Star Chromatic Index of Tensor Product of Graphs

Abstract

A star edge coloring of a graph G is a proper edge-coloring without bichromatic paths or cycles of length 4. The smallest integer k such that G admits a star-edge-coloring with k colors is the star chromatic index of G and is denoted by $\chi'_{st}(G)$. The tensor product of two graphs G and H, denoted by $G \times H$ is the graph with vertex set $V(G) \times V(H)$ and two vertices $u = (u_1, v_1)$, $v = (u_2, v_2)$ are adjacent in $G \times H$ if u_1 is adjacent to u_2 in G and v_1 is adjacent to v_2 in H. In this paper, we obtain the star chromatic index of $P_m \times C_n$ and $C_m \times C_n$ and obtained the following results.
(i) $\chi'_{st}(P_m \times C_n) = 8$, for $m \geq 7, n \geq 7$.
(ii)$\chi'_{st}(C_m \times C_n) = 8$, for $m \geq 3, n \geq 4$.
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1 Introduction

All graphs considered in this paper are finite, simple and undirected. Let P_n and C_n respectively denote a path and a cycle of order n. For a graph G, sG denotes s edge disjoint copies of G. For a graph G and a subgraph H of G, $G \backslash H$ denotes the graph G' in which $V(G') = V(G)$ and $E(G') = E(G) - E(H)$. The tensor product of two graphs G and H, denoted by $G \times H$ is the graph with vertex set $V(G) \times V(H)$ and two vertices $u = (u_1, v_1)$, $v = (u_2, v_2)$ are adjacent in $G \times H$ if $u_1u_2 \in V(G)$ is adjacent to $v_1v_2 \in V(H)$.
An edge coloring of a graph $G = (V, E)$ is a function $f : E \to C \subseteq \mathcal{N}$ in which any two adjacent edges $e, f \in E$ are assigned different colors. The function C is known as the edge coloring function. A graph G for which there exists an edge coloring which requires k colors is called k-edge colorable. The smallest number k for which there exists a k-edge coloring of G is called the edge chromatic index of a graph G and is denoted by $\chi'(G)$. A star edge coloring of a graph G is a proper edge coloring where atleast three distinct colors are used on the edges of every induced paths and cycles of length four in G, i.e., there is neither an induced bichromatic path nor an induced bichromatic cycle of length four in G. The minimum number of colors for which G admits a star edge coloring is called the star chromatic index of G and is denoted by $\chi'_{st}(G)$.
Additional graph theoretic terminologies used in this paper can be found in [1]. The star edge coloring was initiated in 2008 by Liu and Deng [10]. Star chromatic index in product graphs has been done by many researchers. Omooni and Dastjerbi [11]

determined the upper bounds of the star chromatic index of the cartesian product of paths with cycles, d-dimensional grids, d-dimensional hypercubes and d-dimensional toroided grids. Kaliraj et al. [8] obtained the star chromatic index of the corona product of path and wheel graph families. Kavita et al. [9] obtained the star chromatic index of the tensor product of two paths. Deng et al.[3] constructed infinite sequence of cubic graphs G with $\chi'_{st}(G) = 6$ and obtained the star chromatic index of striped maximal outerplanar graph. Yunfeng Tang et al. [13] obtained the star edge coloring of $K_{2,t}$ -free planar graphs . Fernando et al. obtained the star chromatic index of some simple graphs such as pan graphs, tadpole graphs, friendship graphs, ladder graphs, flower graphs and umbrella graphs.
In the following section, we discuss the star chromatic index of the tensor product of graphs such as $P_m \times C_n$ and $C_m \times C_n$ respectively for the given positive integers m and n.

2 Preliminaries

We use the following results in the proof of our main theorem.

Theorem 2.1. *[12] If H is a subgraph of G, then $\chi'_{st}(H) \leq \chi'_{st}(G)$.*

Theorem 2.2. *[7] For the given positive integer n,*

$$\chi'_{st}(P_n) = \begin{cases} 1, & \text{if } n = 2 \\ 3, & \text{if } n \geq 3 \end{cases}$$

$$\chi'_{st}(C_n) = \begin{cases} 3, & \text{if } n \neq 5 \\ 4, & \text{if } n = 5. \end{cases}$$

Theorem 2.3. *[9] For the given positive integers $m \geq 2$ and $n \geq 2$, the star chromatic index of $P_m \times P_n$ is given by*

$$\chi'_{st}(P_m \times P_n) = \begin{cases} 3, & \text{if } m = 2 \text{ and } n \geq 5 \\ 5, & \text{if } m = 3,4 \text{ and } n \geq 5 \\ 6, & \text{if } m \geq 7 \text{ and } n \geq 7 \\ \geq 5 & \text{if } m = 5,6 \text{ and } n \geq 5\ . \end{cases}$$

3 Star Chromatic Index of Tensor Product of Paths and Cycles

Theorem 3.1. *For the given positive integers $m \geq 2$ and $n \geq 3$, the star chromatic index of $P_m \times C_n$ is given by*

$$\chi'_{st}(P_m \times C_n) = \begin{cases} 3, & \text{for } m = 2 \text{ and } n \geq 3 \\ 5, & \text{for } m = 3 \text{ and } n = 3 \\ 6, & \text{for } m = 3 \text{ and } n \geq 4 \text{ or } m = 4 \text{ and } n \geq 3 \text{ or } m = 5 \text{ and } n = 3 \\ 7, & \text{for } m = 5 \text{ and } n \geq 4 \text{ or } m \in \{6, 7\} \text{ and } 3 \leq n \leq 6 \\ 8, & \text{for } m \geq 7 \text{ and } n \geq 7. \end{cases}$$

Proof. Let $G(V, E) = P_m \times C_n$, by the definition of tensor product, the vertex set and the edge set of the graph $P_m \times C_n$ is given by $V(P_m \times C_n) = \{(u_i, v_j) : 0 \leq i \leq m-1,\ 0 \leq j \leq n-1\ \ u_i \in V(P_m) \text{ and } v_j \in V(P_m)\}$.
$E(G) = E_1 \cup E_2 \cup E_3 \cup E_4$ where
$E_1 = \bigcup_{i=0}^{m-1} \{(u_i, v_j)(u_{i+1}, v_{j+1}) : 0 \leq j \leq n-2\}$
$E_2 = \bigcup_{i=0}^{m-1} \{(u_i, v_j)(u_{i+1}, v_{j-1}) : 1 \leq j \leq n-1\}$
$E_3 = \bigcup_{i=0}^{m-1} \{(u_i, v_0)(u_{i+1}, v_j) : j = n-1\}$
$E_4 = \bigcup_{i=0}^{m-1} \{(u_i, v_j)(u_{i+1}, v_0) : j = n-1\}$.
Let f be the function defined by $f : E \rightarrow C \subseteq \mathcal{N} \cup \{0\}$, and C is the set of colors. We obtain the star edge chromatic index of $P_m \times C_n$ according to the given values of m,n as follows:
Case 1: $m = 2$, $n \geq 3$

In this case $G = P_2 \times C_n$, $n \geq 3$ is isomorphic to C_{2n} and by Theorem 2.2, we have $\chi'_{st}(C_{2n}) = 3$ for $n \geq 3$ and therefore $\chi'_{st}(P_2 \times C_n) = 3$.
Case 2: $m = 3$, $n = 3$

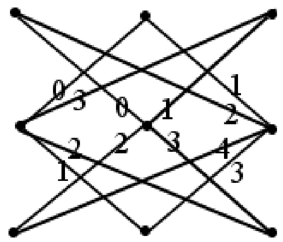

Figure 1: Star edge coloring of $P_3 \times C_3$

In this case $G = P_3 \times C_3$ and the maximum degree of the graph G is 4 and there exists 3 vertices of maximum degree 4. Hence $\chi'_{st}(G) \geq 4$, first we color the edges that are incident at one of these vertices of maximum degree say at v, using 4 colors $\{0, 1, 2, 3\}$. Then to color the edges that are incident at another maximum degree vertex, we can use any of the three colors from the list $\{0, 1, 2, 3\}$ as one edge is already colored from the previous step. But coloring 4 adjacent edges using only 3 colors will not yield a proper coloring. Hence fifth color must be introduced. We can do the permutation of 5 colors to color the adjacent edges that are incident at the

vertices of degree 4 in G. Thus $\chi'_{st}(G) = 5$.
Case 3: $m = 3$, $n \geq 4$
Subcase (i):

In this case $G = P_3 \times C_4$ contains $G_1 = P_3 \times P_4$ as a subgraph, by Theorem 2.3, it is clear that $\chi'_{st}(P_3 \times P_4) = 4$, and it is not possible to color the edges in $G \backslash G_1$ with the same set of colors already used in the coloring of G_1 . So we introduce the fifth color for the edges in between the first and second layers of $G \backslash G_1$. At the same time we cannot use the fifth color for the new edges in between the second and third layer of $G \backslash G_1$. So, we introduce the sixth color and hence $\chi'_{st}(P_3 \times C_4) = 6$.

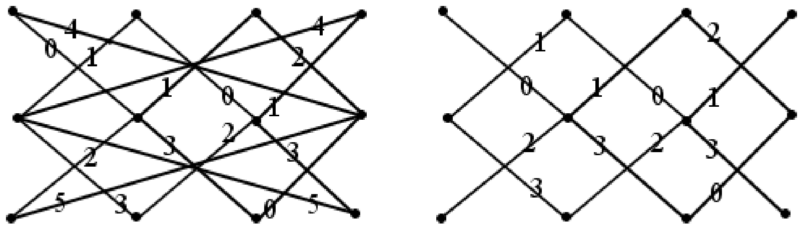

Figure 2: Star edge coloring of the graph $P_3 \times C_4$ and its subgraph $P_3 \times P_4$

Subcase (ii):

In this case $G = P_3 \times C_n$ for $n \geq 5$ contains $G_2 = P_3 \times P_n$ as a subgraph, by Theorem 2.3, it is clear that $\chi'_{st}(P_3 \times P_n) = 5$, for $n \geq 5$. But, it is not possible to color the edges of $G \backslash G_2$ with the same set of colors already used in the coloring of G_2. We introduce the sixth color for the edges in between the second and third layers of $G \backslash G_2$. Therefore, $\chi'_{st}(P_3 \times C_n) = 6$, for $n \geq 5$.
Hence $\chi'_{st}(P_m \times C_n) = 6$, for $m = 3$, $n \geq 4$
Case 4: $m = 4$, $n \geq 3$
Subcase (i):

In this case $G = P_4 \times C_3$ contains $G_3 = P_4 \times P_3$ as a subgraph, By Theorem 2.3, it is clear that $\chi'_{st}(P_4 \times P_3) = 5$. Here every vertex of the graph G is at a distance of one, two or three from the other and hence it is not possible to color the edges of $G \backslash G_3$ with the same set of five colors already used for G_3. So, we introduce the sixth color. Therefore, $\chi'_{st}(P_4 \times C_3) = 6$. Similarly, we can prove that $\chi'_{st}(P_4 \times C_4) = 6$.
The proof of other cases are given in the following table as discussed in cases 1 to 4:

m	n	$G = P_m \times C_n$	G_1	$\chi'_{st}(G_1)$	$\chi'_{st}(G)$
4	$n \geq 4$	$P_4 \times C_n$	$P_4 \times P_n$	5	6
5	3	$P_5 \times C_3$	$P_5 \times P_3$	5	6
5	4	$P_5 \times C_4$	$P_5 \times P_4$	5	7
5	5	$P_5 \times C_5$	$P_5 \times P_5$	≥ 5	7
6,7	$3 \leq n \leq 6$	$P_m \times C_n$	$P_5 \times P_3$	5	7
7	7	$P_7 \times C_7$	$P_7 \times P_7$	6	8

Table 1: $\chi'_{st}(P_m \times C_n)$ for $m \geq 4$, $n \geq 3$.

By using the similar argument,we can prove that $\chi'_{st}(P_m \times C_n) = 8$, for $m \geq 7, n \geq 7$. To develop an algorithmic approach for the star edge coloring of $G = P_m \times C_n$, for large values of $m, n \geq 7$, we give the following coloring pattern. The star edge coloring pattern f of $G = P_m \times C_n$, for $m \geq 7, n \geq 7$ is represented in Tables 2(a) and 2(b) for $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$. Table 2(a) represents the star edge coloring patterns $C_{ij} = f((u_i, v_j)(u_{i+1}, v_{j+1}))$ and $D_{ij} = f((u_i, v_j)(u_{i+1}, v_{j-1}))$ where the rows and columns denote the values of i and j respectively for $i \equiv r_1(mod\ 8)$ where $r_1 \in \{0, 1, 2, ..., 7\}$ and $j \equiv r_2(mod\ 8)$ where $r_2 \in \{0, 1, 2\}$ and the Table 2(b) represents the star edge coloring patterns $E_{ij} = f((u_i, v_0)(u_{i+1}, v_j))$ and $F_{ij} = f((u_i, v_j)(u_{i+1}, v_0))$, for $0 \leq i \leq m-1$ and $j = n-1$.

Table 2 (a):The star edge coloring patterns C_{ij} and D_{ij}

i \ j	0	1	2
0	0	1	2
1	3	4	5
2	6	7	0
3	1	2	3
4	4	5	6
5	7	0	1
6	2	3	4
7	5	6	7

Table 2 (b):The star edge coloring patterns E_{ij} and F_{ij}

	E_{ij}	F_{ij}
For $n \equiv 0(mod\ 3)$	$3i + 2(mod\ 8)$	$3i + 2(mod\ 8)$
For $n \equiv 1(mod\ 3)$	$3i + 1(mod\ 8)$	$3i + 1(mod\ 8)$
For $n \equiv 2(mod\ 3)$	$3i + 1(mod\ 8)$	$3i + 1(mod\ 8)$

This coloring pattern and the following example validate that the star chromatic index of $P_m \times C_n$ is 8.

Example 1: Since f has a repeated star edge coloring pattern for all $m \geq 7$, $n \geq 7$, it sufficies to check that $P_{10} \times C_{10}$ is 8-star edge colorable (see in Figure 3).

□

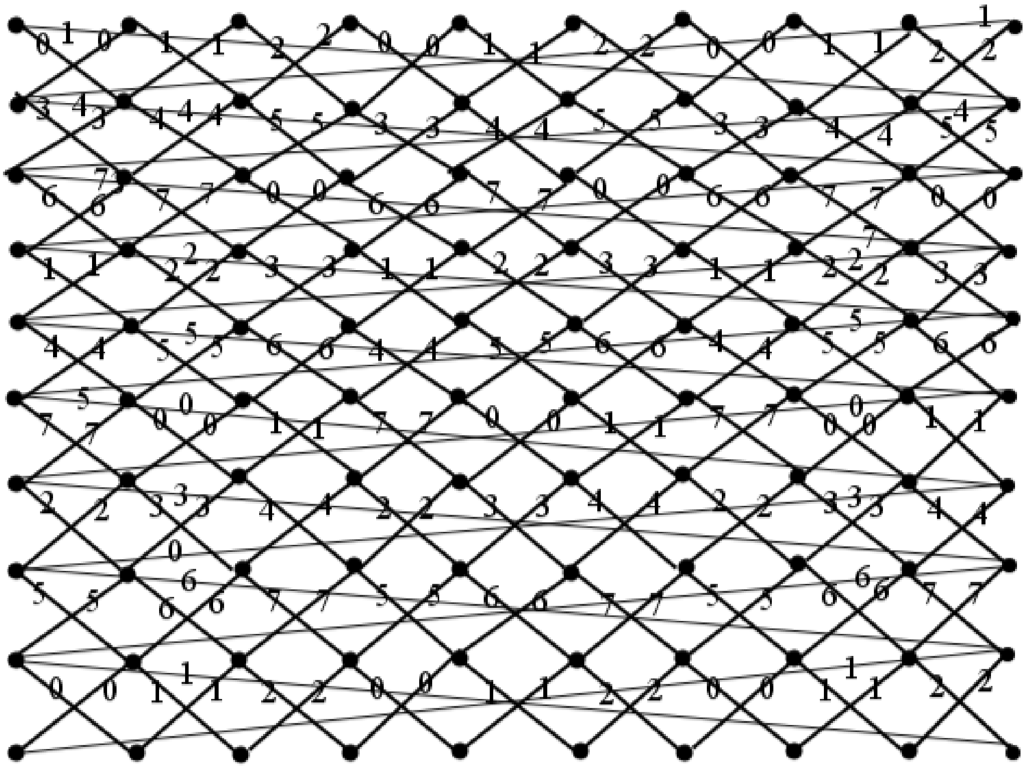

Figure 3: Star edge coloring of $P_{10} \times C_{10}$

4 Star Chromatic Index of Tensor Product of Cycles

Theorem 4.1. *For the given positive integers $m \geq 3$ and $n \geq 3$, the star chromatic index of $C_m \times C_n$ is given by*

$$\chi'_{st}(C_m \times C_n) = \begin{cases} 7, & \textit{for } m = 3, n = 3 \\ 8, & \textit{otherwise} \end{cases}$$

Proof. Let $G = C_m \times C_n$ and by the definition of tensor product, $V(G) = \{(u_i, v_j) : 0 \leq i \leq m-1,\ 0 \leq j \leq n-1\}$, $E(G) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \cup E_8$, where
$E_1 = \bigcup_{i=1}^{m-2}\{(u_i, v_j)(u_{i+1}, v_{j+1}) : 0 \leq j \leq n-2\}$
$E_2 = \bigcup_{i=1}^{m-2}\{(u_i, v_j)(u_{i+1}, v_{j-1}) : 1 \leq j \leq n-1\}$
$E_3 = \bigcup_{i=0}^{m-1}\{(u_i, v_0)(u_{i+1}, v_j) : j = n-1\}$
$E_4 = \bigcup_{i=0}^{m-1}\{(u_i, v_j)(u_{i+1}, v_0) : j = n-1\}$
$E_5 = \bigcup_{j=1}^{n-1}\{(u_0, v_j)(u_i, v_{j-1}) : i = m-1\}$
$E_6 = \bigcup_{j=0}^{n-2}\{(u_0, v_j)(u_i, v_{j+1}) : i = m-1\}$.
$E_7 = \{(u_0, v_j)(u_i, v_0) : i = m-1, j = n-1\}$.
$E_8 = \{(u_0, v_0)(u_i, v_j) : i = m-1, j = n-1\}$.
and j is taken addition modulo n with residues $0, 1, 2, ..., n-1$. Let f be the function defined by $f : E \to C \subseteq \mathcal{N} \cup \{0\}$ and C is the set of colors. We obtain the star edge

chromatic index of $C_m \times C_n$ according to the given values of m,n as follows:

Case 1: $m = 3$, $n = 3$

In this case the graph $G = C_3 \times C_3$ is a 4-regular graph and contains $G_1 = P_3 \times C_3$ as a subgraph, and by Theorem 4.1, it is found that $\chi'_{st}(P_3 \times C_3) = 5$. Also, $G_2 = G \backslash G_1$ is a cycle of length six and by Theorem 2.2, $\chi'_{st}(C_6) = 3$. In the graph G it is not possible to color the edges of G_2 with the same set of colors already used in the coloring of G_1. Since if we choose any vertex v from the first and last layers of G_2 there exists two new adjacent edges at the particular vertex v, which cannot be colored with the colors already used for the edges of G_1 . Hence we need additional two colors to color those two edges. Hence, to color the edges of G_2, we introduce sixth and seventh color along with one more color which is already used to color G_1. Hence it is found that, $\chi'_{st}(C_3 \times C_3) = 7$.

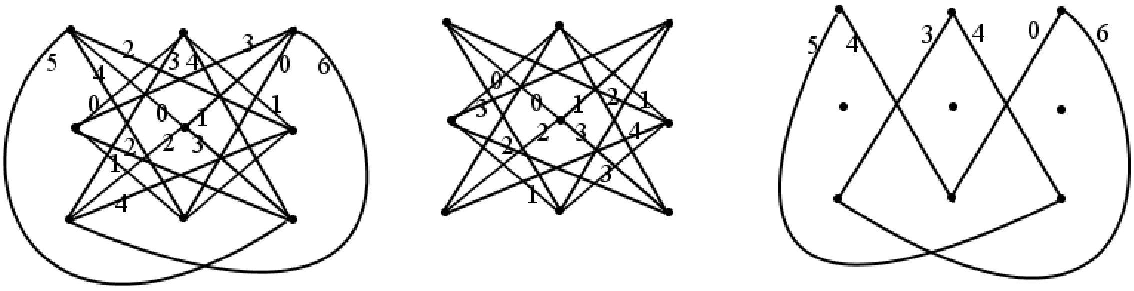

Figure 4: $G = C_3 \times C_3, G_1 = P_3 \times C_3$ and $G_2 = G \backslash G_1$

Case 2: i)$m = 3, n \geq 3$ *and* ii) $m \geq 4, n \geq 3$

When $m = 3$ and $n = 4$, the graph $G = C_3 \times C_4$ is a 4-regular graph that contains $G_3 = P_3 \times C_4$ as a subgraph, and by Theorem 4.1, it is found that $\chi'_{st}(P_3 \times C_4) = 6$. Also, in $G_4 = G \backslash G_3$ we can find two cycles of length six and by Theorem 2.2 $\chi'_{st}(C_6) = 3$. In the graph G_4, it is not possible to color the edges with the same set of colors already used in the coloring of G_3, because if we choose any vertex v in G_4 , two new adjacent edges are added at the particular vertex v, hence we need additional two colors to color those two edges. Therefore, we introduce seventh and eighth color for the edges in G_4. Hence it is found that $\chi'_{st}(C_3 \times C_4) = 8$. By using the same argument we can prove $\chi'_{st}(C_3 \times C_n) = 8$, for $n \geq 4$.
The proof of other cases are discussed based on the star chromatic index of the maximal induced subgraph G_1 of G for which $\chi'_{st}(G_1)$ is already known and that can be used to fix the lower bound of $\chi'_{st}(G)$ as in the following table:

Table (3):The star edge coloring of $G = C_m \times C_n$, for $m \geq 3$, $n \geq 4$

m	n	$C_m \times C_n$	G_1	$\chi'_{st}(G_1)$	$\chi'_{st}(G)$
4	$n \geq 3$	$C_4 \times C_n$	$P_4 \times C_n$	6	8
5	$n \geq 4$	$C_5 \times C_n$	$P_5 \times C_n$	7	8
6,7	$3 \leq n \leq 6$	$C_m \times C_n$	$P_m \times C_n$	7	8
7	7	$C_7 \times C_7$	$P_7 \times P_7$	8	8

By using a similar argument we can prove that $\chi'_{st}(C_m \times C_n) = 8$, for $m \geq 7, n \geq 7$. To develop an algorithmic approach for the star edge coloring pattern of $G = C_m \times C_n$, for large values of $m \geq 3$, $n > 3$, we give the following coloring pattern. And the star edge coloring pattern f of $G = C_m \times C_n$, for $m \geq 3$, $n > 3$ is given below. As we mentioned previously, the edges of the subgraph $G' = P_m \times C_n$ can be colored using the same coloring pattern as given in Theorem 3.1. Here $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$.

Case 1: $m \equiv 0(mod\ 8), n > 3$

$$f((u_0, v_0)(u_{m-1}, v_{1)})) = 5$$

$$f((u_0, v_0)(u_{m-1}, v_{n-1})) = 7$$

$$f((u_0, v_{n-1})(u_{m-1}, v_{n-2})) = \begin{cases} 3, & \text{if } n \equiv 0(mod\ 3) \\ 6, & \text{if } n \equiv 1(mod\ 3) \\ 7, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$$f((u_0, v_{n-1})(u_{m-1}, v_0)) = \begin{cases} 7, & \text{if } n \equiv 0(mod\ 3) \\ 7, & \text{if } n \equiv 1(mod\ 3) \\ 3, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq\ j \leq n-2$,

$$f((u_0, v_j)(u_{m-1}, v_{j-1})) = \begin{cases} 7, & \text{if } n \equiv 0(mod\ 3) \\ 2, & \text{if } n \equiv 1(mod\ 3) \\ 6, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq\ j \leq n-2$,

$$f((u_0, v_j)(u_{m-1}, v_{j+1})) = \begin{cases} 5, & \text{if } n \equiv 0(mod\ 3) \\ 6, & \text{if } n \equiv 1(mod\ 3) \\ 7, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

Case 2: $m \equiv 1(mod\ 8), n > 3$

$$f((u_0, v_0)(u_{m-1}, v_1)) = 3$$

$$f((u_0, v_0)(u_{m-1}, v_{n-1})) = 4$$

$$f((u_0, v_{n-1}), (u_{m-1}, v_{n-2})) = \begin{cases} 3, & \text{if } n \equiv 0(mod\ 3) \\ 4, & \text{if } n \equiv 1(mod\ 3) \\ 5, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$$f((u_0, v_{n-1})(u_{m-1}, v_0)) = \begin{cases} 4, & \text{if } n \equiv 0(mod\ 3) \\ 0, & \text{if } n \equiv 1(mod\ 3) \\ 1, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq\ j \leq n-2$,

$$f((u_0, v_j)(u_{m-1} v_{j-1})) = \begin{cases} 5, & \text{if } n \equiv 0(mod\ 3) \\ 3, & \text{if } n \equiv 1(mod\ 3) \\ 4, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq\ j \leq n-2$,

$$f((u_0, v_j)(u_{m-1}, v_{j+1})) = \begin{cases} 1, & \text{if } n \equiv 0(mod\ 3) \\ 4, & \text{if } n \equiv 1(mod\ 3) \\ 0, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

Case 3: $m \equiv 2(mod\ 8), n > 3$

$$f((u_0, v_0)(u_{m-1}, v_1)) = 3$$

$$f((u_0, v_0)(u_{m-1}, v_{n-1})) = 5$$

$$f((u_0, v_j)(u_{m-1}, v_{n-2})) = \begin{cases} 4, & \text{if } n \equiv 0(mod\ 3) \\ 5, & \text{if } n \equiv 1(mod\ 3) \\ 3, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$$f((u_0, v_{n-1})(u_{m-1}, v_0)) = \begin{cases} 5, & \text{if } n \equiv 0(mod\ 3) \\ 3, & \text{if } n \equiv 1(mod\ 3) \\ 4 & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq\ j \leq n-2$,

$$f((u_0, v_j)(u_{m-1}, v_{j-1})) = \begin{cases} 5, & \text{if } n \equiv 0(mod\ 3) \\ 3, & \text{if } n \equiv 1(mod\ 3) \\ 4, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq\ j \leq n-2$,

$$f((u_0, v_j)(u_{m-1}, v_{j+1})) = \begin{cases} 6, & \text{if } n \equiv 0(mod\ 3) \\ 4, & \text{if } n \equiv 1(mod\ 3) \\ 5, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

Case 4: $m \equiv 3(mod\ 8), n > 3$

$$f((u_0, v_0)(u_{m-1}, v_{1)})) = 7$$

$$f((u_0,v_0)(u_{m-1},v_{n-1})) = \begin{cases} 3, & \text{if } n \equiv 0(mod\ 3) \\ 3, & \text{if } n \equiv 1(mod\ 3) \\ 5, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$$f((u_0,v_{n-1})(u_{m-1},v_{n-2})) = \begin{cases} 6, & \text{if } n \equiv 0(mod\ 3) \\ 1, & \text{if } n \equiv 1(mod\ 3) \\ 7, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$$f((u_0,v_{n-1})(u_{m-1},v_0)) = \begin{cases} 0, & \text{if } n \equiv 0(mod\ 3) \\ 2, & \text{if } n \equiv 1(mod\ 3) \\ 6, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq\ j \leq n-2$

$$f((u_0,v_j)(u_{m-1},v_{j-1})) = \begin{cases} 1, & \text{if } n \equiv 0(mod\ 3) \\ 7, & \text{if } n \equiv 1(mod\ 3) \\ 6, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$1 \leq\ j \leq n-2,$

$$f((u_0,v_j)(u_{m-1},v_{j+1})) = \begin{cases} 7, & \text{if } n \equiv 0(mod\ 3) \\ 6, & \text{if } n \equiv 1(mod\ 3) \\ 0, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

Case 5: $m \equiv 4(mod\ 8), n > 3$

$$f((u_0,v_0)(u_{m-1},v_{1)})) = 3$$

$$f((u_0,v_0)(u_{m-1},v_{n-1})) = \begin{cases} 5, & \text{if } n \equiv 0(mod\ 3) \\ 5, & \text{if } n \equiv 1(mod\ 3) \\ 6, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$$f((u_0,v_{n-1})(u_{m-1},v_{n-2})) = \begin{cases} 4, & \text{if } n \equiv 0(mod\ 3) \\ 5, & \text{if } n \equiv 1(mod\ 3) \\ 3, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$$f((u_0,v_{n-1})(u_{m-1},v_0)) = \begin{cases} 5, & \text{if } n \equiv 0(mod\ 3) \\ 4, & \text{if } n \equiv 1(mod\ 3) \\ 2 & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq\ j \leq n-2$

$$f((u_0,v_j)(u_{m-1},v_{j-1})) = \begin{cases} 5, & \text{if } n \equiv 0(mod\ 3) \\ 3, & \text{if } n \equiv 1(mod\ 3) \\ 4, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$1 \leq\ j \leq n-2$

$$f((u_0,v_j)(u_{m-1},v_{j+1})) = \begin{cases} 1, & \text{if } n \equiv 0(mod\ 3) \\ 4, & \text{if } n \equiv 1(mod\ 3) \\ 5, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

Case 6: $m \equiv 5(mod\ 8), n > 3$

$f((u_0,v_0)(u_{m-1},v_1)) = 5$
$f((u_0,v_0)(u_{m-1},v_{n-1})) = 7$

$$f((u_0,v_{n-1})(u_{m-1},v_{n-2})) = \begin{cases} 6, & \text{if } n \equiv 0(mod\ 3) \\ 4, & \text{if } n \equiv 1(mod\ 3) \\ 5, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$$f((u_0,v_{n-1})(u_{m-1},v_0)) = \begin{cases} 5, & \text{if } n \equiv 0(mod\ 3) \\ 0, & \text{if } n \equiv 1(mod\ 3) \\ 6, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq\ j \leq n-2$,

$$f((u_0,v_j)(u_{m-1},v_{j-1})) = \begin{cases} 1, & \text{if } n \equiv 0(mod\ 3) \\ 5, & \text{if } n \equiv 1(mod\ 3) \\ 6, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq\ j \leq n-2$,

$$f((u_0,v_j)(u_{m-1},v_{j+1})) = \begin{cases} 7, & \text{if } n \equiv 0(mod\ 3) \\ 6, & \text{if } n \equiv 1(mod\ 3) \\ 4, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $m \equiv 6(mod\ 8), n > 3$
$f((u_0,v_0)(u_{m-1},v_1)) = 7$
$f((u_0,v_0)(u_{m-1},v_{n-1})) = 3$

$$f((u_0,v_{n-1})(u_{m-1},v_{n-2})) = \begin{cases} 7, & \text{if } n \equiv 0(mod\ 3) \\ 6, & \text{if } n \equiv 1(mod\ 3) \\ 6, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

$$f((u_0,v_{n-1})(u_{m-1},v_0)) = \begin{cases} 2, & \text{if } n \equiv 0(mod\ 3) \\ 0, & \text{if } n \equiv 1(mod\ 3) \\ 3, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq\ j \leq n-2$

$$f((u_0,v_j)(u_{m-1},v_{j-1})) = \begin{cases} 1, & \text{if } n \equiv 0(mod\ 3) \\ 0, & \text{if } n \equiv 1(mod\ 3) \\ 7, & \text{if } n \equiv 2(mod\ 3) \end{cases}$$

For $1 \leq \ j \leq n-2$

$$f((u_0, v_j)(u_{m-1}, v_{j+1})) = \begin{cases} 7, & \text{if } n \equiv 0 (mod\ 3) \\ 4, & \text{if } n \equiv 1 (mod\ 3) \\ 3, & \text{if } n \equiv 2 (mod\ 3) \end{cases}$$

Case 7: $m \equiv 7(mod\ 8), n > 3$

$f((u_0, v_0)(u_{m-1}, v_{1)})) = 3$

$f((u_0, v_0)(u_{m-1}, v_{n-1})) = 6$

$$f((u_0, v_{n-1})(u_{m-1}, v_{n-2})) = \begin{cases} 3, & \text{if } n \equiv 0 (mod\ 3) \\ 4, & \text{if } n \equiv 1 (mod\ 3) \\ 5, & \text{if } n \equiv 2 (mod\ 3) \end{cases}$$

$f((u_0, v_{n-1})(u_{m-1}, v_0)) = 6$ for all n

For $1 \leq \ j \leq n-2$

$$f((u_0, v_{n-1})(u_{m-1}, v_{j-1})) = \begin{cases} 5, & \text{if } n \equiv 0 (mod\ 3) \\ 3, & \text{if } n \equiv 1 (mod\ 3) \\ 4, & \text{if } n \equiv 2 (mod\ 3) \end{cases}$$

For $1 \leq \ j \leq n-2$

$$f((u_0, v_j)(u_{m-1}, v_{j+1})) = \begin{cases} 1, & \text{if } n \equiv 0 (mod\ 3) \\ 4, & \text{if } n \equiv 1 (mod\ 3) \\ 5, & \text{if } n \equiv 2 (mod\ 3) \end{cases}$$

This coloring pattern and the following example validate that the star chromatic index of $C_m \times C_n$ is 8.

Example 2: Since f has a repeated star edge coloring pattern for all $m \geq 3, n > 3$ it sufficies to check that $C_{10} \times C_{10}$ has a 8-star edge coloring as shown in Figure 5.

□

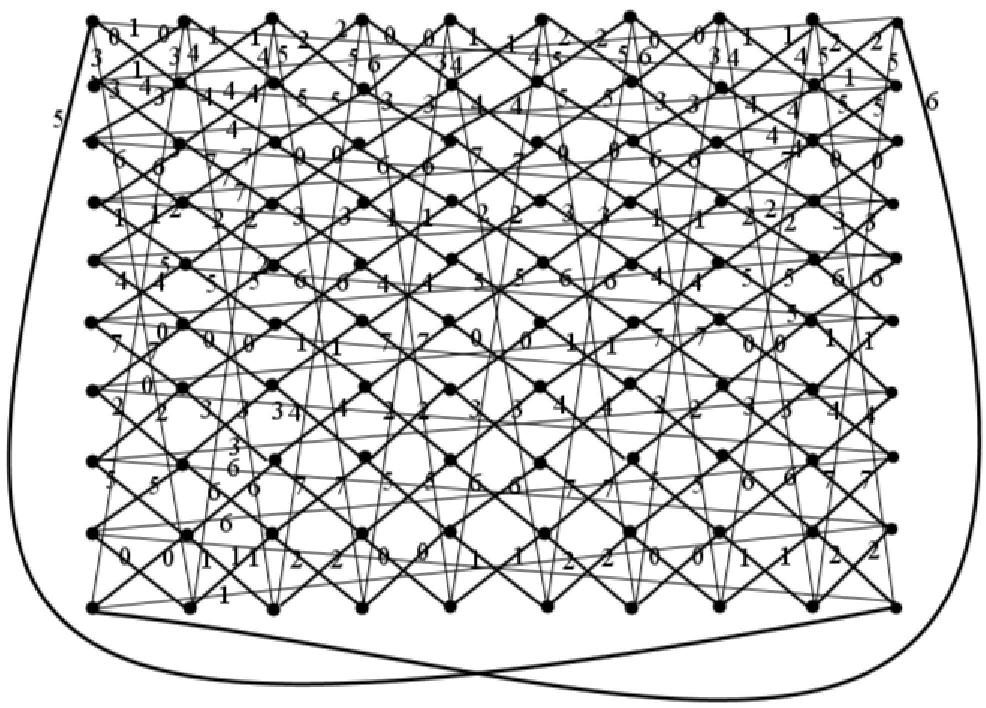

Figure 5: Star edge coloring of $C_{10} \times C_{10}$

5 Conclusion

This research has explored the star chromatic index of $P_m \times C_n$ and $C_m \times C_n$. Further, coloring patterns were given to star edge color any such graph and that can be used as an algorithmic tool to obtain the star edge coloring of large graph structure such as given product graphs, for various values of m and n. This work can be extended further to find the star edge chromatic index of the other product graphs.

6 Declaration

- Conflict of Interest- The authors declare that they have no conflict of interest.

- Ethics Approval- Not Applicable

- Consent for Publication- The authors hereby provide consent for the publication of this manuscript upon acceptance.

- Data availability- Not Applicable

- Material availability- Not Applicable

- Code availabilty- Not Applicable

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