Interpolative Fixed Point Theory in Perturbed Metric Spaces via Exact Metric Decomposition

Abstract

Perturbed metric spaces naturally model situations in which observed distances are influenced by systematic or measurement-induced distortions. Motivated by the perturbation–decomposition paradigm, where a measured distance is expressed as the sum of an exact metric and an explicit perturbation term, we develop a new interpolative fixed point theory governed directly by the induced exact metric. Several families of interpolative contractions are introduced, including Kannan-type, Reich–Rus–Çirić-type, Suzuki-triggered, and a unified superclass covering all these cases. For each class, existence, uniqueness, and global convergence of Picard iterations are established in complete perturbed metric spaces. The constructed examples demonstrate that the proposed hypotheses are strictly weaker than those required in classical metric and b-metric settings. As applications, we obtain well-posedness results for a nonlinear integral equation and for a Bellman-type functional equation arising in dynamic programming.

Keywords: perturbed metric space; exact metric; interpolative contraction; fixed point theorem; Picard iteration; integral equation; dynamic programming.
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1 Introduction

Metric fixed point theory traces its origin to the celebrated contraction principle of Banach [1], which established a simple yet powerful criterion for existence and uniqueness of fixed points in complete metric spaces. Since its inception, the theory has evolved along two major axes: the relaxation or modification of contractive conditions, and the enlargement of the underlying distance structure.

On the contractive side, influential generalizations include Kannan-type mappings [2], Reich-type inequalities [3], and the comprehensive quasi-contractive framework introduced by Çirić [4]. These developments demonstrated that fixed point phenomena persist well beyond strict Lipschitz contractions. From a geometric perspective, numerous extensions of metric spaces have been proposed, such as b-metric spaces [5], generalized metric-type spaces [6], partial metric spaces [7, 8], and control-function based generalized metrics [9]. Each of these frameworks has expanded the applicability of fixed point methods to broader analytical contexts.

A more recent and increasingly influential direction is the *interpolative approach*, in which contractive conditions are formulated through weighted or multiplicative combinations of distances involving points and their images. This paradigm has proved particularly effective in unifying several classical fixed point principles while yielding improved convergence flexibility [10, 11, 13, 12]. Independently, a perturbation–decomposition methodology has been introduced to address situations where observed distances are distorted by measurement errors or modeling imperfections. In this setting, the measured distance is decomposed into an exact metric component and an explicit perturbation term, and fixed point results are derived through the induced exact metric [22, 23]. Such an approach naturally arises in discretized integral models [18, 19] and in optimization and dynamic programming problems [20, 21].

The aim of the present work is to integrate these two modern viewpoints by developing fixed point principles for interpolative contractions in perturbed metric spaces, with the contractive control explicitly expressed in terms of the induced exact metric. The results established herein ensure existence, uniqueness, and global convergence of Picard-type iterations. Classical interpolative forms related to Banach and Kannan contractions are recovered as particular cases [10, 11], while the overall framework remains compatible with perturbation decompositions [22]. Furthermore, illustrative examples are provided to show that the imposed hypotheses may be satisfied even when standard metric or b-metric techniques are not readily applicable [5, 9]. Finally, applications are indicated for a nonlinear integral equation and for a Bellman-type functional equation, following the classical operator-theoretic route in fixed point analysis [18, 17].

1.1 Some classical and recent fixed point principles

For completeness and comparison, we recall several fundamental fixed point results that motivate and contextualize the perturbed–interpolative framework developed in this paper.

Theorem 1.1 (Banach contraction principle [1]). *Let (Ω, δ) be a complete metric space and let $\mathcal{T} : \Omega \to \Omega$. If there exists $\lambda \in [0, 1)$ such that*

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda\, \delta(\xi, \eta) \quad \text{for all } \xi, \eta \in \Omega,$$

then $\mathcal{T}$ admits a unique fixed point $\xi^* \in \Omega$, and for every $\xi_0 \in \Omega$ the sequence $\xi_{\nu+1} = \mathcal{T}\xi_\nu$ converges to ξ^*.

Theorem 1.2 (Kannan fixed point theorem [2]). *Let (Ω, δ) be a complete metric space and let $\mathcal{T} : \Omega \to \Omega$. Assume that there exists $\lambda \in [0, \frac{1}{2})$ such that*

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda\big(\delta(\xi, \mathcal{T}\xi) + \delta(\eta, \mathcal{T}\eta)\big) \quad \text{for all } \xi, \eta \in \Omega.$$

Then $\mathcal{T}$ has a unique fixed point in Ω, and the Picard iteration converges to it from any initial point.

Theorem 1.3 (Reich contraction [3]). *Let (Ω, δ) be a complete metric space and let $\mathcal{T} : \Omega \to \Omega$. If there exist $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$ such that*

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \alpha\, \delta(\xi, \eta) + \beta\, \delta(\xi, \mathcal{T}\xi) + \gamma\, \delta(\eta, \mathcal{T}\eta) \quad \text{for all } \xi, \eta \in \Omega,$$

then $\mathcal{T}$ admits a unique fixed point, and the Picard sequence converges to it.

Theorem 1.4 (Çirić quasi-contraction [4]). *Let (Ω, δ) be a complete metric space and let $\mathcal{T} : \Omega \to \Omega$. If there exists $\lambda \in [0, 1)$ such that for all $\xi, \eta \in \Omega$,*

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda \max\Big\{\delta(\xi, \eta),\, \delta(\xi, \mathcal{T}\xi),\, \delta(\eta, \mathcal{T}\eta),\, \delta(\xi, \mathcal{T}\eta),\, \delta(\eta, \mathcal{T}\xi)\Big\},$$

then $\mathcal{T}$ has a unique fixed point in Ω, and the Picard iteration converges to it.

Theorem 1.5 (Banach principle in perturbed metric spaces [22]). *Let* $(\mathit{\Omega}, \Delta, \Pi)$ *be a complete perturbed metric space and define the induced exact metric*

$$\delta(\xi, \eta) := (\Delta - \Pi)(\xi, \eta).$$

If $\mathcal{T} : \mathit{\Omega} \to \mathit{\Omega}$ *satisfies*

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda\, \delta(\xi, \eta) \quad \text{for all } \xi, \eta \in \mathit{\Omega}$$

for some $\lambda \in [0, 1)$*, then* $\mathcal{T}$ *has a unique fixed point* $\xi^* \in \mathit{\Omega}$ *and* $\mathcal{T}^{\nu}\xi \to \xi^*$ *for every* $\xi \in \mathit{\Omega}$ *with respect to* δ*.*

Theorem 1.6 (Interpolative Kannan-type contraction [10, 13]). *Let* $(\mathit{\Omega}, \delta)$ *be a complete metric space and let* $\mathcal{T} : \mathit{\Omega} \to \mathit{\Omega}$*. Assume that there exist* $\lambda \in [0, 1)$ *and* $\alpha \in (0, 1)$ *such that*

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda\, [\delta(\xi, \mathcal{T}\xi)]^{\alpha}\, [\delta(\eta, \mathcal{T}\eta)]^{1-\alpha}$$

for all $\xi, \eta \in \mathit{\Omega}$ *with* $\mathcal{T}\xi \neq \xi$ *or* $\mathcal{T}\eta \neq \eta$*. Then* $\mathcal{T}$ *admits a unique fixed point and Picard iteration converges to it.*

Theorem 1.7 (Interpolative Çirić–Reich–Rus contraction [12]). *Let* $(\mathit{\Omega}, \delta)$ *be a complete metric space and let* $\mathcal{T} : \mathit{\Omega} \to \mathit{\Omega}$*. If there exist* $\lambda \in [0, 1)$ *and* $\alpha, \beta, \gamma \in (0, 1)$ *with* $\alpha + \beta + \gamma < 1$ *such that*

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda\, [\delta(\xi, \mathcal{T}\xi)]^{\alpha}\, [\delta(\eta, \mathcal{T}\eta)]^{\gamma}\, [\delta(\xi, \eta)]^{\beta},$$

then $\mathcal{T}$ *has a unique fixed point in* $\mathit{\Omega}$*, and the Picard iteration converges to it.*

2 Preliminaries

In this section, we recall the basic concepts and terminology required for the development of our main results.

Definition 2.1 (Perturbed metric space [22]). *Let* $\mathit{\Omega}$ *be a nonempty set and let* $\Delta, \Pi : \mathit{\Omega} \times \mathit{\Omega} \to [0, \infty)$ *be two mappings. We say that* Δ *is a* perturbed metric *on* $\mathit{\Omega}$ *with respect to the perturbation mapping* Π *if the function*

$$\delta(\xi, \eta) := (\Delta - \Pi)(\xi, \eta) = \Delta(\xi, \eta) - \Pi(\xi, \eta), \quad \xi, \eta \in \mathit{\Omega},$$

is a classical metric on $\mathit{\Omega}$*. The function* Π *is referred to as the* perturbation term*, while* δ *is called the* exact metric *associated with* Δ*. The triple* $(\mathit{\Omega}, \Delta, \Pi)$ *is then called a* perturbed metric space*.*

Remark 2.1. *In general, the measured distance* Δ *itself is not required to satisfy the axioms of a metric. In particular,* Δ *may fail to satisfy the triangle inequality, symmetry, or both. The key requirement is that once the perturbation* Π *is removed, the resulting mapping* $\delta = \Delta - \Pi$ *restores the classical metric structure. This decomposition distinguishes perturbed metric spaces from other metric-type generalizations, such as b-metric spaces [5], where the triangle inequality is weakened multiplicatively, and from control-function generalized metric spaces [9], where deviations are absorbed into an external control function.*

Remark 2.2. *From an applied perspective,* $\Delta(\xi, \eta)$ *may be interpreted as an observed or computed distance, while* $\Pi(\xi, \eta)$ *represents accumulated numerical error, modeling uncertainty, or systematic distortion. The exact metric* δ *captures the intrinsic geometric structure underlying the perturbed measurement. This interpretation motivates the use of* δ *as the primary tool for convergence and completeness arguments, as emphasized in [22].*

Definition 2.2 (Perturbed convergence [22]). *Let $(\mathit{\Omega}, \Delta, \Pi)$ be a perturbed metric space with exact metric δ. A sequence $\{\xi_\nu\} \subset \mathit{\Omega}$ is said to* converge *to $\xi \in \mathit{\Omega}$ if*

$$\delta(\xi_\nu, \xi) \longrightarrow 0 \quad \text{as } \nu \to \infty.$$

In this case, we write $\xi_\nu \xrightarrow{\delta} \xi$.

Definition 2.3 (Perturbed Cauchy sequence [22]). *A sequence $\{\xi_\nu\} \subset \mathit{\Omega}$ is called* perturbed Cauchy *if it is a Cauchy sequence with respect to the exact metric δ, that is,*

$$\delta(\xi_\nu, \xi_\mu) \longrightarrow 0 \quad \text{as } \nu, \mu \to \infty.$$

Definition 2.4 (Completeness [22]). *A perturbed metric space $(\mathit{\Omega}, \Delta, \Pi)$ is said to be* complete *if every perturbed Cauchy sequence converges in $\mathit{\Omega}$ with respect to the exact metric δ. Equivalently, $(\mathit{\Omega}, \Delta, \Pi)$ is complete if and only if the metric space $(\mathit{\Omega}, \delta)$ is complete.*

Remark 2.3. *The equivalence stated in Definition 2.4 allows one to transfer classical completeness arguments from metric fixed point theory directly to the perturbed setting. In particular, no additional assumptions on the perturbation Π are required beyond those ensuring that δ is a metric.*

Remark 2.4. *Limits with respect to the exact metric δ are unique. Consequently, any sequence in a perturbed metric space can have at most one limit point in the sense of Definition 2.2.*

Remark 2.5. *If a sequence generated by an iterative procedure is Cauchy in the exact metric δ, then convergence is guaranteed whenever $(\mathit{\Omega}, \Delta, \Pi)$ is complete. This observation is central to the Picard iteration technique used in fixed point theory, both in classical metric spaces [1, 3] and in perturbed metric spaces [22, 23].*

Throughout the remainder of this paper, all notions of convergence, Cauchy property, and completeness are understood with respect to the exact metric δ, unless explicitly stated otherwise. This convention ensures consistency with standard metric fixed point arguments while fully incorporating the perturbation structure.

3 New interpolative contractions in perturbed metric spaces

Throughout, $(\mathit{\Omega}, \Delta, \Pi)$ is a perturbed metric space with induced exact metric $\delta = \Delta - \Pi$.

Definition 3.1 (Interpolative perturbed Kannan contraction (IPKC)). *A mapping $\mathcal{T} : \mathit{\Omega} \to \mathit{\Omega}$ is an* IPKC *if there exist constants $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that*

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \le \lambda\, [\delta(\xi, \mathcal{T}\xi)]^{\alpha}\, [\delta(\eta, \mathcal{T}\eta)]^{1-\alpha} \tag{1}$$

for all $\xi, \eta \in \mathit{\Omega}$ with $\mathcal{T}\xi \neq \xi$ or $\mathcal{T}\eta \neq \eta$. This interpolative form is inspired by the metric interpolative Kannan program [10, 13].

Definition 3.2 (Interpolative perturbed Reich–Rus–Çirić contraction (IPRRC)). *A mapping $\mathcal{T} : \mathit{\Omega} \to \mathit{\Omega}$ is an* IPRRC *if there exist $\lambda \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ such that*

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \le \lambda\, [\delta(\xi, \eta)]^{\beta}\, [\delta(\xi, \mathcal{T}\xi)]^{\alpha}\, [\delta(\eta, \mathcal{T}\eta)]^{\gamma} \tag{2}$$

for all $\xi, \eta \in \mathit{\Omega}$. This interpolative regime is motivated by Reich–Rus–Çirić-type contractions [3, 4, 18] and the interpolative approach [11, 12].

Definition 3.3 (Perturbed interpolative Suzuki trigger (PIST))**.** *A mapping $\mathcal{T} : \Omega \to \Omega$ satisfies* PIST *if there exist $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that for all $\xi, \eta \in \Omega$,*

$$\frac{1}{2}\,\delta(\xi, \mathcal{T}\xi) \le \delta(\xi, \eta) \quad \Rightarrow \quad \delta(\mathcal{T}\xi, \mathcal{T}\eta) \le \lambda\,[\delta(\xi, \mathcal{T}\xi)]^{\alpha}\,[\delta(\eta, \mathcal{T}\eta)]^{1-\alpha}. \tag{3}$$

This combines Suzuki-type activation ideas (in metric settings) with interpolation [10, 11].

Definition 3.4 (Generalized interpolative perturbed contractive mapping)**.** *Let (Ω, Δ, Π) be a perturbed metric space with induced exact metric*

$$\delta(\xi, \eta) := (\Delta - \Pi)(\xi, \eta), \qquad \xi, \eta \in \Omega.$$

A mapping $\mathcal{T} : \Omega \to \Omega$ is called a generalized interpolative perturbed contractive mapping *(GIPCM) if there exist constants $\lambda \in [0, 1)$ and exponents $\alpha, \beta, \gamma \in [0, 1)$ with*

$$\alpha + \beta + \gamma < 1,$$

such that for all $\xi, \eta \in \Omega$ satisfying

$$\frac{1}{2}\,\delta(\xi, \mathcal{T}\xi) \le \delta(\xi, \eta), \tag{4}$$

the following inequality holds:

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \le \lambda\,[\delta(\xi, \mathcal{T}\xi)]^{\alpha}\,[\delta(\eta, \mathcal{T}\eta)]^{\gamma}\,[\delta(\xi, \eta)]^{\beta}. \tag{5}$$

4 Main results

Throughout this section (Ω, Δ, Π) denotes a perturbed metric space with induced exact metric $\delta = \Delta - \Pi$. All convergence and completeness are understood with respect to δ.

Theorem 4.1 (Fixed point theorem for IPKC)**.** *Let (Ω, Δ, Π) be complete. If $\mathcal{T} : \Omega \to \Omega$ is an interpolative perturbed Kannan contraction (IPKC), then $\mathcal{T}$ has a unique fixed point $\xi^* \in \Omega$. Moreover, for every $\xi_0 \in \Omega$, the Picard sequence $\xi_{\nu+1} = \mathcal{T}\xi_\nu$ converges to ξ^* in the metric space (Ω, δ).*

Proof. Fix $\xi_0 \in \Omega$ and set $\xi_{\nu+1} = \mathcal{T}\xi_\nu$ for $\nu \in \mathbb{N}$. If $\xi_{\nu_0} = \mathcal{T}\xi_{\nu_0}$ for some ν_0, then $\xi_\nu = \xi_{\nu_0}$ for all $\nu \ge \nu_0$ and the claim follows. Assume henceforth that $\xi_\nu \ne \xi_{\nu+1}$ for all ν, so that $a_\nu := \delta(\xi_\nu, \xi_{\nu+1}) > 0$.

The IPKC condition provides $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \le \lambda\,[\delta(\xi, \mathcal{T}\xi)]^{\alpha}\,[\delta(\eta, \mathcal{T}\eta)]^{1-\alpha} \quad (\xi, \eta \in \Omega).$$

Substituting $(\xi, \eta) = (\xi_\nu, \xi_{\nu-1})$ and using $\xi_{\nu+1} = \mathcal{T}\xi_\nu$ gives

$$a_\nu = \delta(\xi_{\nu+1}, \xi_\nu) \le \lambda\, a_\nu^{\alpha}\, a_{\nu-1}^{1-\alpha}.$$

Dividing by a_ν^{α} yields $a_\nu^{1-\alpha} \le \lambda\, a_{\nu-1}^{1-\alpha}$, and therefore

$$a_\nu \le q\, a_{\nu-1} \quad \text{for all } \nu \ge 1, \qquad q := \lambda^{\frac{1}{1-\alpha}} \in (0, 1).$$

Induction implies $a_\nu \le q^{\nu} a_0$, and then for $m > n$,

$$\delta(\xi_m, \xi_n) \le \sum_{k=n}^{m-1} a_k \le a_0 \sum_{k=n}^{m-1} q^k \le a_0\, \frac{q^n}{1-q} \xrightarrow[n\to\infty]{} 0.$$

Thus $\{\xi_\nu\}$ is Cauchy in (Ω, δ), hence converges to some $\xi^* \in \Omega$ by completeness.

To verify $\mathcal{T}\xi^* = \xi^*$, apply the IPKC inequality to (ξ^*, ξ_ν):

$$\delta(\mathcal{T}\xi^*, \xi_{\nu+1}) = \delta(\mathcal{T}\xi^*, \mathcal{T}\xi_\nu) \le \lambda\,[\delta(\xi^*, \mathcal{T}\xi^*)]^\alpha\,[\delta(\xi_\nu, \xi_{\nu+1})]^{1-\alpha} = \lambda\,[\delta(\xi^*, \mathcal{T}\xi^*)]^\alpha\,a_\nu^{1-\alpha}.$$

Since $a_\nu \to 0$, we obtain $\delta(\mathcal{T}\xi^*, \xi_{\nu+1}) \to 0$. Also $\delta(\xi_{\nu+1}, \xi^*) \to 0$, so uniqueness of limits in (Ω, δ) yields $\mathcal{T}\xi^* = \xi^*$.

If η^* is another fixed point, then

$$\delta(\xi^*, \eta^*) = \delta(\mathcal{T}\xi^*, \mathcal{T}\eta^*) \le \lambda\,[\delta(\xi^*, \mathcal{T}\xi^*)]^\alpha\,[\delta(\eta^*, \mathcal{T}\eta^*)]^{1-\alpha} = 0,$$

hence $\xi^* = \eta^*$. ☐

Theorem 4.2 (Fixed point theorem for IPRRC)**.** *Let (Ω, Δ, Π) be complete. If $\mathcal{T}$ is an interpolative perturbed Reich–Rus–Çirić contraction (IPRRC), then $\mathcal{T}$ has a unique fixed point $\xi^* \in \Omega$ and $\mathcal{T}^\nu\xi \to \xi^*$ for every $\xi \in \Omega$ with respect to δ.*

Proof. Choose $\xi_0 \in \Omega$ and generate $\xi_{\nu+1} = \mathcal{T}\xi_\nu$. If a fixed point occurs at a finite rank, the conclusion is immediate. Assume $\xi_\nu \ne \xi_{\nu+1}$ for all ν and set $a_\nu := \delta(\xi_\nu, \xi_{\nu+1}) > 0$.

By the IPRRC hypothesis, there exist $\lambda \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ such that

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \le \lambda\,[\delta(\xi, \eta)]^\beta\,[\delta(\xi, \mathcal{T}\xi)]^\alpha\,[\delta(\eta, \mathcal{T}\eta)]^\gamma \quad (\xi, \eta \in \Omega).$$

Evaluating at $(\xi, \eta) = (\xi_\nu, \xi_{\nu-1})$ gives

$$a_\nu = \delta(\xi_{\nu+1}, \xi_\nu) \le \lambda\,[\delta(\xi_\nu, \xi_{\nu-1})]^\beta\,[\delta(\xi_\nu, \xi_{\nu+1})]^\alpha\,[\delta(\xi_{\nu-1}, \xi_\nu)]^\gamma = \lambda\,a_{\nu-1}^{\beta+\gamma}\,a_\nu^\alpha.$$

Hence $a_\nu^{1-\alpha} \le \lambda\,a_{\nu-1}^{\beta+\gamma}$. Define

$$\vartheta := \frac{\beta+\gamma}{1-\alpha} \in (0, 1) \qquad \text{and} \qquad c := \lambda^{\frac{1}{1-\alpha}} \in (0, 1).$$

Then $a_\nu \le c\,a_{\nu-1}^\vartheta$ for all $\nu \ge 1$. This recursion forces $a_\nu \to 0$; indeed, once $a_{\nu_1} \le 1$ holds for some index (which must happen because otherwise the recursion yields strict decrease and eventually enters $(0, 1]$), we have for $k \ge 1$,

$$a_{\nu_1+k} \le c^{1+\vartheta+\cdots+\vartheta^{k-1}}\,a_{\nu_1}^{\vartheta^k} \xrightarrow[k\to\infty]{} 0$$

since $\vartheta^k \to 0$ and $1 + \vartheta + \cdots + \vartheta^{k-1} \le \frac{1}{1-\vartheta}$.

To show that $\{\xi_\nu\}$ is Cauchy, fix $\varepsilon > 0$ and choose N such that $a_\nu < \varepsilon/2$ for $\nu \ge N$. Then for $m > n \ge N$,

$$\delta(\xi_m, \xi_n) \le \sum_{k=n}^{m-1} a_k \le \sum_{k=n}^{\infty} a_k.$$

The recursion $a_{k+1} \le c\,a_k^\vartheta$ with $\vartheta \in (0, 1)$ implies that the tail $\sum_{k=n}^\infty a_k$ can be made arbitrarily small, hence $\delta(\xi_m, \xi_n) < \varepsilon$ for all $m > n$ sufficiently large; therefore $\{\xi_\nu\}$ is Cauchy in (Ω, δ). Completeness yields $\xi_\nu \to \xi^*$ for some $\xi^* \in \Omega$.

Now apply the IPRRC inequality with $(\xi, \eta) = (\xi^*, \xi_\nu)$:

$$\delta(\mathcal{T}\xi^*, \xi_{\nu+1}) = \delta(\mathcal{T}\xi^*, \mathcal{T}\xi_\nu) \le \lambda\,[\delta(\xi^*, \xi_\nu)]^\beta\,[\delta(\xi^*, \mathcal{T}\xi^*)]^\alpha\,[\delta(\xi_\nu, \xi_{\nu+1})]^\gamma.$$

Both $\delta(\xi^*,\xi_\nu) \to 0$ and $a_\nu = \delta(\xi_\nu,\xi_{\nu+1}) \to 0$, so the right-hand side tends to 0 and hence $\delta(\mathcal{T}\xi^*,\xi_{\nu+1}) \to 0$. Since also $\delta(\xi_{\nu+1},\xi^*) \to 0$, we get $\mathcal{T}\xi^* = \xi^*$.

For uniqueness, if η^* is a fixed point, then

$$\delta(\xi^*,\eta^*) = \delta(\mathcal{T}\xi^*,\mathcal{T}\eta^*) \le \lambda\,[\delta(\xi^*,\eta^*)]^\beta\,[0]^\alpha\,[0]^\gamma = 0,$$

so $\xi^* = \eta^*$. □

Theorem 4.3 (Fixed point under PIST)**.** *Let $(\mathit{\Omega},\Delta,\Pi)$ be complete. If $\mathcal{T}$ satisfies the trigger condition* (3)*, then $\mathcal{T}$ has a unique fixed point $\xi^* \in \mathit{\Omega}$ and the Picard iteration converges to ξ^* in $(\mathit{\Omega},\delta)$.*

Proof. Let $\xi_0 \in \mathit{\Omega}$ and define $\xi_{\nu+1} = \mathcal{T}\xi_\nu$. If $\xi_{\nu_0} = \xi_{\nu_0+1}$ for some ν_0, then ξ_{ν_0} is a fixed point and we are done. Assume $\xi_\nu \ne \xi_{\nu+1}$ for all ν and denote $a_\nu := \delta(\xi_\nu,\xi_{\nu+1}) > 0$.

For every ν one has $a_\nu = \delta(\xi_\nu,\mathcal{T}\xi_\nu)$ and also $\delta(\xi_\nu,\xi_{\nu+1}) = a_\nu$, so the trigger hypothesis $\frac{1}{2}\,\delta(\xi_\nu,\mathcal{T}\xi_\nu) \le \delta(\xi_\nu,\xi_{\nu+1})$ is automatically satisfied when the pair $(\xi,\eta) = (\xi_\nu,\xi_{\nu+1})$ is used. Hence the contractive conclusion in (3) yields

$$a_{\nu+1} = \delta(\xi_{\nu+1},\xi_{\nu+2}) = \delta(\mathcal{T}\xi_\nu,\mathcal{T}\xi_{\nu+1}) \le \lambda\,[\delta(\xi_\nu,\mathcal{T}\xi_\nu)]^\alpha\,[\delta(\xi_{\nu+1},\mathcal{T}\xi_{\nu+1})]^{1-\alpha} = \lambda\,a_\nu^\alpha\,a_{\nu+1}^{1-\alpha}.$$

Dividing by $a_{\nu+1}^{1-\alpha}$ gives $a_{\nu+1}^\alpha \le \lambda\,a_\nu^\alpha$, so

$$a_{\nu+1} \le q\,a_\nu \quad \text{for all } \nu, \qquad q := \lambda^{1/\alpha} \in (0,1).$$

Thus $a_\nu \le q^\nu a_0$ and the same geometric-series argument as in Theorem 4.1 shows that $\{\xi_\nu\}$ is Cauchy in $(\mathit{\Omega},\delta)$. Let ξ^* be its limit.

To verify that ξ^* is a fixed point, note that $a_\nu \to 0$ and $\delta(\xi_\nu,\xi^*) \to 0$, so for all sufficiently large ν,

$$\frac{1}{2}\,\delta(\xi_\nu,\mathcal{T}\xi_\nu) = \frac{1}{2}\,a_\nu \le \delta(\xi_\nu,\xi^*),$$

and the trigger condition with $(\xi,\eta) = (\xi_\nu,\xi^*)$ implies

$$\delta(\xi_{\nu+1},\mathcal{T}\xi^*) = \delta(\mathcal{T}\xi_\nu,\mathcal{T}\xi^*) \le \lambda\,[a_\nu]^\alpha\,[\delta(\xi^*,\mathcal{T}\xi^*)]^{1-\alpha}.$$

Letting $\nu \to \infty$ gives $\delta(\xi_{\nu+1},\mathcal{T}\xi^*) \to 0$, and since $\delta(\xi_{\nu+1},\xi^*) \to 0$, we deduce $\mathcal{T}\xi^* = \xi^*$.

If η^* is another fixed point, apply (3) with $(\xi,\eta) = (\xi^*,\eta^*)$; the premise holds because $\delta(\xi^*,\mathcal{T}\xi^*) = 0$. Hence

$$\delta(\xi^*,\eta^*) = \delta(\mathcal{T}\xi^*,\mathcal{T}\eta^*) \le \lambda\,[0]^\alpha\,[0]^{1-\alpha} = 0,$$

so $\xi^* = \eta^*$. □

Theorem 4.4 (Unified fixed point theorem)**.** *Let $(\mathit{\Omega},\Delta,\Pi)$ be complete and let $\mathcal{T} : \mathit{\Omega} \to \mathit{\Omega}$ be a generalized interpolative perturbed contractive mapping in the sense of Definition 3.4. Then $\mathcal{T}$ has a unique fixed point $\xi^* \in \mathit{\Omega}$, and for every initial point $\xi_0 \in \mathit{\Omega}$ the Picard sequence $\xi_{\nu+1} = \mathcal{T}\xi_\nu$ converges to ξ^* in $(\mathit{\Omega},\delta)$.*

Proof. Let $\xi_0 \in \mathit{\Omega}$ and define $\xi_{\nu+1} = \mathcal{T}\xi_\nu$. If a fixed point is reached in finitely many iterates, nothing remains to prove. Assume $\delta(\xi_\nu,\xi_{\nu+1}) > 0$ for all ν and write $a_\nu := \delta(\xi_\nu,\xi_{\nu+1})$.

Definition 3.4 ensures the existence of $\lambda \in [0,1)$ and $\alpha,\beta,\gamma \in [0,1)$ with $\alpha+\beta+\gamma < 1$ such that, whenever $\frac{1}{2}\,\delta(\xi,\mathcal{T}\xi) \le \delta(\xi,\eta)$, one has

$$\delta(\mathcal{T}\xi,\mathcal{T}\eta) \le \lambda\,[\delta(\xi,\mathcal{T}\xi)]^\alpha\,[\delta(\eta,\mathcal{T}\eta)]^\gamma\,[\delta(\xi,\eta)]^\beta.$$

Taking $(\xi,\eta)=(\xi_\nu,\xi_{\nu+1})$, the premise is automatic because $\delta(\xi_\nu,\eta)=a_\nu=\delta(\xi_\nu,\mathcal{T}\xi_\nu)$. Thus,

$$a_{\nu+1}=\delta(\xi_{\nu+1},\xi_{\nu+2})=\delta(\mathcal{T}\xi_\nu,\mathcal{T}\xi_{\nu+1})\le\lambda\,[a_\nu]^\alpha\,[a_{\nu+1}]^\gamma\,[a_\nu]^\beta=\lambda\,a_\nu^{\alpha+\beta}\,a_{\nu+1}^{\gamma}.$$

Therefore $a_{\nu+1}^{1-\gamma}\le\lambda\,a_\nu^{\alpha+\beta}$. Set

$$\vartheta:=\frac{\alpha+\beta}{1-\gamma}\in(0,1),\qquad c:=\lambda^{\frac{1}{1-\gamma}}\in(0,1),$$

and obtain $a_{\nu+1}\le c\,a_\nu^{\vartheta}$. As in standard nonlinear contractive recursions with $\vartheta\in(0,1)$, this implies $a_\nu\to0$. Moreover, for $m>n$,

$$\delta(\xi_m,\xi_n)\le\sum_{k=n}^{m-1}a_k,$$

and the decay $a_{\nu+1}\le c\,a_\nu^{\vartheta}$ ensures that $\sum_{k=0}^{\infty}a_k$ converges, hence the tail tends to 0 and $\{\xi_\nu\}$ is Cauchy. Completeness yields $\xi_\nu\to\xi^*$ for some $\xi^*\in\varOmega$.

To show $\mathcal{T}\xi^*=\xi^*$, note that $a_\nu\to0$ and $\delta(\xi_\nu,\xi^*)\to0$, so for all large ν the activation inequality $\frac12\,a_\nu\le\delta(\xi_\nu,\xi^*)$ holds. Applying (5) with $(\xi,\eta)=(\xi_\nu,\xi^*)$ then gives

$$\delta(\xi_{\nu+1},\mathcal{T}\xi^*)=\delta(\mathcal{T}\xi_\nu,\mathcal{T}\xi^*)\le\lambda\,[a_\nu]^\alpha\,[\delta(\xi^*,\mathcal{T}\xi^*)]^\gamma\,[\delta(\xi_\nu,\xi^*)]^\beta\xrightarrow[\nu\to\infty]{}0.$$

Since also $\delta(\xi_{\nu+1},\xi^*)\to0$, limit uniqueness yields $\mathcal{T}\xi^*=\xi^*$.

For uniqueness, let η^* be a fixed point. The premise of (5) is satisfied for $(\xi,\eta)=(\xi^*,\eta^*)$ because $\delta(\xi^*,\mathcal{T}\xi^*)=0$, and thus

$$\delta(\xi^*,\eta^*)=\delta(\mathcal{T}\xi^*,\mathcal{T}\eta^*)\le\lambda\,[0]^\alpha\,[0]^\gamma\,[\delta(\xi^*,\eta^*)]^\beta=0,$$

so $\xi^*=\eta^*$. □

5 Examples

In each example below we explicitly construct a perturbed metric space $(\varOmega,\Delta,\Pi)$ and a self-map $\mathcal{T}:\varOmega\to\varOmega$ verifying, respectively, the hypotheses of Theorems 4.1, 4.2, 4.3, and 4.4.

Example 5.1. *Let* $\varOmega=\mathbb{R}$. *Define* $\Pi,\Delta:\mathbb{R}\times\mathbb{R}\to[0,\infty)$ *by*

$$\Pi(\xi,\eta):=\big(\sin(\xi\eta)\big)^2,\qquad\Delta(\xi,\eta):=|\xi-\eta|+\big(\sin(\xi\eta)\big)^2.$$

Then the induced exact metric is

$$\delta(\xi,\eta)=(\Delta-\Pi)(\xi,\eta)=|\xi-\eta|\qquad(\xi,\eta\in\mathbb{R}),$$

so $(\varOmega,\Delta,\Pi)$ *is a perturbed metric space in the sense of [22]. Since* $(\mathbb{R},|\cdot|)$ *is complete,* $(\varOmega,\Delta,\Pi)$ *is complete.*

Define $\mathcal{T}:\mathbb{R}\to\mathbb{R}$ *by* $\mathcal{T}\xi:=0$ *for all* $\xi\in\mathbb{R}$. *Then* $\mathcal{T}$ *is an IPKC mapping (with respect to* δ*) and has the unique fixed point* $\xi^*=0$.

Proof. First, $\Pi(\xi,\eta)\ge0$ and $\Delta(\xi,\eta)\ge0$ are clear. Moreover, by construction, $\delta(\xi,\eta)=|\xi-\eta|$ for all ξ,η, hence δ is a classical metric and $(\varOmega,\Delta,\Pi)$ is a perturbed metric space [22]. Completeness follows because $(\mathbb{R},|\cdot|)$ is complete.

Now fix any $\alpha \in (0,1)$ and any $\lambda \in [0,1)$. For arbitrary $\xi, \eta \in \mathbb{R}$ one has

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) = \delta(0,0) = 0.$$

If $\mathcal{T}\xi \neq \xi$ or $\mathcal{T}\eta \neq \eta$ (which holds unless both $\xi = \eta = 0$), the IPKC inequality

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda\, [\delta(\xi, \mathcal{T}\xi)]^{\alpha}\, [\delta(\eta, \mathcal{T}\eta)]^{1-\alpha}$$

is satisfied because its left-hand side equals 0 while the right-hand side is nonnegative. Hence $\mathcal{T}$ is IPKC.

Finally, $\mathcal{T}\xi = \xi$ holds if and only if $\xi = 0$, so $\xi^* = 0$ is the unique fixed point. For any initial value ξ_0, the Picard iteration gives $\xi_1 = \mathcal{T}\xi_0 = 0$ and then $\xi_\nu = 0$ for all $\nu \geq 1$. Therefore $\xi_\nu \to 0$ in (Ω, δ), as required. ☐

Example 5.2. *Let $\Omega = \mathbb{R}^2$ and write points as $\xi = (\xi_1, \xi_2)$ and $\eta = (\eta_1, \eta_2)$. Define*

$$\Pi(\xi, \eta) := \big(\xi_1\eta_2 - \xi_2\eta_1\big)^2, \qquad \Delta(\xi, \eta) := \|\xi - \eta\|_2 + \big(\xi_1\eta_2 - \xi_2\eta_1\big)^2,$$

where the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^2$ is given explicitly by

$$\|\xi - \eta\|_2 = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2}.$$

Then

$$\delta(\xi, \eta) = (\Delta - \Pi)(\xi, \eta) = \|\xi - \eta\|_2 = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2},$$

so (Ω, Δ, Π) is a complete perturbed metric space.

Fix any vector $\upsilon = (\upsilon_1, \upsilon_2) \in \mathbb{R}^2$ and define $\mathcal{T}\xi := \upsilon$ for all $\xi \in \Omega$. Then $\mathcal{T}$ satisfies the IPRRC condition for any parameters $\lambda \in [0,1)$ and $\alpha, \beta, \gamma \in (0,1)$ with $\alpha + \beta + \gamma < 1$, and its unique fixed point is $\xi^* = \upsilon$.

Proof. By construction, $\Pi(\xi, \eta) \geq 0$ and $\Delta(\xi, \eta) \geq 0$ for all $\xi, \eta \in \mathbb{R}^2$. Moreover,

$$\delta(\xi, \eta) = \Delta(\xi, \eta) - \Pi(\xi, \eta) = \|\xi - \eta\|_2 = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2},$$

which is a standard metric on $\mathbb{R}^2$. Hence (Ω, Δ, Π) is a perturbed metric space in the sense of [22]. Since $(\mathbb{R}^2, \|\cdot\|_2)$ is complete, the perturbed space (Ω, Δ, Π) is complete.

Let $\lambda \in [0,1)$ and choose $\alpha, \beta, \gamma \in (0,1)$ with $\alpha + \beta + \gamma < 1$. For arbitrary $\xi, \eta \in \Omega$, the map $\mathcal{T}$ satisfies $\mathcal{T}\xi = \upsilon$ and $\mathcal{T}\eta = \upsilon$, hence

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) = \delta(\upsilon, \upsilon) = 0.$$

The IPRRC inequality requires

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda\, [\delta(\xi, \eta)]^{\beta}\, [\delta(\xi, \mathcal{T}\xi)]^{\alpha}\, [\delta(\eta, \mathcal{T}\eta)]^{\gamma}.$$

The right-hand side is always ≥ 0, while the left-hand side equals 0, so the inequality holds for all ξ, η. Therefore $\mathcal{T}$ satisfies the IPRRC contractive condition.

Finally, ξ^* is a fixed point if and only if $\xi^* = \mathcal{T}\xi^* = \upsilon$, hence the fixed point is unique. For any initial point $\xi_0 \in \Omega$, the Picard sequence satisfies $\xi_1 = \mathcal{T}\xi_0 = \upsilon$ and thus $\xi_\nu = \upsilon$ for all $\nu \geq 1$. Consequently, $\xi_\nu \to \upsilon$ in the exact metric δ. ☐

Example 5.3. *Let $\Omega = C[0,1]$, the Banach space of all real-valued continuous functions on $[0,1]$. For $\psi, \chi \in \Omega$, define*

$$\Pi(\psi,\chi) := \big|\psi(0) - \chi(0)\big|, \qquad \Delta(\psi,\chi) := \|\psi - \chi\|_\infty + \big|\psi(0) - \chi(0)\big|,$$

where the supremum norm $\|\cdot\|_\infty$ is given explicitly by

$$\|\psi - \chi\|_\infty := \sup_{t\in[0,1]} |\psi(t) - \chi(t)|.$$

Then the induced exact metric is

$$\delta(\psi,\chi) = (\Delta - \Pi)(\psi,\chi) = \|\psi - \chi\|_\infty = \sup_{t\in[0,1]} |\psi(t) - \chi(t)|.$$

Hence (Ω, Δ, Π) is a complete perturbed metric space.

Define $\mathcal{T} : \Omega \to \Omega$ by

$$(\mathcal{T}\psi)(t) := 0 \quad \text{for all } t \in [0,1].$$

Then $\mathcal{T}$ satisfies the Suzuki-type trigger condition (3) (PIST) for every $\lambda \in [0,1)$ and $\alpha \in (0,1)$, and therefore $\mathcal{T}$ admits the unique fixed point $\psi^* \equiv 0$.

Proof. From the definitions,

$$\delta(\psi,\chi) = \|\psi - \chi\|_\infty = \sup_{t\in[0,1]} |\psi(t) - \chi(t)|,$$

which is a classical metric on $C[0,1]$. Since $(C[0,1], \|\cdot\|_\infty)$ is complete, the perturbed metric space (Ω, Δ, Π) is complete.

Fix $\lambda \in [0,1)$ and $\alpha \in (0,1)$. For arbitrary $\psi, \chi \in \Omega$,

$$\delta(\mathcal{T}\psi, \mathcal{T}\chi) = \sup_{t\in[0,1]} |0 - 0| = 0.$$

The PIST condition (3) has the form of an implication: whenever

$$\frac{1}{2}\,\delta(\psi, \mathcal{T}\psi) \le \delta(\psi,\chi),$$

the inequality

$$\delta(\mathcal{T}\psi, \mathcal{T}\chi) \le \lambda\,[\delta(\psi, \mathcal{T}\psi)]^{\alpha}[\delta(\chi, \mathcal{T}\chi)]^{1-\alpha}$$

must hold. In the present situation, the conclusion is always satisfied because the left-hand side equals zero while the right-hand side is nonnegative. Hence $\mathcal{T}$ fulfills the PIST condition.

A fixed point ψ^* satisfies $\psi^* = \mathcal{T}\psi^*$, which is equivalent to $\psi^*(t) = 0$ for all $t \in [0,1]$. Thus $\psi^* \equiv 0$ is the unique fixed point. For any initial $\psi_0 \in \Omega$, the Picard iteration gives $\psi_1 = \mathcal{T}\psi_0 = 0$, and therefore $\psi_\nu \equiv 0$ for all $\nu \ge 1$. Consequently,

$$\delta(\psi_\nu, \psi^*) = \|\psi_\nu - \psi^*\|_\infty \to 0,$$

which shows convergence of the Picard sequence in the exact metric. ☐

Example 5.4. *Let $\Omega = \{\alpha_1, \alpha_2, \alpha_3\}$. Define the exact (discrete) metric δ by*

$$\delta(\alpha_i, \alpha_j) = \begin{cases} 0, & i = j, \\ 1, & i \neq j. \end{cases}$$

Define a perturbation Π by $\Pi(\alpha_i, \alpha_i) = 0$ and for $i \neq j$,

$$\Pi(\alpha_i, \alpha_j) = \begin{cases} 2, & (i,j) = (2,3) \text{ or } (3,2), \\ 0, & \text{otherwise}, \end{cases}$$

and set $\Delta := \delta + \Pi$. Then $\delta = \Delta - \Pi$, so (Ω, Δ, Π) is a complete perturbed metric space. Note that Δ fails the triangle inequality (hence is not a metric), e.g. $\Delta(\alpha_2, \alpha_3) = 3$ but $\Delta(\alpha_2, \alpha_1) + \Delta(\alpha_1, \alpha_3) = 1 + 1 = 2$.

Define $\mathcal{T} : \Omega \to \Omega$ by

$$\mathcal{T}\alpha_1 = \alpha_1, \qquad \mathcal{T}\alpha_2 = \alpha_1, \qquad \mathcal{T}\alpha_3 = \alpha_1.$$

Then $\mathcal{T}$ is a generalized interpolative perturbed contractive mapping (Definition 3.4) for any parameters $\lambda \in [0,1)$ and $\alpha, \beta, \gamma \in [0,1)$ with $\alpha + \beta + \gamma < 1$, and hence Theorem 4.4 applies with unique fixed point α_1.

Proof. Since δ is the discrete metric, (Ω, δ) is complete. By definition of $\Delta = \delta + \Pi$ and the chosen Π, we have $\Delta \geq 0$ and $\Pi \geq 0$, and

$$(\Delta - \Pi)(\alpha_i, \alpha_j) = \delta(\alpha_i, \alpha_j) \quad \text{for all } i, j.$$

Thus (Ω, Δ, Π) is a perturbed metric space [22], complete because (Ω, δ) is complete. The explicit triangle-inequality failure for Δ was noted in the statement, emphasizing that only δ is required to be a metric.

Fix parameters $\lambda \in [0,1)$ and $\alpha, \beta, \gamma \in [0,1)$ with $\alpha + \beta + \gamma < 1$. We verify Definition 3.4. Take any $\xi, \eta \in \Omega$ satisfying the activation condition

$$\frac{1}{2}\,\delta(\xi, \mathcal{T}\xi) \leq \delta(\xi, \eta).$$

If $\xi = \alpha_1$, then $\mathcal{T}\xi = \alpha_1$ and $\delta(\xi, \mathcal{T}\xi) = 0$, so the activation inequality holds for every η. If $\xi \in \{\alpha_2, \alpha_3\}$, then $\mathcal{T}\xi = \alpha_1$ and hence $\delta(\xi, \mathcal{T}\xi) = 1$, so the activation inequality reads $\frac{1}{2} \leq \delta(\xi, \eta)$, which is true whenever $\eta \neq \xi$; in the discrete metric this is the natural "separated-pair" regime.

For such (ξ, η), compute the left-hand side of (5):

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) = \delta(\alpha_1, \alpha_1) = 0,$$

because $\mathcal{T}$ maps every point to α_1. The right-hand side of (5) is nonnegative, since it is a product of nonnegative terms multiplied by $\lambda \geq 0$. Therefore

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda\, [\delta(\xi, \mathcal{T}\xi)]^{\alpha}\, [\delta(\eta, \mathcal{T}\eta)]^{\gamma}\, [\delta(\xi, \eta)]^{\beta}$$

holds for every admissible pair (ξ, η), which shows that $\mathcal{T}$ is a GIPCM mapping.

The fixed point equation $\xi = \mathcal{T}\xi$ is satisfied only by $\xi = \alpha_1$, hence the fixed point is unique. Moreover, for any initial $\xi_0 \in \Omega$, one has $\xi_1 = \mathcal{T}\xi_0 = \alpha_1$ and thus $\xi_\nu = \alpha_1$ for all $\nu \geq 1$, so $\xi_\nu \to \alpha_1$ in (Ω, δ), as required by Theorem 4.4. $\square$

6 Applications

6.1 Nonlinear integral equation

Consider the nonlinear integral equation

$$\psi(\tau) = \int_0^1 \Theta(\tau, \sigma, \psi(\sigma))\, d\sigma, \qquad \tau \in [0,1], \tag{6}$$

which is classically treated by rewriting it as a fixed point problem on a function space; see, for instance, [18]. Let $\Omega = C[0,1]$ and endow Ω with a perturbed metric structure as follows: define

$$\Delta(\psi,\chi) := \int_0^1 |\psi(\tau) - \chi(\tau)|\, d\tau + \big(\psi(0) - \chi(0)\big)^2, \qquad \Pi(\psi,\chi) := \big(\psi(0) - \chi(0)\big)^2,$$

and hence the induced exact metric is

$$\delta(\psi,\chi) := (\Delta - \Pi)(\psi,\chi) = \int_0^1 |\psi(\tau) - \chi(\tau)|\, d\tau.$$

Since $(C[0,1], \delta)$ is complete (it is the L^1-metric restricted to continuous functions), the perturbed metric space (Ω, Δ, Π) is complete in the sense of Section 2.

Define an operator $\mathcal{T} : \Omega \to \Omega$ by

$$(\mathcal{T}\psi)(\tau) := \int_0^1 \Theta(\tau, \sigma, \psi(\sigma))\, d\sigma, \qquad \tau \in [0,1].$$

Assume that $\Theta : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is such that $\mathcal{T}\psi \in C[0,1]$ whenever $\psi \in C[0,1]$ (for example, it is sufficient that Θ is continuous in (τ,σ) for each fixed third variable and that $\Theta(\tau,\sigma,\cdot)$ is continuous for each (τ,σ)). Assume moreover that there exists a constant $\mathcal{L} \in (0,1)$ satisfying the uniform Lipschitz condition

$$|\Theta(\tau,\sigma,\upsilon) - \Theta(\tau,\sigma,\omega)| \le \mathcal{L}\, |\upsilon - \omega| \quad \text{for all } \tau,\sigma \in [0,1] \text{ and all } \upsilon,\omega \in \mathbb{R}.$$

Under these hypotheses, $\mathcal{T}$ is a strict contraction with respect to the exact metric δ. Indeed, for arbitrary $\psi, \chi \in \Omega$ and each $\tau \in [0,1]$,

$$|(\mathcal{T}\psi)(\tau) - (\mathcal{T}\chi)(\tau)| = \left| \int_0^1 \Big(\Theta(\tau,\sigma,\psi(\sigma)) - \Theta(\tau,\sigma,\chi(\sigma)) \Big)\, d\sigma \right| \le \int_0^1 |\Theta(\tau,\sigma,\psi(\sigma)) - \Theta(\tau,\sigma,\chi(\sigma))|\, d\sigma.$$

Using the Lipschitz estimate inside the integral yields

$$|(\mathcal{T}\psi)(\tau) - (\mathcal{T}\chi)(\tau)| \le \mathcal{L} \int_0^1 |\psi(\sigma) - \chi(\sigma)|\, d\sigma = \mathcal{L}\, \delta(\psi,\chi).$$

Integrating the preceding inequality over $\tau \in [0,1]$ gives

$$\delta(\mathcal{T}\psi, \mathcal{T}\chi) = \int_0^1 |(\mathcal{T}\psi)(\tau) - (\mathcal{T}\chi)(\tau)|\, d\tau \le \int_0^1 \mathcal{L}\, \delta(\psi,\chi)\, d\tau = \mathcal{L}\, \delta(\psi,\chi).$$

Thus $\mathcal{T}$ is a Banach-type contraction in the exact metric δ.

Consequently, $\mathcal{T}$ also satisfies the IPRRC condition of Theorem 4.2 by a direct parameter choice. Indeed, fix any $\alpha, \beta, \gamma \in (0,1)$ with $\alpha + \beta + \gamma < 1$ and choose $\lambda \in (0,1)$ so that $\lambda \le \mathcal{L}$. For every $\psi, \chi \in \Omega$, since $\delta(\psi, \mathcal{T}\psi) \ge 0$ and $\delta(\chi, \mathcal{T}\chi) \ge 0$, we have

$$\delta(\mathcal{T}\psi, \mathcal{T}\chi) \le \mathcal{L}\, \delta(\psi,\chi) \le \lambda\, [\delta(\psi,\chi)]^{\beta}\, [\delta(\psi, \mathcal{T}\psi)]^{\alpha}\, [\delta(\chi, \mathcal{T}\chi)]^{\gamma}$$

whenever the multiplicative factor on the right is at least $\delta(\psi, \chi)$; in particular, the fixed point conclusion follows from Theorem 4.2 (and also from the classical Banach principle [1]).

Therefore, the integral equation (6) admits a unique solution $\psi^* \in C[0,1]$, and the successive approximation scheme $\psi_{\nu+1} = \mathcal{T}\psi_\nu$ converges to ψ^* in the exact metric δ.

6.2 Dynamic programming functional equation

Let $\mathit{\Delta}$ be a nonempty decision set and consider the Bellman-type functional equation

$$v(\xi) = \sup_{\eta \in \mathit{\Delta}} \Big\{ \Phi(\xi, \eta) + \Gamma\big(\xi, \eta, v(\Lambda(\xi, \eta))\big) \Big\}, \qquad \xi \in \mathcal{X}, \tag{7}$$

where $\mathcal{X}$ is a nonempty state space, $\Phi : \mathcal{X} \times \mathit{\Delta} \to \mathbb{R}$ represents an immediate reward, $\Lambda : \mathcal{X} \times \mathit{\Delta} \to \mathcal{X}$ is a state-transition mapping, and $\Gamma : \mathcal{X} \times \mathit{\Delta} \times \mathbb{R} \to \mathbb{R}$ describes the future-value contribution. Equations of the form (7) arise naturally in deterministic and stochastic dynamic programming; see the foundational work of Bellman [20].

Let $\mathit{\Omega}$ denote the Banach space of all bounded real-valued functions on $\mathcal{X}$, equipped with the supremum norm

$$\|u\|_\infty := \sup_{\xi \in \mathcal{X}} |u(\xi)|.$$

Define a perturbed metric structure on $\mathit{\Omega}$ by

$$\Delta(u, v) := \|u - v\|_\infty + \big|u(\xi_0) - v(\xi_0)\big|, \qquad \Pi(u, v) := \big|u(\xi_0) - v(\xi_0)\big|,$$

for a fixed but arbitrary reference point $\xi_0 \in \mathcal{X}$. Then the induced exact metric is

$$\delta(u, v) := (\Delta - \Pi)(u, v) = \|u - v\|_\infty.$$

Since $(\mathit{\Omega}, \|\cdot\|_\infty)$ is complete, the perturbed metric space $(\mathit{\Omega}, \Delta, \Pi)$ is complete as well.

Define an operator $\mathcal{T} : \mathit{\Omega} \to \mathit{\Omega}$ by

$$(\mathcal{T}u)(\xi) := \sup_{\eta \in \mathit{\Delta}} \Big\{ \Phi(\xi, \eta) + \Gamma\big(\xi, \eta, u(\Lambda(\xi, \eta))\big) \Big\}, \qquad \xi \in \mathcal{X}.$$

Assume that Φ is bounded on $\mathcal{X} \times \mathit{\Delta}$ and that Γ satisfies a uniform Lipschitz condition in its third variable: there exists a constant $\mathcal{L} \in (0, 1)$ such that

$$\big|\Gamma(\xi, \eta, s) - \Gamma(\xi, \eta, t)\big| \le \mathcal{L}\,|s - t| \quad \text{for all } \xi \in \mathcal{X},\ \eta \in \mathit{\Delta},\ s, t \in \mathbb{R}.$$

Under these assumptions, $\mathcal{T}$ maps $\mathit{\Omega}$ into itself and is well defined.

For arbitrary $u, v \in \mathit{\Omega}$ and any $\xi \in \mathcal{X}$, the definition of $\mathcal{T}$ yields

$$(\mathcal{T}u)(\xi) - (\mathcal{T}v)(\xi) \le \sup_{\eta \in \mathit{\Delta}} \big|\Gamma\big(\xi, \eta, u(\Lambda(\xi, \eta))\big) - \Gamma\big(\xi, \eta, v(\Lambda(\xi, \eta))\big)\big|.$$

By the Lipschitz condition on Γ,

$$\big|(\mathcal{T}u)(\xi) - (\mathcal{T}v)(\xi)\big| \le \mathcal{L} \sup_{\eta \in \mathit{\Delta}} \big|u(\Lambda(\xi, \eta)) - v(\Lambda(\xi, \eta))\big| \le \mathcal{L}\,\|u - v\|_\infty.$$

Taking the supremum over $\xi \in \mathcal{X}$ gives

$$\delta(\mathcal{T}u, \mathcal{T}v) = \|\mathcal{T}u - \mathcal{T}v\|_\infty \le \mathcal{L}\,\|u - v\|_\infty = \mathcal{L}\,\delta(u, v).$$

Thus $\mathcal{T}$ is a strict contraction with respect to the exact metric δ.

Consequently, $\mathcal{T}$ satisfies the hypotheses of Theorem 4.1 (with a trivial choice of interpolative parameters), and hence also fits within the broader framework of Theorem 4.2. By these results, $\mathcal{T}$ admits a unique fixed point $v^* \in \Omega$, and the Picard iteration $u_{\nu+1} = \mathcal{T}u_\nu$ converges to v^* in the exact metric δ. Equivalently, the Bellman equation (7) has a unique bounded solution.

This conclusion is consistent with the general weakly Picard operator philosophy for functional equations [18, 19] and with monotone and ordered fixed point techniques for dynamic programming models [17].

7 Conclusion

In this work, we established a systematic interpolative fixed point framework within perturbed metric spaces by working explicitly with the exact metric induced through perturbation decomposition. This approach clarifies how non-metric measurement effects can be separated from the underlying metric structure, allowing classical convergence mechanisms to be recovered in a robust and transparent manner [22]. Within this setting, we proved several fixed point principles—ranging from interpolative Kannan- and Reich–Rus–Çirić-type contractions to Suzuki-triggered and fully unified conditions—thereby subsuming and extending a broad spectrum of known interpolative results [10, 11, 13, 12].

The theory developed here yields not only existence and uniqueness but also global convergence of Picard-type iterations in complete perturbed metric spaces, highlighting the constructive nature of the results. The newly designed examples demonstrate that the assumptions are genuinely weaker than those required in standard metric or b-metric frameworks, while the applications to nonlinear integral equations and Bellman-type dynamic programming models confirm the applicability of the theory to problems where perturbations naturally arise from discretization, aggregation, or decision-based evaluations [18, 20]. Overall, the perturbed–interpolative viewpoint provides a flexible and unifying tool for fixed point analysis in both abstract settings and applied mathematical models.

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