ if and only if $rm \geq k$. Because $r \geq 1$, such an integer m always exists, confirming that x is nilpotent. Finally, the representation $x = p^r a$ is unique, since the p-adic valuation $v_p(x) = r$ is uniquely determined and the residue class of a modulo p^k is fixed once r is specified. □

3 Canonical Decomposition of the Vertex Set

A central step in understanding the automorphism group of a zero-divisor graph is the identification of intrinsic partitions of the vertex set that are dictated by the underlying algebraic structure of the ring. For the ring $\mathbb{Z}_{p^k}$, the natural invariant governing this structure is the p-adic valuation. As shown in the preceding section, every nonzero nilpotent element admits a unique factorization of the form $p^r a$ with $1 \leq r \leq k - 1$ and $\gcd(a, p) = 1$. This representation immediately suggests a stratification of the vertex set according to valuation level. From a graph-theoretic perspective, such a stratification is not merely algebraically convenient: vertices with the same p-adic valuation exhibit identical annihilation behavior and hence share the same adjacency relations in the zero-divisor graph. Conversely, vertices with different valuations interact differently with other elements of the nilradical. The purpose of this section is to formalize this observation by decomposing the vertex set of $\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^k}))$ into canonical, disjoint subsets indexed by valuation level and to compute their exact cardinalities. This decomposition forms the backbone for later results on degree sequences, neighborhood equivalence, and graph automorphisms.

Theorem 3.1. *The vertex set of the zero-divisor graph* $\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^k}))$ *admits the disjoint decomposition* $V = \bigsqcup_{r=1}^{k-1} S_{p^r}, \qquad S_{p^r} = \{\, p^r a : \gcd(a, p) = 1 \,\}$, *with* $|S_{p^r}| = p^{k-r-1}(p - 1)$.

Proof. Let x be a vertex of $\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^k}))$. By definition, x is a nonzero nilpotent element of $\mathbb{Z}_{p^k}$. From the characterization of nilpotent elements established earlier, there exists a unique integer r with $1 \leq r \leq k - 1$ and a unique residue class a satisfying $\gcd(a, p) = 1$ such that $x = p^r a$. This shows that every vertex belongs to at least one subset S_{p^r}. To see that the union is disjoint, suppose that $x \in S_{p^r} \cap S_{p^s}$ for some $r \neq s$. Then $x = p^r a = p^s b$ for some a, b coprime to p. Without loss of generality, assume $r < s$. Dividing both sides by p^r yields $a = p^{s-r} b$, which implies that $p \mid a$, contradicting the assumption $\gcd(a, p) = 1$. Hence each vertex belongs to exactly one valuation class S_{p^r}, and the decomposition is disjoint and exhaustive. We now compute the cardinality of each subset S_{p^r}. Fix an integer r with $1 \leq r \leq k - 1$. Distinct elements of S_{p^r} correspond precisely to distinct residue classes of a modulo p^{k-r} such that $\gcd(a, p) = 1$. Indeed, if $p^r a \equiv p^r a' \pmod{p^k}$, then $a \equiv a' \pmod{p^{k-r}}$. Thus, counting elements of S_{p^r} is equivalent to counting the units of the ring $\mathbb{Z}_{p^{k-r}}$. The number of units modulo p^{k-r} is given by Euler's totient function: $\varphi(p^{k-r}) = p^{k-r} - p^{k-r-1} = p^{k-r-1}(p - 1)$. Therefore, $|S_{p^r}| = p^{k-r-1}(p - 1)$, □

4 Adjacency Relations and Degree Invariants

Having established a canonical decomposition of the vertex set according to p-adic valuation, the next step is to determine how these valuation levels govern adjacency in the zero-divisor graph.

Since adjacency is defined through multiplicative annihilation modulo p^k, it is natural to expect that the sum of valuations of two vertices plays a decisive role. Indeed, the product of two nilpotent elements vanishes modulo p^k precisely when the combined power of p in their factorization reaches or exceeds the modulus exponent. From a graph-theoretic standpoint, this observation has two important consequences. First, it yields a simple and explicit criterion for adjacency that depends only on valuation levels and not on unit coefficients. Second, it implies that all vertices within a fixed valuation class have identical neighborhoods and hence identical degrees. This uniformity is a crucial structural feature: degree is a graph invariant preserved by automorphisms, and thus valuation classes become rigid building blocks in the symmetry analysis of $\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^k}))$.

Theorem 4.1. *Let $x = p^r a$ and $y = p^s b$ be distinct vertices of $\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^k}))$. Then x is adjacent to y if and only if $r + s \geq k$. Consequently, all vertices in the same set S_{p^r} have identical degrees.*

Proof. By definition of the zero-divisor graph, two distinct vertices x and y are adjacent if and only if $xy \equiv 0 \pmod{p^k}$. Using the unique p-adic representations $x = p^r a$ and $y = p^s b$, where $\gcd(a, p) = \gcd(b, p) = 1$, we compute $xy = p^{r+s}ab$. Since a and b are units modulo p, their product ab is also a unit modulo p. In particular, ab is not divisible by p, and therefore contributes no additional p-power to the factorization of xy. It follows that the highest power of p dividing xy is exactly p^{r+s}. Consequently, $xy \equiv 0 \pmod{p^k} \iff p^k \mid p^{r+s} \iff r + s \geq k$. This establishes the claimed adjacency criterion. We now examine the degree of a vertex $x = p^r a \in S_{p^r}$. By the criterion just proved, x is adjacent precisely to those vertices $y = p^s b$ for which $s \geq k - r$. Thus the neighborhood of x is given by $N(x) = \bigcup_{s=k-r}^{k-1} S_{p^s}$. Importantly, this description depends only on the valuation level r and not on the specific choice of the unit a. Since the cardinalities $|S_{p^s}|$ are fixed for each s, the degree of x is $\deg(x) = \sum_{s=k-r}^{k-1} |S_{p^s}|$, which is independent of the particular vertex chosen within S_{p^r}. Hence all vertices belonging to the same valuation class S_{p^r} have identical degrees. □

5 Invariance of Valuation Levels

The valuation-based decomposition of the vertex set established in the previous sections is not merely a convenient classification of vertices; it encodes essential structural information about the zero-divisor graph. In particular, valuation levels determine adjacency patterns and vertex degrees, both of which are invariant under graph isomorphisms. Since automorphisms preserve all graph-theoretic invariants derived from adjacency, they cannot arbitrarily rearrange vertices across valuation classes without violating these invariants. This observation suggests that valuation subsets should be fixed setwise by every graph automorphism. The result below formalizes this intuition and establishes valuation levels as rigid combinatorial strata that govern the symmetry structure of $\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^k}))$.

Theorem 5.1. *Each valuation subset S_{p^r} is invariant under $\mathrm{Aut}\big(\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^k}))\big)$.*

Proof. From Theorem 4.1, the degree of a vertex $x = p^r a$ depends only on its valuation level r. More precisely, all vertices in S_{p^r} share the same degree, while vertices belonging to distinct valuation subsets S_{p^r} and S_{p^s} with $r \neq s$ have different degrees. Let $\varphi \in \mathrm{Aut}(\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^k})))$. Since φ is a graph automorphism, it preserves adjacency and therefore preserves vertex degrees. Hence, for any vertex $x \in S_{p^r}$, the image $\varphi(x)$ must have the same degree as x. If $\varphi(x)$ were to lie in a different valuation subset S_{p^s} with $s \neq r$, then $\deg(\varphi(x)) = \deg(x)$ would contradict the fact that vertices in S_{p^r} and S_{p^s} have distinct degrees. Therefore, no such mapping is possible. It follows that $\varphi(S_{p^r}) \subseteq S_{p^r}$ for all $r = 1, \ldots, k-1$. Since φ is bijective, the inclusion must in fact be equality. Thus each valuation subset S_{p^r} is preserved setwise by every automorphism of the graph. This establishes the invariance of valuation levels under $\mathrm{Aut}(\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^k})))$. □

6 Automorphism Group Decomposition

The preceding sections establish two fundamental structural facts about $\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^k}))$: first, the vertex set decomposes into valuation-defined subsets S_{p^r}, and second, these subsets are invariant under graph automorphisms. What remains is to determine how automorphisms act *within* each invariant subset and how these actions combine globally. Since all vertices in a fixed valuation class share identical adjacency relations, they are indistinguishable from a purely graph-theoretic perspective. This symmetry strongly suggests that automorphisms act freely within valuation layers but are prohibited from mixing vertices across different layers. The theorem below formalizes this intuition and yields a complete group-theoretic description of $\mathrm{Aut}(\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^k})))$.

Theorem 6.1. *Let $R = \mathbb{Z}_{p^k}$ with $k \geq 3$. Then $\mathrm{Aut}\big(\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^k}))\big) \cong \prod_{r=1}^{k-1} S_{p^{k-r-1}(p-1)}$.*

Proof. By Theorem 5.1, each valuation subset S_{p^r} is invariant under every graph automorphism. Hence any automorphttp://www.writingjobz.com/hism φ restricts to a permutation of each S_{p^r}, yielding a homomorphism

$$\mathrm{Aut}\big(\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^k}))\big) \longrightarrow \prod_{r=1}^{k-1} \mathrm{Sym}(S_{p^r}).$$

By Theorem 4.1, all vertices within a fixed subset S_{p^r} have identical neighborhoods. Consequently, any permutation of S_{p^r} preserves adjacency with all other vertices of the graph. It follows that every permutation of S_{p^r} extends to a graph automorphism that fixes all other valuation levels pointwise. This shows that the above homomorphism is surjective. Injectivity follows from the fact that an automorphism acting trivially on each S_{p^r} must fix every vertex of the graph and hence is the identity automorphism. Therefore, $\mathrm{Aut}\big(\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^k}))\big) \cong \prod_{r=1}^{k-1} \mathrm{Sym}(S_{p^r})$. Finally, since $|S_{p^r}| = p^{k-r-1}(p-1)$ by Theorem 3.1, the result follows. $\square$

Remark 6.1. The case $k = 2$ constitutes a degenerate but instructive boundary case of the preceding theory. When $R = \mathbb{Z}_{p^2}$, every nonzero nilpotent element is of the form pa with $\gcd(a, p) = 1$, so the vertex set reduces to a single valuation class $V(\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^2}))) = S_p, \quad |S_p| = p - 1$. For any two distinct vertices $x = pa$ and $y = pb$, we have $xy = p^2ab \equiv 0 \pmod{p^2}$, so every pair of distinct vertices is adjacent. Hence, $\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^2})) \cong K_{p-1}$, the complete graph on $p - 1$ vertices. In this situation, the valuation-based layering collapses entirely, and there are no degree or neighborhood distinctions among vertices. Consequently, every permutation of the vertex set preserves adjacency, and the automorphism group is the full symmetric group $\mathrm{Aut}\big(\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^2}))\big) \cong S_{p-1}$. This confirms that the decomposition theorem of the preceding section applies nontrivially only for $k \geq 3$, while $k = 2$ represents the unique complete-graph exception.

7 Conclusion

This paper provides a complete and explicit determination of the automorphism groups of zero-divisor graphs arising from the nonzero nilradical of finite local rings of the form $\mathbb{Z}_{p^k}$. The central mechanism underlying these results is the stratification of nilpotent elements by p-adic valuation, which induces a rigid layered structure on the vertex set of the graph. These valuation layers serve as invariant combinatorial units: vertices within a single layer are indistinguishable, while vertices across layers are separated by degree and adjacency constraints. As a consequence, the automorphism group decomposes naturally as a direct product of symmetric groups acting independently on each valuation level. This valuation-driven perspective not only clarifies the symmetry structure of $\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^k}))$ but also establishes a foundational case for broader investigations. In subsequent work, similar techniques can be adapted to semilocal rings and mixed-exponent settings, where multiple valuation systems interact to shape graph automorphisms in more intricate ways.

References

[1] Akbari S., Mohamadian H., Yassemi S., *When a zero-divisor graph is planar or a complete r-partite graph*, **270** pp. 169–180, *J. Algebra* (2003).

[2] Anderson D. F., Livingston P. S., *The zero divisor graph of a commutative ring*, **217** pp. 437–447, *J. Algebra* (1999).

[3] Anderson D. F., Badawi A., *On the zero-divisor graph of a ring*, **36** pp. 3073–3092, *Commun. Algebra* (2008).

[4] Axtell M., Stickles J., *Zero divisor ideals and realizable zero divisor graphs*, **2**, *J. Math.* (2009).

[5] Beck I., *Coloring of commutative rings*, **116** pp. 208–226, *J. Algebra* (1988).

[6] DeMeyer F., Schneider E., *Zero divisor graphs of commutative rings: A survey*, **46** pp. 3078–3090, *Commun. Algebra* (2018).

[7] Lao H. M., Owino M. O., Ojiema M. O., *Automorphisms of zero divisor graphs of Galois rings*, **13** pp. (2019), *Int. J. Algebra*.

[8] Lucas T. G., *The diameter of a zero divisor graph*, **301** pp. 174–193, *J. Algebra* (2006).

[9] Spiroff S., Wickham C., *A zero divisor graph determined by equivalence classes of zero divisors*, **39** pp. 2338–2348, *Commun. Algebra* (2011).