Original Research Article

CONNECTED DOM-k-FORCING SETS IN GRAPHS

ABSTRACT. A vertex subset D_{kf} in a graph G is referred to as a dom-k-forcing set if it satisfies the properties of both a k-forcing set and a dominating set. The dom-k-forcing number of a graph G, denoted as $F_{dk}(G)$, represents the smallest possible size of a dom-k-forcing set. A connected dom-k-forcing set of a graph G, is a dom-k-forcing set of G that induces a sub graph of G which is connected. The connected dom-k-forcing number of G, $F_{cdk}(G)$, is the minimum size of a connected dom-k-forcing set. In this paper, we introduce the study of connected dom-k-forcing sets in graphs. We characterize connected dom-k-forcing sets in certain special graphs, deriving formulae for connected dom-k-forcing number based on these characterizations. Additionally, we explore the relationships between this parameter and other well-known graph theory parameters.

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Key Words: Connected domination number, k-forcing number, dom-k-forcing number, connected dom-k-forcing number.

1. Introduction

The domination number of a graph G, denoted by $\gamma(G)$, is the minimum number of vertices in a dominating set. A dominating set is a subset S of the graph's vertex set such that every vertex not in S is adjacent to at least one vertex in S. In other words, all vertices outside the set are directly connected to vertices within it. The domination number reflects how effectively a subset of vertices can oversee or influence the entire graph [1].

If the subgraph induced by a dominating set is connected, then the set is known as a connected dominating set. Among all such sets, a minimum connected dominating set is one with the smallest possible number of vertices. The size of this set is called the connected domination number, denoted by $\gamma_c(G)$ [2].

A subset Z_k of vertices in a graph is called a k-forcing set if its vertices are initially assigned a color while the remaining vertices start as uncolored. The graph undergoes a color propagation process based on the following rule: a colored vertex with at most k uncolored neighbors forces each of those neighbors to become colored. This process continues until all vertices in the graph are colored. The k-forcing number of a graph, denoted as $Z_k(G)$, represents the smallest possible size of a k-forcing set see[8].

A connected k-forcing set of a graph G, is a k-forcing set of G that induces a sub graph of G which is connected. The connected k-forcing number of G, $Z_{ck}(G)$, is the minimum size of connected k-forcing sets. Any connected k-forcing set of order $Z_{ck}(G)$ is called a minimum connected k-forcing set [4].

A vertex subset D_{kf} in a graph G is referred to as a dom-k-forcing set if it satisfies the properties of both a k-forcing set and a dominating set. The dom-k-forcing number of a graph G, denoted as $F_{dk}(G)$, represents the smallest possible size of a dom-k-forcing set.

This study integrates the concepts of connected domination and connected k-forcing to introduce a new graph-theoretic parameter: the connected dom-k-forcing set.

Definition 1.1. *For any $k \in \mathcal{N}, 1 \leq k \leq \Delta$, a connected dom-k-forcing forcing set of a graph G, is a dom-k-forcing set of G that induces a sub graph of G which is connected. The connected dom-k-forcing number of G, $F_{cdk}(G)$, is the minimum size of a connected dom-k- forcing set.*

For instance, consider the graph G illustrated in Figure 1. The set $D_{fk} = \{v_2, v_5\}$ is a valid 2-forcing set as well as dominating set, the sub graph induced by D_{fk} is connected. Here the connected dom-2-forcing number of G is 2.

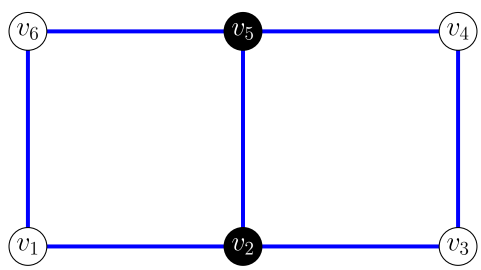

FIGURE 1. $G(V, E)$

When k = 1, connected dom-k-forcing number is equivalent to the connected dom-forcing number, denoted by $F_{cd}(G)$ See [6]. When $k = 2$, this is known as the connected dom-2-forcing number and is denoted by $F_{cd2}(G)$.

In this paper, we introduce the study of connected dom-k-forcing sets in graphs. We characterize connected dom-k-forcing sets in certain special graphs, deriving formulas for connected dom-k-forcing number based on these characterizations. Additionally, we explore the relationships between this parameter and other well-known graph theory parameters.

2. Motivation

The study of the connected dom-k-forcing number of graphs is at the intersection of domination theory and dynamic graph processes, motivated by both theoretical curiosity and practical applications. In graph theory, domination parameters help to characterise how influence, control or monitoring can be established over a network. k-forcing processes, a generalisation of the zero forcing process, model how information, disease or opinions spread through a system, subject to constraints given by the number k of forced neighbours.

By combining these two concepts—connected domination and k-forcing—we can investigate how an initial set of "influencing" nodes can control the spread in a graph and remain connected throughout the process. This is crucial in situations where continuous communication or cohesion among influencers is necessary, such as in wireless sensor networks, distributed computing or coordinated response systems. Studying the connected dom-k-forcing number not only enriches the theoretical framework of graph parameters but also leads to efficient algorithms and heuristics for real-world network management tasks, including fault detection, resource allocation and containment strategies. Understanding the complexity, bounds and characterisations of this parameter opens up new avenues in combinatorics and applied network science.

3. Bounds for connected dom-k-forcing number

Within this section, we fond some bounds for connected dom-k-forcing number. Also investigating about the graph having $F_{cdk}(G) = \gamma_c(G)$.

The definition makes it evident that the combination of a connected k- forcing set and a connected dominating set constitutes a dom-k-forcing set. Therefore, the following relationship holds:

Proposition 3.1. *For a graph G*
i) $Z_{ck}(G) \leq F_{cdk}(G) \leq Z_{ck}(G) + \gamma_c(G)$
ii) $\gamma_c(G) \leq F_{cdk}(G) \leq Z_{ck}(G) + \gamma_c(G)$

From the definition it is also evident that, for a graph G every connected dom-k-forcing set is also a connected dom-$k+1$-forcing set. Hence we have the following result.

Proposition 3.2. *Let G be a graph. Then $F_{cd(k+1)}(G) \leq F_{cdk}(G)$, for all $k \in \mathbb{N}$.*

Also from the definition every connected dom-k-forcing set is a dom-k-forcing set. Hence we have the following result.

Proposition 3.3. *For any connected graph G,*

$$F_{dk}(G) \leq F_{cdk}(G).$$

Theorem 3.4. [3] *If G is a connected graph with $\Delta(G) \leq 2$, then $Z_2(G) = 1$.*

Theorem 3.5. *Let G be a connected graph with $\Delta(G) \leq 2$. Then $F_{cd2}(G) = \gamma_c(G)$.*

Proof. Let G be a graph with $\Delta(G) \leq 2$. Then any single vertex in $V(G)$ 2-forces the graph. Hence every connected dominating set is also a connected dom-2-forcing set. Therefore $F_{cd2}(G) = \gamma_c(G)$. □

There are graphs with $Z_{c2}(G) = 1$ and $F_{d2}(G) > \gamma_c(G)$. Consider the graph given below. The set $Z_2 = \{v_1\}$ is a connected 2- forcing set, but it cannot dominates G. The set $S = \{v_2\}$ is a dominating set for G, but it is not a 2-forcing set. $Z_2 \cup S$ forms a 2-forcing as well as dominating set for G. Hence G is a graph with $Z_{c2}(G) = 1$, $\gamma_c(G) = 1$ and $F_{cd2}(G) = 2$

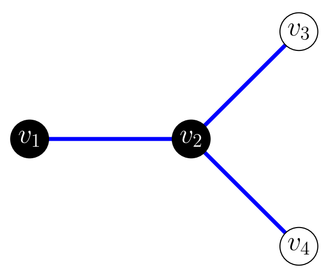

FIGURE 2. A graph G having $Z_{c2}(G) = 1$, $\gamma_c(G) = 1$ and $F_{cd2}(G) = 2$.

From the above example we can see that the upper bound given in Proposition 3.1 is sharp. We can easily observe the following result.

Proposition 3.6. *If G is a connected graph with minimum degree $\delta \geq k+1$, then $F_{cdk}(G) > 1$.*

In general we have the following result

Theorem 3.7. *Let G be a graph with $F_{cdp}(G) = \gamma_c(G)$ for some positive integer p. Then $F_{cdk}(G) = \gamma_c(G)$ for all $k \geq p$.*

4. Graphs where $F_{dk}(G) = F_{cdk}(G)$

There are graphs with the dom-k-forcing number and the connected dom-k-forcing number that are the same.

The friendship graph F_n is formed by connecting n copies of the complete graph K_3 to a single common vertex see [9].

Proposition 4.1. [7] *Let F_p denotes the friendship graph with $p \geq 2$ triangles. Then $F_{d2}(F_p) = p$.*

Now we discuss the case when $k = 2$.

Proposition 4.2. *Let F_p denotes the friendship graph with $p \geq 2$ triangles. Then $F_{d2}(F_p) = F_{cd2}(F_p) = p$.*

Proof. Let $v, v_1, v_1', \cdots, v_p, v_p'$ be the vertices of F_p with common vertex v. Consider the subset $A = \{v, v_1, \cdots, v_{p-1}\}$ of $V(F_p)$. Then A be a connected dom-2-forcing set, which is minimum and $F_{d2}(F_p) = p$. Hence $F_{d2}(F_p) = F_{cd2}(F_p) = p$. □

Let $G = K_n$ be a complete graph with n vertices. Then $F_{dk}(K_n) = \max\{n-k, 1\}$, for any positive integer k see [7].

Theorem 4.3. *For the complete graph K_n, $F_{dk}(K_n) = F_{cdk}(K_n) = \max\{n-k, 1\}$, where k is any positive integer.*

Proof. For any complete graph K_n, every dom-k-forcing set is connected. Hence K_n, $F_{dk}(K_n) = F_{cdk}(K_n) = \max\{n-k, 1\}$, for any positive integer k. □

Let $K_{m,n}$ be a complete bipartite where $n, m \geq 3$. Then we have $F_{d2}(K_{m,n}) = m + n - 4$ see [7]. Here every dom-2-forcing set of $K_{m,n}$ is connected, hence we get the following result.

Theorem 4.4. *Let $K_{m,n}$ be a complete bipartite where $n, m \geq 3$. Then $F_{d2}(K_{m,n}) = F_{cd2}(K_{m,n}) = m + n - 4$.*

The wheel graph, W_n, is a graph obtained by connecting a single vertex to all the vertices of a cycle graph C_{n-1}. Also we have $F_{dk}(W_n) = 2$ for $2 \leq k < n-1$ see [7].

Theorem 4.5. *For the wheel graph W_n, $F_{dk}(W_n) = F_{cdk}(W_n) = 2$, where $2 \leq k < n-1$.*

Proof. Let $v_1, v_2, \cdots, v_n$ be the vertices of the wheel graph W_n. Assume that v_1 has degree $d(v_1) = n-1$ and all other vertices have degree 3. Then for $2 \leq k < n-1$ every minimum dom-k- forcing set contains v_1 and the set connected. Therefore, $F_{dk}(W_n) = F_{cdk}(W_n) = 2$, for $2 \leq k < n-1$. □

A helm graph H_m is a graph that is created by attaching a pendant edge to each vertex of an n-wheel graph's cycle, where $m \geq 4$ [10].

Theorem 4.6. [7] *Let H_m be the helm graph. Then $F_{d2}(H_m) = m$.*

Theorem 4.7. *Let H_m be the helm graph. Then $F_{d2}(H_m) = F_{cd2}(H_m) = m$.*

The prism graph, also known as the circular ladder graph, is formed by taking the Cartesian product of a cycle C_n and the complete graph K_2. It is commonly represented as $C_n \square K_2$ see [16].

Let G be the prism graph $C_n \square K_2$, where $v_1, v_2, \ldots, v_n$ are the vertices of the cycle C_n and $v_1', v_2', \ldots, v_n'$ be the corresponding vertices in $C_n \square K_2$. Now, consider a graph H obtained by subdividing each edge $v_i v_i'$ (for $i = 1, 2, \ldots, n$) exactly once (see [3]).

Theorem 4.8. *Under the above assumptions let H be the graph obtained from the prism graph by subdividing each edge $v_i v_i'$ (for $i = 1, 2, \ldots, n$) exactly once. Then, $F_{d2}(H) = F_{cd2}(H) = n$.*

Proof. Let $w_1, w_2, \cdots, w_n$ be the vertices which subdivides edges $v_iv'_i$ (for $i = 1, 2, \ldots, n$), respectively, exactly once. Since w_i is adjacent to both v_i and v'_i, at least one of the vertices must belong to every dominating set of H. Hence $\gamma(H) \geq n$. Also $D = \{w_1, w_2, \cdots, w_n\}$ dominates H, $\gamma(H) = n$. Here the set D, 2-forces the graph H and the subgraph induced by D is connected. Therefore, $F_{d2}(H) = F_{cd2}(H) = n$. □

Theorem 4.9. *Let G be the graph $C_n \Box P_3$, $n > 3$. Then $F_{cd2}(G) = \gamma_c(G) = n$.*

Proof. Let $u_1, \cdots, u_n, v_1, \cdots, v_n, w_1, \cdots, w_n$ be the vertices of three copies of C_n in $C_n \Box P_3$, with $deg(u_i) = deg(w_i) = 3$ and $deg(v_i) = 4$. The set $A = \{v_1, \cdots, v_n\}$ be a connected dominating set for G. Here $|A|$ is minimum since if we remove any one of the v_i in A, it affect the domination property. Hence $\gamma_c(G) = n$, the set A 2-forces the entire graph. Therefore $F_{cd2}(G) = \gamma_c(G) = n$. □

5. Graphs where $F_{dk}(G) < F_{cdk}(G)$

There are graphs with $F_{dk}(G) < F_{cdk}(G)$. From Theorem 3.5 we have the following result.

Theorem 5.1.

(1) *For a path P_n of order n, $F_{cd2}(P_n) = \gamma_c(P_n) = n - 1$.*
(2) *For a cycle C_n of order n, $F_{cd2}(C_n) = \gamma_c(C_n) = n - 2$.*

Now we consider the connected dom-k-forcing number of trees.

Definition 5.2. *Let T be a tree, consider the vertices which are adjacent to the leaf (pendant) vertex and call these vertices as parent leaf vertices.*

For instance, consider the tree T illustrated in figure 3. Here the parent leaf vertices of T are v_3 and v_4.

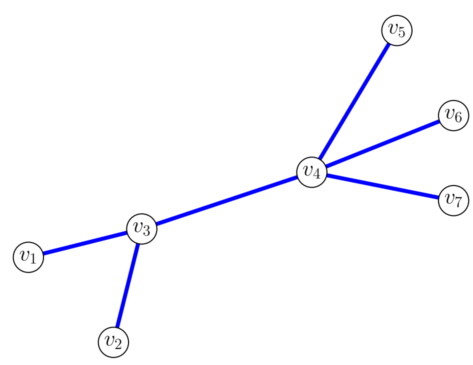

Figure 3. The tree T

Theorem 5.3. *Let T be a tree and let x_i be the number of leaves adjacent to each parent leaf vertex in T. Then $x_i \leq k$ for all i if and only if $F_{cdk}(T) = \gamma_c(T) = n - m$, where n is the number of vertices in T and m is the number of leaves in T.*

Proof. First, we assume that the total number of leaves adjacent to each parent leaf vertex of the tree T is at most k, ie $x_i \leq k$ for all i. We know that all vertices of T except the pendant vertices form a unique minimum connected dominating set, which k-forces the graph T. Hence $F_{cdk}(T) = \gamma_c(T) = n - m$, where n is the number of vertices in T and m is the number of leaves in T.

Conversely, assume that $F_{cdk}(T) = \gamma_c(T) = n - m$ where n is the number of of vertices in T and m is the number of leaves in T. Let H be the subset of the vertex set of T that does not contain the pendant vertices of T. Hence $|H| = n - m$. If possible, assume that there exists a parent leaf vertex, say v, which is adjacent to at least $k+1$ pendant vertices. All the vertices except the pendant vertices are in the unique minimum connected dominating set of T. Also, from v we cannot k-force all the pendant vertices adjacent to v. Therefore, the set H cannot be a connected dom-k-forcing set of T, a contradiction. Hence, the total number of leaves adjacent to each parent leaf vertex of the tree T is at most k. □

We know the cartesian product of two graphs G and H, is denoted by $G\square H$, is the graph with vertex set $V(G) \times V(H)$ such that (u, v) is adjacent to (u', v') if and only if
(1) $u = u'$ and $vv' \in E(H)$, or
(2) $v = v'$ and $uu' \in E(G)$.
The Cartesian product of two path graphs is known as the grid graph.

Let p and q be positive integers such that $p \leq q$. A $p \times q$ grid graph [15] $G_{p,q} = (V, E)$ is a graph where

$$V(G(p, q)) = \{(i, j)|1 \leq i \leq p, 1 \leq j \leq q\},$$

$$E(G(p, q)) = \{\{(i, j), (i, j + 1)\}|1 \leq i \leq p, 1 \leq j \leq q - 1\} \cup\{\{(i, j), (i + 1, j)\}|1 \leq i \leq p - 1, 1 \leq j \leq q\}.$$

The ladder graph L_n is obtained by taking the cartesian product of path P_n with the complete graph K_2. It is known that for a ladder graph L_n of order n, $F_{cd}(L_n) = n = \gamma_c(L_n)$ for $n \geq 4$ (see [6]). Therefore we have the following result.

Proposition 5.4. *Let L_n be the Ladder graph of order $n \geq 4$. Then $F_{cdk}(L_n) = n$, for all $k \in \mathbb{N}$.*

Let us recall the following result from [15] to prove the succeeding theorem.

Theorem 5.5. [15] *The connected domination number of $3 \times p$ grid graph $G_{3,p}$ is $\gamma_c(G_{3,p}) = p$.*

Theorem 5.6. *The connected dom-2-forcing number of $3 \times p$ grid graph $G_{3,p}$ is*

$$F_{cd2}(G_{3,p}) = p.$$

Proof. We can see that the set $A = \{(2, 1),(2, 2), \cdots ,(2, p)\}$ be a minimum connected domination set, and is unique. The set A 2-forces the graph $G_{3,p}$. Hence

$$F_{cd2}(G_{3,p}) = p.$$

□

Theorem 5.7. [21] *The connected domination number of* $p \times q$ $(p \geq 4, q \geq 4)$ *grid graph* $G_{p,q}$ *is*

$$\gamma_c(G_{p,q}) = \frac{pq - a'_{p,q}}{3} + \bar{r}'_{p,q} + c'_{p,q}.$$

in which

$$a'_{p,q} = p(mod\ 3).q(mod\ 3)$$

$$\bar{r}'_{p,q} = \begin{cases} 3 & p(mod\ 3).q(mod\ 3) = 4 \\ 2 & p(mod\ 3).q(mod\ 3) = 2 \\ 1 & p(mod\ 3).q(mod\ 3) = 1 \\ 0 & p(mod\ 3).q(mod\ 3) = 0 \end{cases}$$

$$c'_{p,q} = \begin{cases} \min\{\frac{p}{3}, \frac{q}{3}\} & p(mod\ 3) = 0 \ and \ q(mod\ 3) = 0 \\ \frac{p}{3} & p(mod\ 3) = 0 \ and \ q(mod\ 3) \neq 0 \\ \frac{q}{3} & p(mod\ 3) \neq 0 \ and \ q(mod\ 3) = 0 \\ \lfloor \frac{p}{3} \rfloor + \lfloor \frac{q}{3} \rfloor - 1 & p(mod\ 3) \neq 0 \ and \ q(mod\ 3) \neq 0 \end{cases}$$

Theorem 5.8. *Let* $G_{p,q}$ *be a* $p \times q$ $(p \geq 4, q \geq 4)$ *grid graph. Then*

$$F_{cd2}(G_{p,q}) = \gamma_c(G_{p,q}).$$

Proof. Consider the following cases.

Case 1: For any q and p (mod 3) = 0. We consider the vertex set $S = A \cup B \cup C$ in which $A = \{(1, 2), (2, 2), \cdots, (p, 2)\}, B = \{(2, 3), (2, 4), \cdots, (2, q)\}$, and $C = \{(x, 3), \cdots, (x, q)|x = 5, 8, \cdots, p - 1\}$. Theorem 5.7 says that S is a minimum connected dominating set with cardinality $\frac{p(q+1)}{3}$.

Case 2: For any p and q (mod 3)= 0, we consider the vertex set $S = A \cup B \cup C'$ in which $C' = \{(3, y), \cdots, (p, y)|y = 5, 8, \cdots, q-1\}$. Then S is a minimum connected dominating set with cardinality $\frac{(p+1)q}{3}$ (By Theorem 5.7).

Case 3: For p (mod 3) = 1 and q (mod 3) = 1. We consider the vertex set $S = A \cup B_1 \cup C_1 \cup E_1$ for $p = 4$ and $S = A \cup B \cup C_1 \cup D_1 \cup E_1$ for $p > 4$, in which $B_1 = \{(2, 3), (2, 4), \cdots, (2, q - 2)\}$, $C_1 = \{(3, y), \cdots, (p, y)|y = 5, 8, \cdots, q - 2\}$, $D_1 = \{(x, q - 1), (x, q)|x = 5, 8, \cdots, p - 5\}$ and $E_1 = \{(p - 1, q - 1), (p - 1, q), (p - 2, q)\}$. Then S is a minimum connected dominating set with cardinality $\frac{pq+p+q-3}{3}$ (By Theorem 5.7).

Case 4: For p (mod 3) = 1 and q (mod 3) = 2. We consider the vertex set $S = A \cup B \cup C_2 \cup D_2 \cup E_2$, in which $C_2 = \{(3, y), \cdots, (p, y)|y = 5, 8, \cdots, q - 3\}$, $D_2 = \{(x, q - 2), (x, q - 1), (x, q)|x = 5, 8, \cdots, p - 2\}$ and $E_2 = \{(p, q - 2), (p, q - 1)\}$. Then S is a minimum connected dominating set with cardinality $\frac{pq+p+q-2}{3}$ (By Theorem 5.7).

Case 5: For p (mod 3) = 2 and q (mod 3) = 1: We consider the vertex set $S = A \cup B \cup C_3 \cup D_3 \cup E_3$, in which $C_3 = \{(3, y), \cdots, (p, y)|y = 5, 8, \cdots, q - 2\}$, $D_3 = \{(x, q-2), (x, q-1), (x, q)|x = 5, 8, \cdots, p-3\}$ and $E_3 = \{(p, q-1), (p, q)\}$.

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Then S is a minimum connected dominating set with cardinality $\frac{pq+p+q-2}{3}$ (By Theorem 5.7).

Case 6: For p (mod 3) = 2 and q (mod 3) = 2. We consider the vertex set $S = A \cup B \cup C_4 \cup D_4 \cup E_4$, in which $C_4 = \{(3, y), \cdots, (p, y) | y = 5, 8, \cdots, q-3\}$, $D_4 = \{(x, q-2), (x, q-1), (x, q) | x = 5, 8, \cdots, p-3\}$ and $E_4 = \{(p, q-2), (p, q-1), (p, q)\}$.Then S is a minimum connected dominating set with cardinality $\frac{pq+p+q-2}{3}$ (By Theorem 5.7).

In all the cases the set S 2-forces the entire graph. Hence

$$F_{cd2}(G_{p,q}) = \gamma_c(G_{p,q}).$$

□

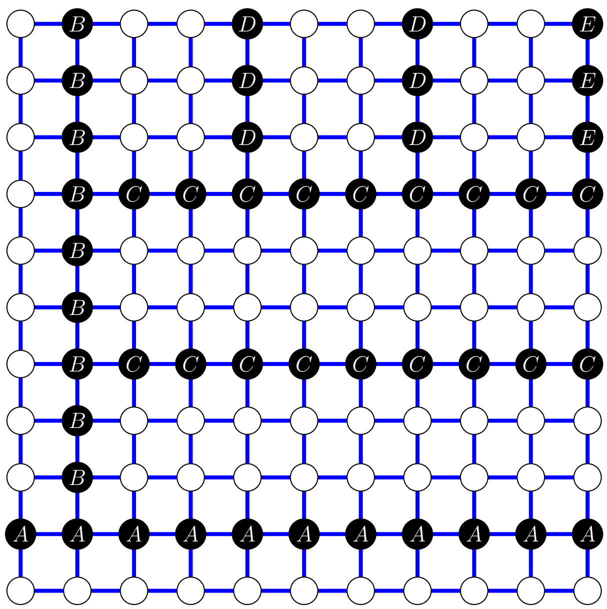

FIGURE 4. Connected dom-2-forcing set as well as connected dominating set for $G_{11,11}$

Now consider the prism graph $C_n \Box K_2$.

Theorem 5.9. [22] *Let G be a prism graph $C_n \Box K_2$, $n > 3$. Then $\gamma_c(G) = n$.*

Theorem 5.10. *Let G be a prism graph $C_n \Box K_2$, $n > 3$. Then $F_{cdk}(G) = \gamma_c(G) = n$ for all positive integer $1 \leq k \leq 3$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n and let $v'_1, v'_2, \ldots, v'_n$ be the corresponding vertices in $C_n \Box K_2$. Consider the set $A = \{v_1, v_2, \ldots, v_n\}$, which dominates G. The set A is a minimum connected dominating set by above theorem. We can see that A forces the entire graph G. Hence $F_{cdk}(G) = \gamma_c(G) = n$ for all positive integer $1 \leq k \leq 3$. □

An antiprism graph [23] is a graph having $2n$ vertices on two cycles labeled with $\{v_1, v_2, \ldots, v_n\}$ in the inner cycle, $\{v_{n+1}, v_{n+2}, \ldots, v_{2n}\}$ in the outer cycle. Joining these two cycles with edges of the form $(v_i v_j : 1 \leq i \leq n, j = n + i), (v_1 v_{2n})$ and $(v_i v_{j-1} : 2 \leq i \leq n, j = n + i)$ such that v_i and $v_{i+1(modn)}$are adjacent. It is denoted as A_n.

Theorem 5.11. [22] *Let A_n be an antiprism graph with $2n$ vertices. Then $\gamma_c(A_n) = n - 1$.*

Theorem 5.12. *Let A_n, $n > 3$ be an antiprism graph. Then $F_{cd2}(A_n) = \gamma_c(A_n) = n - 1$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the inner cycle and let $\{v_{n+1}, v_{n+2}, \ldots, v_{2n}\}$ be the corresponding vertices in A_n. Consider the set $A = \{v_1, v_2, \ldots, v_{n-1}\}$, which dominates A_n. The set A is a minimum connected dominating set by above theorem. We can see that the set A 2-forces the entire graph A_n by starting the forcing process from the vertex v_2. Hence $F_{cd2}(A_n) = \gamma_c(A_n) = n - 1$. □

In the case of A_8, above theorem is illustrated in Figure 5.

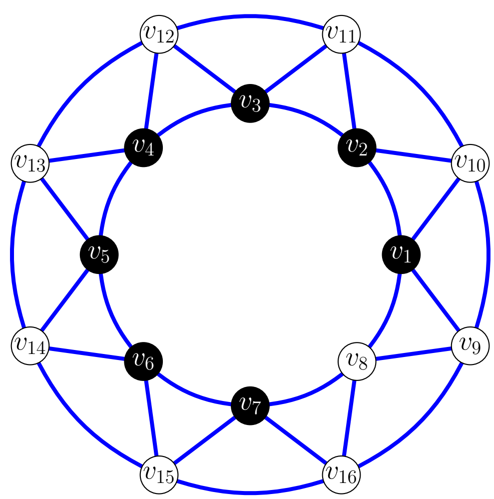

FIGURE 5. The antiprism graph A_8, connected dom-2-forcing set $A = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $F_{cd2}(A_8) = 7$.

The n-crossed prism graph [22], R_n, of positive even n vertices is the graph obtained by taking two disjoint cycle graphs C_n labeled with $\{v_1, v_2, \ldots, v_n\}$ in the inner cycle, $\{v'_1, v'_2, \ldots, v'_n\}$ in the outer cycle. Adding edges (v_k, v'_{2k+1}) and (v_{k+1}, v'_{2k}) for $k = 1, 3, \cdots, (n - 1)$. We can see that R_n is a 3-regular graph.

Theorem 5.13. [22] *Let R_n be the n-crossed prism graph with $2n$ vertices. Then $\gamma_c(R_n) = n$.*

Theorem 5.14. *Let R_n be the n-crossed prism graph. Then $F_{cdk}(R_n) = \gamma_c(R_n) = n$ for all positive integer k.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the inner cycle and let $\{v'_1, v'_2, \ldots, v'_n\}$ be the corresponding vertices of the outer circle. Consider the set $A = \{v_1, v_2, \ldots, v_n\}$, which dominates R_n. The set A is a minimum connected dominating set by above theorem. We can see that the set A forces the entire graph R_n. Hence $F_{cdk}(R_n) = \gamma_c(R_n) = n$ for all positive integer k. □

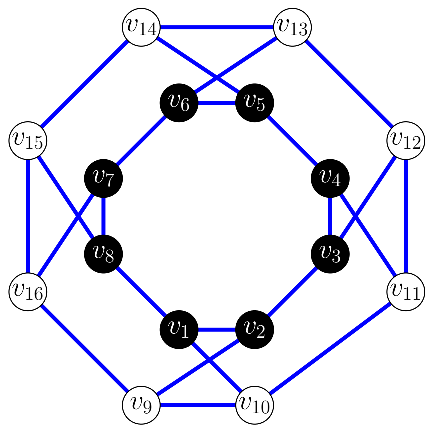

FIGURE 6. The 8-crossed prism graph R_8, connected dom-2-forcing set $A = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and $F_{cd2}(A_8) = 8$.

The gear graph is a wheel graph with a vertex added between each pair of adjacent vertices of the outer cycle. The gear graph G_n has $2n+1$ vertices and $3n$ edges.

Theorem 5.15. [22] *For the gear graph G_n, $\gamma_c(G_n) = 1 + \lceil n/2 \rceil$.*

Theorem 5.16. *For the gear graph G_n,*

$$F_{cdk}(G_n) = \begin{cases} 2 + \lceil n/2 \rceil & for \quad k = 1 \\ 1 + \lceil n/2 \rceil & for \quad k > 1 \end{cases}$$

Proof. Let $v, v_1, v_2, \ldots, v_n$ be the vertices of the wheel graph with $deg(v) = n$ and $u_1, u_2, \cdots, u_n$ be the vertices added to get the gear graph G_n. Now consider $A = \{v, v_1, v_3, \ldots, v_n\}$ if n is odd and $A = \{v, v_1, v_3, \ldots, v_{n-1}\}$ if n is even. In both cases the set A is the minimum connected dominating set by above theorem and which is unique up to isomorphism. We can see that the set A 2-forces the graph G_n. Since every vertex in the set A has two white neighbours, it cannot zero forces the graph G_n. But $A \cup \{u_1\}$ zero forces G_n. Hence

$$F_{cdk}(G_n) = \begin{cases} 2 + \lceil n/2 \rceil & for \quad k = 1 \\ 1 + \lceil n/2 \rceil & for \quad k > 1 \end{cases}$$

□

The above theorem is illustrated in Figure 7.

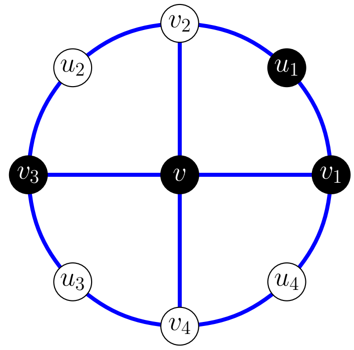

FIGURE 7. The gear graph G_4, $A = \{v, v_1, u_1, v_3\}$ be a minimum connected dom-forcing set and $B = \{v, v_1, v_3\}$ be a minimum connected dom-2-forcing set. Hence $F_{cd}(G_4) = 4$ and $F_{cd2}(G_4) = 3$.

6. Connected dom-k-forcing number of splitting graph of a graph

The splitting graph of a graph G is the graph $S(G)$ obtained by taking a vertex v' corresponding to each vertex $v \in G$ and join v' to all vertices of G adjacent to v [17].

Theorem 6.1. [20] *Let G be a connected graph of order n. Then*

$$\gamma_c[S(G)] = \begin{cases} 2 & if \quad \gamma_c(G) = 1 \\ \gamma_c(G) & if \quad \gamma_c(G) \geq 2 \end{cases}$$

Theorem 6.2. *Let $S(P_n)$, $n \geq 4$ be the splitting graph of the path P_n. Then $F_{cd2}[S(P_n)] = \gamma_c[S(P_n)] = n - 2$.*

Proof. Let $v_1, v_2, \cdots, v_n$ be the vertices of the path graph P_n in $S(P_n)$, and let $u_1, u_2, \cdots, u_n$ be the vertices corresponding to $v_1, v_2, \cdots, v_n$ which are added to obtain $S(P_n)$. Then the set $A = \{v_2, \cdots, v_{n-1}\}$ be a minimum connected dominating set by Theorem 6.1. We can see that this set 2-forces $S(P_n)$. Hence $F_{cd}[S(P_n)] = n - 2$. □

Remark: The graph $S(P_2)$, which is isomorphic to P_4, hence $F_{cd2}[S(P_2)] = \gamma_c[S(P_2)] = 2$. For the graph $S(P_3)$, $A = \{v_1, v_2\}$ be a minimum connected dominating set which 2-forces the entire graph. Hence $F_{cd2}[S(P_3)] = \gamma_c[S(P_3)] = 2$. Therefore in all cases $F_{cd2}[S(P_n)] = \gamma_c[S(P_n)]$ for all n.

Theorem 6.3. *Let $S(C_n)$, $n \geq 5$ be the splitting graph of the cycle C_n. Then*

$$F_{cd2}[S(C_n)] = \gamma_c[S(C_n)] = n - 2.$$

Proof. Let $v_1, v_2, \cdots, v_n$ be the vertices of the cycle graph C_n in $S(C_n)$, and let $u_1, u_2, \cdots, u_n$ be the vertices corresponding to $v_1, v_2, \cdots, v_n$ which are added to obtain $S(C_n)$. Then $A = \{v_1, \cdots, v_{n-2}\}$ be a minimum connected dominating set by Theorem 6.1. We can see that this set 2-forces $S(C_n)$. Hence $F_{cd}[S(C_n)] = n - 2$. □

In the case of $S(C_3)$, $A = \{v_1, u_1\}$ be a minimum connected dominating set which 2-forces the entire graph. Hence $F_{cd2}[S(C_3)] = \gamma_c[S(C_3)] = 2$. For $S(C_4)$, $A = \{v_1, v_2\}$ be a minimum connected dominating set and is unique. We can see that this set is not a 2-forcing set. But $A' = \{v_1, v_2, u_1\}$ be a minimum connected dom-2-forcing set. Hence $F_{cd2}[S(C_3)] = 3$.

Theorem 6.4. *For $n \geq 2$, $F_{cd3}[S(L_n)] = \gamma_c[S(L_n)] = n$.*

Proof. Let $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ be the vertices of the ladder graph L_n and let $\{u'_1, u'_2, \cdots, u'_n, v'_1, v'_2, \cdots, v'_n\}$ be the vertices corresponding to $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ which are added to obtain $S(L_n)$. Then $A = \{v_1, \cdots, v_n\}$ be a minimum connected dominating set by Theorem 6.1. We can see that the set A 3-forces the entire graph. Hence $F_{cd3}[S(L_n)] = n$. □

7. Conclusion and open problems

This article deals with the problem of determining the connected dom-k-forcing number of graphs. Section 3 deals with some bounds for the connected dom-k-forcing number. Also, in this section, we determined some graphs with the property $F_{cdk}(G) = \gamma_c(G)$. In Section 4, we identified some graphs whose dom-k-forcing number and connected dom-k-forcing number are the same. In particular, Section 5 characterizes the trees for which $F_{cdk}(T) = n - m$. Additionally, the connected dom-k-forcing number of the splitting graph of a graph is determined in Section 6. Determining the connected dom-k-forcing number of a splitting graph for an arbitrary graph is still an open question.

References

[1] C. Berge, *Theory of graphs and its applications*, Dunod, Paris, (1958).

[2] Sampathkumar E., Walikar HB , *The connected domination number of a graph*, J. Math. Phys. Sci, 13 (1979).

[3] K. P. Premodkumar, Charles Dominic, Baby Chacko, *K- Forcing Number of Some Graphs and Their Splitting Graphs*, International Journal of Scientific Research in Mathematical and Statistical Sciences, Volume 6, Issue 3, (2019), pp. 121 -127

[4] K. P. Premodkumar, Charles Dominic, and Baby Chacko, *Connected k- Forcing Sets of Graphs and Splitting Graphs*, Journal of Mathematical and Computational Science, 10 (2020), Number 3, pp. 656 - 680.

[5] Susanth P, Charles Dominic, and K. P. Premodkumar, *Dom-forcing sets in graphs*, (Communicated).

[6] Susanth P, Charles Dominic, and K. P. Premodkumar, *Connected dom-forcing sets in graphs*, Discrete Mathematics, Algorithms and Applications (Communicated)

[7] Susanth P, Charles Dominic, and K. P. Premodkumar, *Dom-k-forcing sets in graphs*, (Communicated)

[8] David Amos, Yair Caro, Randy Davila and Ryan Pepper, *Upper Bounds on the k-Forcing Number of a Graph*, Discrete Applied Mathematics, 181 (2015), pp. 1 - 10.

[9] J. A. Gallian, *A Dynamic Survey of Graph Labelling*, The Electronics Journal of Combinatorics, 17 (2014), DS6.

[10] A N Hayyu, Dafik, I M Tirta, R Adawiyah, R M Prihandini *Resolving domination number of helm graph and it's operation* Journel of Physics: Conference series 1465(2020)

[11] Dr. A. Sugumaran, E. Jayachandran, *Domination number of some graphs*, IJSDR 3 (2018).

[12] David Amos, Yair Caro, Randy Davila, Ryan Pepper *Upper bounds on the k-forcing number of a graph*,Discrete Applied Mathematics. Volume 181,(2015), pp 1-10.

[13] F. Harary, *Graph Theory*, Narosa Publishing Home, NewDelhi, 1969.

[14] J.Vernold Vivina, M. Vekatachalam *On b-chromatic number of sun let graph and wheel graph families* Journal of the Egyptian Mathematical Society(2014). http://dx.doi.org/10.1016/j.joems.2014.05.011

[15] P. C. Lie and M. Toulouse. *Maximum leaf spanning tree problem for grid graphs*. Journal of Combinatorial Mathematics and Combinatorial Computing, 73:181–193, (2010).

[16] Hladnik. M, Marusic. D, and Pisanski, *Cyclic Haar Graphs*, Discrete Mathematics, 244, (2002), pp. 137 - 153.

[17] E. Sampathkumar and H.B. Walikaer, *On the splitting graph of a graph*, Journal of Karnatak University Science 25, (1981)

[18] Baby Chacko, Charles Dominic, and K. P. Premodkumar, *On the zero forcing number of graphs and their splitting graphs*, Algebra and Discrete Mathematics 28 (2019), 29-43

[19] Samir K. Vaidya1 and Nirang J. Kothari, *Domination Integrity of Splitting Graph of Path and Cycle*,Hindawi Publishing Corporation 2013 (2013).

[20] J. Deepalakshmi, G. Marimuthu, A. Somasundaram, and S. Arumugam,*Domination parameters of the splitting graph of a graph*, Communications in Combinatorics and Optimization Vol. 8, No. 4 (2023) pp. 631-637.

[21] Masahisa Goto, Koji M. Kobayashi, *Connected domination in grid graphs*, arxiv 2021.

[22] Weisstein, Eric W. *Connected Domination Number*, MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/ConnectedDominationNumber.html

[23] S. N. Daoud and K. Mohamed, *The complexity of some families of cycle-related graphs*, J. Taibah Univ. Sci., 11(2)(2017), 205–228. https://doi.org/10.1016/j.jtusci.2016.04.002