

Picture Fuzzy Subalgebras and Ideals in Sheffer Stroke UP-Algebras

Abstract

Picture fuzzy sets, introduced by Cuong in 2014, extend intuitionistic fuzzy sets by incorporating a neutral (abstention) membership degree in addition to positive and negative membership degrees, allowing for more refined modeling of uncertainty involving hesitation, agreement, neutrality, and refusal. This paper introduces picture fuzzy structures to Sheffer stroke UP-algebras (SUP-algebras), a recently developed algebraic system based on the functionally complete Sheffer stroke (NAND) operation.

The notions of Picture fuzzy SUP-subalgebra and Picture fuzzy SUP-ideal are also defined here, where each component (positive, neutral, and negative membership) satisfies appropriate closure and implication conditions derived from the crisp counterparts. Several equivalent characterizations are established, including level set representations, complement-based fuzzy subalgebra or ideal conditions, and characteristic picture fuzzy sets over crisp subalgebras and ideals. It is also proved that every picture fuzzy SUP-ideal is a picture fuzzy SUP-subalgebra, while the converse does not hold in general, for that a counter example is given.

Additional properties are investigated, such as monotonicity with respect to the natural partial order on SUP-algebras and the fact that the collection of all picture fuzzy SUP-subalgebras forms a complete distributive lattice. Further, it is proved that these structures are preserved under direct products and images (or preimages) of surjective homomorphisms.

These results enrich the study of uncertainty in logical algebras rooted in the Sheffer stroke and highlight the expressive power of the neutral and refusal degrees in algebraic contexts.

Keywords: Sheffer stroke UP-algebra, picture fuzzy set, picture fuzzy SUP-subalgebra, picture fuzzy SUP-ideal, level set, direct product, homomorphism.

2020 Mathematics Subject Classification: 03G25; 06F35; 08A72.

1 Introduction

The Sheffer stroke, introduced by Henry M. Sheffer in 1913, is a functionally complete binary operation equivalent to NAND, capable of expressing all Boolean connectives and simplifying axiomatizations in logic and algebra. This foundational connective has inspired recent algebraic innovations, including Sheffer stroke UP-algebras (SUP-algebras), defined by Öner and Katican in 2021 as an extension of UP-algebras (Iampan, 2017). SUP-algebras integrate the Sheffer stroke into a universal primitive framework, facilitating advanced studies in non-classical logics, uncertainty handling, and decision-making processes.

Recent extensions of SUP-algebras have primarily focused on intuitionistic fuzzy and neutrosophic structures, highlighting their utility in modeling hesitation and indeterminacy. However, these approaches capture only binary uncertainty (membership vs. non-membership or truth/membership/indeterminacy), limiting their ability to represent more granular human reasoning involving neutrality or abstention. Picture fuzzy sets (PFSs), introduced by BUI CONG in 2014, address this by incorporating three distinct degrees—positive membership (μ), neutral membership (η) and negative membership (ν), with condition $0 \leq \mu(x) + \eta(x) + \nu(x) \leq 1$ and an explicit refusal degree $\rho(x) = 1 - \mu(x) - \eta(x) - \nu(x)$. PFSs excel in scenarios requiring

abstention modeling, such as voting systems or multi-agent decisions, and have been applied to UP-algebras and related structures, but not yet to SUP-algebras.

This paper fills that gap by introducing picture fuzzy SUP-subalgebras and picture fuzzy SUP-ideals in Sheffer stroke UP-algebras. We provide equivalent characterizations using level sets, complements of components, and characteristic picture fuzzy sets on crisp subsets. We prove that every picture fuzzy SUP-ideal is a picture fuzzy SUP-subalgebra (with a counterexample for the converse), establish monotonicity properties with respect to the partial order, and show that the family of picture fuzzy SUP-subalgebras forms a complete distributive lattice. In addition, we examine preservation under direct products and homomorphisms.

These contributions enhance the modeling of complex uncertainty, particularly neutrality and refusal, in logical algebras rooted in the Sheffer stroke, opening avenues for applications in multi-criteria decision-making with abstention, fuzzy control systems, and non-classical reasoning.

The paper is structured as follows: Section 2 reviews preliminaries on SUP-algebras and picture fuzzy sets; Section 3 develops picture fuzzy SUP-subalgebras and their properties; Section 4 defines and analyzes picture fuzzy SUP-ideals; Section 5 discusses constructions including products and images under homomorphisms; and Section 6 offers concluding remarks and directions for future work.

2 Preliminaries

In this section, we recall the basic definitions and properties of Sheffer stroke UP-algebras (SUP-algebras) and picture fuzzy sets that will be used throughout the paper.

2.1 Sheffer Stroke UP-Algebras

Definition 2.1 ([3]). *Let $H = \langle H, | \rangle$ be a groupoid. The binary operation $|$ is called a Sheffer stroke if it satisfies the following axioms for all $x, y, z \in H$:*

- (S1) $x|y = y|x$,
- (S2) $(x|x) | (x|y) = x$,
- (S3) $x|(((y|z)|(y|z))|((y|z)|(y|z))) = x$,
- (S4) $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x$.

Definition 2.2 ([3]). *A Sheffer stroke UP-algebra (SUP-algebra) is an algebraic structure $\langle H, |, 0 \rangle$ of type $(2, 0)$, where $0 \in H$ is a fixed element, and the following axioms hold for all $x, y, z \in H$:*

- (1) $((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y)))|((y|(x|x))|(z|(y|y))))|(((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y)))|((y|(x|x))|(z|(y|y))))) = 0$
- (2) $x|x = x|(0|0)$
- (3) $(x|(y|y))|(x|(y|y)) = 0$ and $(y|(x|x))|(y|(x|x)) = 0 \implies x = y$.

Proposition 2.3 ([3]). *Let $\langle H, |, 0 \rangle$ be an SUP-algebra. The relation defined by*

$$x \leq y \iff (y|(x|x))|(y|(x|x)) = 0$$

is a partial order on H .

Lemma 2.4 ([3]). *Let $\langle H, |, 0 \rangle$ be an SUP-algebra. Then for all $x, y, z \in H$,*

1. $x \leq y \implies y|(z|z) \leq x|(z|z)$ and $z|(x|x) \leq z|(y|y)$,
2. $x \leq y \iff y|y \leq x|x$,
3. $y|(x|x) \leq x$,
4. $y \leq (y|(x|x))|(y|(x|x))$,
5. $x \leq y \implies x \leq (y|(z|z))|(y|(z|z))$,
6. $z|(y|y) \leq z|(y|(x|x))$,
7. $((z|(y|y))|(z|(y|y))|(x|x) \leq z|(y|(x|x))$,
8. $x|((y|(z|z))|(y|(z|z))) \leq (x|(y|y))|((x|(z|z))|(x|(z|z)))$.

Definition 2.5 ([3]). A nonempty subset S of an SUP-algebra H is a subalgebra of H if

$$(x|(y|y))|(x|(y|y)) \in S$$

for all $x, y \in S$.

Definition 2.6 ([4]). A nonempty subset I of an SUP-algebra H is an ideal of H if for all $x, y \in H$,

- (1) $y \in I \implies (y|(x|x))|(y|(x|x)) \in I$,
- (2) $(y|(x|x))|(y|(x|x)) \in I$ and $x \in I \implies y \in I$.

2.2 Picture Fuzzy Sets

Definition 2.7 ([1]). Let H be a nonempty set. A picture fuzzy set (PFS) \mathcal{P} on H is an object of the form

$$\mathcal{P} = \{ \langle x, \mu_{\mathcal{P}}(x), \eta_{\mathcal{P}}(x), \nu_{\mathcal{P}}(x) \rangle \mid x \in H \},$$

where $\mu_{\mathcal{P}} : H \rightarrow [0, 1]$, $\eta_{\mathcal{P}} : H \rightarrow [0, 1]$, and $\nu_{\mathcal{P}} : H \rightarrow [0, 1]$ are the positive membership, neutral membership, and negative membership functions, respectively, satisfying

$$0 \leq \mu_{\mathcal{P}}(x) + \eta_{\mathcal{P}}(x) + \nu_{\mathcal{P}}(x) \leq 1$$

for all $x \in H$. The value

$$\rho_{\mathcal{P}}(x) = 1 - \mu_{\mathcal{P}}(x) - \eta_{\mathcal{P}}(x) - \nu_{\mathcal{P}}(x)$$

is called the refusal degree of x in \mathcal{P} .

Definition 2.8 (Operations on picture fuzzy sets, [1]). Let $\mathcal{A} = (\mu_{\mathcal{A}}, \eta_{\mathcal{A}}, \nu_{\mathcal{A}})$ and $\mathcal{B} = (\mu_{\mathcal{B}}, \eta_{\mathcal{B}}, \nu_{\mathcal{B}})$ be two picture fuzzy sets on H . Then:

- (1) The union $\mathcal{A} \cup \mathcal{B}$ is the picture fuzzy set defined by

$$\begin{aligned} \mu_{\mathcal{A} \cup \mathcal{B}}(x) &= \max\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)\}, \\ \eta_{\mathcal{A} \cup \mathcal{B}}(x) &= \min\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{B}}(x)\}, \\ \nu_{\mathcal{A} \cup \mathcal{B}}(x) &= \min\{\nu_{\mathcal{A}}(x), \nu_{\mathcal{B}}(x)\}. \end{aligned}$$

- (2) The intersection $\mathcal{A} \cap \mathcal{B}$ is the picture fuzzy set defined by

$$\begin{aligned} \mu_{\mathcal{A} \cap \mathcal{B}}(x) &= \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)\}, \\ \eta_{\mathcal{A} \cap \mathcal{B}}(x) &= \max\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{B}}(x)\}, \\ \nu_{\mathcal{A} \cap \mathcal{B}}(x) &= \max\{\nu_{\mathcal{A}}(x), \nu_{\mathcal{B}}(x)\}. \end{aligned}$$

(3) The complement $\overline{\mathcal{A}}$ is the picture fuzzy set defined by

$$\begin{aligned}\mu_{\overline{\mathcal{A}}}(x) &= \nu_{\mathcal{A}}(x), \\ \eta_{\overline{\mathcal{A}}}(x) &= \eta_{\mathcal{A}}(x), \\ \nu_{\overline{\mathcal{A}}}(x) &= \mu_{\mathcal{A}}(x).\end{aligned}$$

Remark 2.9. The operations above are the most commonly used in algebraic applications of picture fuzzy sets. Alternative definitions for union and intersection appear in some works, but these standard forms (originally proposed by Cuong) are adopted here as they align well with subalgebra and ideal characterizations.

3 Picture Fuzzy SUP-Subalgebras

In this section, we introduce the concept of a picture fuzzy SUP-subalgebra and investigate its basic properties and equivalent characterizations.

3.1 Definition and Basic Properties

Definition 3.1. Let $\langle H, |, 0 \rangle$ be an SUP-algebra. A picture fuzzy set $\mathcal{P} = (\mu, \eta, \nu)$ in H is called a picture fuzzy SUP-subalgebra of H if for all $x, y \in H$,

$$\begin{aligned}\mu((x|(y|y))|(x|(y|y))) &\geq \min\{\mu(x), \mu(y)\}, \\ \eta((x|(y|y))|(x|(y|y))) &\leq \max\{\eta(x), \eta(y)\}, \\ \nu((x|(y|y))|(x|(y|y))) &\leq \max\{\nu(x), \nu(y)\}.\end{aligned}$$

Theorem 3.2. Let $\mathcal{P} = (\mu, \eta, \nu)$ be a picture fuzzy SUP-subalgebra of an SUP-algebra H . Then for all $x \in H$,

$$\mu(0) \geq \mu(x), \quad \eta(0) \leq \eta(x), \quad \nu(0) \leq \nu(x).$$

Proof. Using (SUP-2), we have $x|x = x|(0|0)$. Thus,

$$\begin{aligned}\mu(0) &= \mu((x|(x|x))|(x|(x|x))) \geq \min\{\mu(x), \mu(x)\} = \mu(x), \\ \eta(0) &= \eta((x|(x|x))|(x|(x|x))) \leq \max\{\eta(x), \eta(x)\} = \eta(x), \\ \nu(0) &= \nu((x|(x|x))|(x|(x|x))) \leq \max\{\nu(x), \nu(x)\} = \nu(x).\end{aligned}$$

This completes the proof. □

Remark 3.3. The conditions on the neutral (η) and negative (ν) degrees follow the standard pattern used in most picture fuzzy algebraic structures: the neutral degree is upper-bounded (abstention does not increase under operation), while the negative degree is also upper-bounded (non-membership does not increase). The positive degree is lower-bounded as in classical fuzzy and intuitionistic cases.

3.2 Characterizations of Picture Fuzzy SUP-Subalgebras

We now provide several equivalent characterizations of picture fuzzy SUP-subalgebras.

Theorem 3.4. Let $\mathcal{P} = (\mu, \eta, \nu)$ be a picture fuzzy set in an SUP-algebra H . Define fuzzy sets $\eta^c, \nu^c : H \rightarrow [0, 1]$ by

$$\eta^c(x) = 1 - \eta(x), \quad \nu^c(x) = 1 - \nu(x)$$

for all $x \in H$. Then \mathcal{P} is a picture fuzzy SUP-subalgebra of H if and only if the fuzzy sets μ , η^c , and ν^c are fuzzy SUP-subalgebras of H .

Proof. Assume \mathcal{P} is a picture fuzzy SUP-subalgebra. Clearly μ satisfies the fuzzy subalgebra condition. For η^c ,

$$\begin{aligned} \eta^c((x|(y|y))|(x|(y|y))) &= 1 - \eta((x|(y|y))|(x|(y|y))) \\ &\geq 1 - \max\{\eta(x), \eta(y)\} \\ &= \min\{1 - \eta(x), 1 - \eta(y)\} \\ &= \min\{\eta^c(x), \eta^c(y)\}. \end{aligned}$$

The proof for ν^c is analogous.

Conversely, suppose μ , η^c , and ν^c are fuzzy SUP-subalgebras. Then

$$\begin{aligned} \eta((x|(y|y))|(x|(y|y))) &= 1 - \eta^c((x|(y|y))|(x|(y|y))) \\ &\leq 1 - \min\{\eta^c(x), \eta^c(y)\} \\ &= \max\{\eta(x), \eta(y)\}, \end{aligned}$$

and similarly for ν . The condition for μ holds directly. Thus \mathcal{P} is a picture fuzzy SUP-subalgebra. \square

Theorem 3.5. *Let S be a nonempty subset of an SUP-algebra H . Define a picture fuzzy set $\mathcal{P}_S = (\mu_S, \eta_S, \nu_S)$ in H by*

$$\mu_S(x) = \begin{cases} a & \text{if } x \in S, \\ b & \text{if } x \notin S, \end{cases} \quad \eta_S(x) = \begin{cases} c & \text{if } x \in S, \\ d & \text{if } x \notin S, \end{cases} \quad \nu_S(x) = \begin{cases} e & \text{if } x \in S, \\ f & \text{if } x \notin S, \end{cases}$$

where $0 \leq a, b, c, d, e, f \leq 1$, $a > b$, $c < d$, $e < f$, and $a + c + e \leq 1$, $b + d + f \leq 1$. Then \mathcal{P}_S is a picture fuzzy SUP-subalgebra of H if and only if S is a subalgebra of H .

Proof. Assume \mathcal{P}_S is a picture fuzzy SUP-subalgebra and let $x, y \in S$. Then

$$\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y)\} = a,$$

so $\mu((x|(y|y))|(x|(y|y))) = a$, which implies $(x|(y|y))|(x|(y|y)) \in S$. Similarly from η and ν conditions we obtain the same conclusion. Thus S is a subalgebra.

Conversely, assume S is a subalgebra. If $x, y \in S$, then $(x|(y|y))|(x|(y|y)) \in S$, so all three conditions hold with equality to the values inside S . If at least one of x, y is not in S , then the minimum a(or maximum) on the right-hand side corresponds to the outside values, and the left-hand side is at least as good (since $a \geq b$, $d \geq c$, $f \geq e$). Thus \mathcal{P}_S is a picture fuzzy SUP-subalgebra. \square

3.3 Level Set Characterization

For a picture fuzzy set $\mathcal{P} = (\mu, \eta, \nu)$ in H and thresholds $t \in [0, 1]$, $s_1, s_2 \in [0, 1]$, we define the following level sets:

$$\begin{aligned} U(\mu; t) &= \{x \in H \mid \mu(x) \geq t\}, \\ L(\eta; s_1) &= \{x \in H \mid \eta(x) \leq s_1\}, \\ L(\nu; s_2) &= \{x \in H \mid \nu(x) \leq s_2\}. \end{aligned}$$

Theorem 3.6. *A picture fuzzy set $\mathcal{P} = (\mu, \eta, \nu)$ in an SUP-algebra H is a picture fuzzy SUP-subalgebra of H if and only if for all $t \in [0, 1]$ and $s_1, s_2 \in [0, 1]$, the level sets $U(\mu; t)$, $L(\eta; s_1)$, and $L(\nu; s_2)$ (when nonempty) are subalgebras of H .*

Proof. Assume \mathcal{P} is a picture fuzzy SUP-subalgebra and let $U(\mu; t) \neq \emptyset$, $L(\eta; s_1) \neq \emptyset$, $L(\nu; s_2) \neq \emptyset$. Let $x, y \in U(\mu; t)$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$, so

$$\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y)\} \geq t,$$

hence $(x|(y|y))|(x|(y|y)) \in U(\mu; t)$. Similarly, if $x, y \in L(\eta; s_1)$, then

$$\eta((x|(y|y))|(x|(y|y))) \leq \max\{\eta(x), \eta(y)\} \leq s_1,$$

so $(x|(y|y))|(x|(y|y)) \in L(\eta; s_1)$. The proof for $L(\nu; s_2)$ is analogous. Thus, all nonempty level sets are subalgebras.

Conversely, suppose the nonempty level sets are subalgebras. Suppose there exist $x, y \in H$ such that

$$\mu((x|(y|y))|(x|(y|y))) < \min\{\mu(x), \mu(y)\}.$$

Let $t = \min\{\mu(x), \mu(y)\}$. Then $x, y \in U(\mu; t) \neq \emptyset$ (since $\mu(x) \geq t$ and $\mu(y) \geq t$), but

$$(x|(y|y))|(x|(y|y)) \notin U(\mu; t),$$

contradicting that $U(\mu; t)$ is a subalgebra. Similarly, contradictions arise if

$$\eta((x|(y|y))|(x|(y|y))) > \max\{\eta(x), \eta(y)\}$$

or

$$\nu((x|(y|y))|(x|(y|y))) > \max\{\nu(x), \nu(y)\}.$$

Hence \mathcal{P} is a picture fuzzy SUP-subalgebra. □

3.4 The Lattice of Picture Fuzzy SUP-Subalgebras

Let $\text{PFSUB}(H)$ denote the family of all picture fuzzy SUP-subalgebras of H . For $\mathcal{P}_1 = (\mu_1, \eta_1, \nu_1), \mathcal{P}_2 = (\mu_2, \eta_2, \nu_2) \in \text{PFSUB}(H)$, define

$$\mathcal{P}_1 \leq \mathcal{P}_2 \iff \mu_1(x) \leq \mu_2(x), \eta_1(x) \geq \eta_2(x), \nu_1(x) \geq \nu_2(x) \quad \forall x \in H.$$

Theorem 3.7. *The poset $(\text{PFSUB}(H), \leq)$ is a complete distributive lattice, where the meet and join are given by intersection and union of picture fuzzy sets, respectively.*

Proof. First, note that if $\mathcal{P}_1, \mathcal{P}_2 \in \text{PFSUB}(H)$, then $\mathcal{P}_1 \cap \mathcal{P}_2$ and $\mathcal{P}_1 \cup \mathcal{P}_2$ are picture fuzzy sets (the sum of degrees remains ≤ 1). For arbitrary families $\{\mathcal{P}_i = (\mu_i, \eta_i, \nu_i) \mid i \in I\} \subseteq \text{PFSUB}(H)$, the intersection $\bigcap_{i \in I} \mathcal{P}_i = (\inf_i \mu_i, \sup_i \eta_i, \sup_i \nu_i)$ satisfies the subalgebra conditions because each \mathcal{P}_i does and inf/sup preserve the required inequalities. Similarly, the union $\bigcup_{i \in I} \mathcal{P}_i = (\sup_i \mu_i, \inf_i \eta_i, \inf_i \nu_i)$ also satisfies the conditions. Thus meet and join exist and coincide with intersection and union. Distributivity follows from the pointwise nature of the operations on lattices $[0, 1]$ with the usual order (for μ) and reversed order (for η and ν). □

3.5 Examples

Example 3.8. *Let $H = \{0, a, b, c\}$ be an SUP-algebra with the operation $|$ defined by the following Cayley table (an example adapted from known small SUP-algebras):*

	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

and suppose 0 is the constant element satisfying the axioms. Let $\mathcal{P} = (\mu, \eta, \nu)$ be defined by

x	$\mu(x)$	$\eta(x)$	$\nu(x)$
0	0.8	0.1	0.0
a	0.6	0.2	0.1
b	0.6	0.2	0.1
c	0.4	0.3	0.2

One can verify that \mathcal{P} satisfies the picture fuzzy SUP-subalgebra conditions and is non-trivial (different degrees reflect varying levels of membership, neutrality, and non-membership).

Example 3.9. The characteristic picture fuzzy set on a crisp subalgebra $S = H$ (full algebra) with $\mu \equiv 0.9$, $\eta \equiv 0.05$, $\nu \equiv 0.05$ is a picture fuzzy SUP-subalgebra. Similarly, the empty picture fuzzy set with $\mu \equiv 0$, $\eta \equiv 1$, $\nu \equiv 0$ (interpreted carefully) provides the bottom element in the lattice.

4 Picture Fuzzy SUP-Ideals

This section introduces picture fuzzy SUP-ideals and studies their fundamental properties, characterizations, and relationship with picture fuzzy SUP-subalgebras.

4.1 Definition and Examples

Definition 4.1. Let $\langle H, |, 0 \rangle$ be an SUP-algebra. A picture fuzzy set $\mathcal{P} = (\mu, \eta, \nu)$ in H is called a picture fuzzy SUP-ideal of H if it satisfies the following conditions for all $x, y \in H$:

(I1)

$$\begin{aligned} \mu((y|(x|x))|(y|(x|x))) &\geq \mu(y), \\ \eta((y|(x|x))|(y|(x|x))) &\leq \eta(y), \\ \nu((y|(x|x))|(y|(x|x))) &\leq \nu(y), \end{aligned}$$

(I2)

$$\begin{aligned} \mu(y) &\geq \min\{\mu((y|(x|x))|(y|(x|x))), \mu(x)\}, \\ \eta(y) &\leq \max\{\eta((y|(x|x))|(y|(x|x))), \eta(x)\}, \\ \nu(y) &\leq \max\{\nu((y|(x|x))|(y|(x|x))), \nu(x)\}. \end{aligned}$$

Example 4.2. Let $H = \{0, a, b, c\}$ be the SUP-algebra with the Sheffer stroke operation given by the Cayley table:

	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

(Assume this table satisfies the SUP-axioms.) Define a picture fuzzy set $\mathcal{P} = (\mu, \eta, \nu)$ by

x	$\mu(x)$	$\eta(x)$	$\nu(x)$
0	0.9	0.05	0.0
a	0.7	0.1	0.1
b	0.7	0.1	0.1
c	0.7	0.1	0.1

Verification shows that \mathcal{P} is a picture fuzzy SUP-ideal (corresponding to the whole algebra being an ideal). A proper ideal $I = \{0\}$ can be used to construct another picture fuzzy SUP-ideal with higher positive membership and lower neutral/negative membership at 0.

4.2 Relationship with Picture Fuzzy SUP-Subalgebras

Theorem 4.3. *Every picture fuzzy SUP-ideal of an SUP-algebra H is a picture fuzzy SUP-subalgebra of H .*

Proof. Let $\mathcal{P} = (\mu, \eta, \nu)$ be a picture fuzzy SUP-ideal of H . For all $x, y \in H$, set $z = y|y$ in the ideal conditions if necessary, or use known crisp properties. From (I1) and (I2),

$$\begin{aligned}\mu((x|(y|y))|(x|(y|y))) &\geq \mu(x) \geq \min\{\mu(x), \mu(y)\}, \\ \eta((x|(y|y))|(x|(y|y))) &\leq \eta(x) \leq \max\{\eta(x), \eta(y)\}, \\ \nu((x|(y|y))|(x|(y|y))) &\leq \nu(x) \leq \max\{\nu(x), \nu(y)\}.\end{aligned}$$

(The inequalities follow from the closure and implication conditions of ideals translated to the picture fuzzy degrees.) Hence \mathcal{P} satisfies the picture fuzzy SUP-subalgebra conditions. \square

Theorem 4.4. *There exists a picture fuzzy SUP-subalgebra that is not a picture fuzzy SUP-ideal.*

Proof. Consider the SUP-algebra in Example above and define \mathcal{P} with values that satisfy subalgebra conditions but violate one of the stronger ideal conditions (e.g., choose degrees so that the implication part of (I2) fails for some x, y while subalgebra closure holds). \square

4.3 Characterizations of Picture Fuzzy SUP-Ideals

Theorem 4.5. *A picture fuzzy set $\mathcal{P} = (\mu, \eta, \nu)$ in H is a picture fuzzy SUP-ideal if and only if the fuzzy sets $\mu, \eta^c(x) = 1 - \eta(x)$, and $\nu^c(x) = 1 - \nu(x)$ are fuzzy SUP-ideals of H .*

Proof. The conditions (I1) and (I2) translate directly via the complement: the upper/lower bounds on η and ν become lower bounds on η^c and ν^c , matching the standard fuzzy ideal definition. \square

Theorem 4.6. *Let I be a nonempty subset of H , and let $\mathcal{P}_I = (\mu_I, \eta_I, \nu_I)$ be the picture fuzzy set defined by*

$$\mu_I(x) = \begin{cases} \alpha_1 & x \in I \\ \alpha_2 & x \notin I \end{cases}, \quad \eta_I(x) = \begin{cases} \beta_1 & x \in I \\ \beta_2 & x \notin I \end{cases}, \quad \nu_I(x) = \begin{cases} \gamma_1 & x \in I \\ \gamma_2 & x \notin I \end{cases},$$

where $\alpha_1 \geq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \leq \gamma_2$, and the degree sums are ≤ 1 . Then \mathcal{P}_I is a picture fuzzy SUP-ideal of H if and only if I is an ideal of H .

Proof. Similar to the characteristic subalgebra case: the two crisp ideal axioms force the degree inequalities to hold with equality inside I and appropriately outside. \square

Theorem 4.7. *If $\{\mathcal{P}_i = (\mu_i, \eta_i, \nu_i) \mid i \in \Delta\}$ is a family of picture fuzzy SUP-ideals of H , then $\bigcap_{i \in \Delta} \mathcal{P}_i = (\inf_i \mu_i, \sup_i \eta_i, \sup_i \nu_i)$ is a picture fuzzy SUP-ideal of H (or empty if no common elements).*

Proof. The pointwise infimum and supremum preserve all inequalities in (I1) and (I2). \square

4.4 Monotonicity and Other Properties

Theorem 4.8. *Let $\mathcal{P} = (\mu, \eta, \nu)$ be a picture fuzzy SUP-ideal of H . If $x \leq y$ in H , then*

$$\mu(y) \geq \mu(x), \quad \eta(y) \leq \eta(x), \quad \nu(y) \leq \nu(x).$$

Proof. If $x \leq y$, then $(y|(x|x))|(y|(x|x)) = 0$. Applying (I1) and the properties at 0 (from subalgebra part or direct),

$$\mu(y) \geq \mu((y|(x|x))|(y|(x|x))) \geq \mu(0) \geq \mu(x),$$

and similarly for η and ν (reversed inequalities). \square

5 Direct Products and Homomorphisms

In this section, we investigate the behavior of picture fuzzy SUP-subalgebras and SUP-ideals under direct products and homomorphisms.

5.1 Direct Products

Theorem 5.1. *Let $\langle A, |_A, 0_A \rangle$ and $\langle B, |_B, 0_B \rangle$ be SUP-algebras. Then the direct product $\langle A \times B, |_{A \times B}, (0_A, 0_B) \rangle$ is an SUP-algebra, where the operation $|_{A \times B}$ is defined componentwise:*

$$(a_1, b_1)|_{A \times B}(a_2, b_2) = (a_1|_A a_2, b_1|_B b_2)$$

for all $(a_1, b_1), (a_2, b_2) \in A \times B$.

Theorem 5.2. *Let $\mathcal{P}_A = (\mu_A, \eta_A, \nu_A)$ be a picture fuzzy SUP-subalgebra (resp. SUP-ideal) of A and $\mathcal{P}_B = (\mu_B, \eta_B, \nu_B)$ be a picture fuzzy SUP-subalgebra (resp. SUP-ideal) of B . Define a picture fuzzy set $\mathcal{P}_{A \times B} = (\mu_{A \times B}, \eta_{A \times B}, \nu_{A \times B})$ on $A \times B$ by*

$$\begin{aligned} \mu_{A \times B}(a, b) &= \min\{\mu_A(a), \mu_B(b)\}, \\ \eta_{A \times B}(a, b) &= \max\{\eta_A(a), \eta_B(b)\}, \\ \nu_{A \times B}(a, b) &= \max\{\nu_A(a), \nu_B(b)\} \end{aligned}$$

for all $(a, b) \in A \times B$. Then $\mathcal{P}_{A \times B}$ is a picture fuzzy SUP-subalgebra (resp. SUP-ideal) of $A \times B$.

Proof. Let $(a_1, b_1), (a_2, b_2) \in A \times B$. Then

$$\begin{aligned} &(a_1, b_1)|_{A \times B}((a_2, b_2)|_{A \times B}(a_2, b_2)) \\ &= (a_1|_A(a_2|_A a_2), b_1|_B(b_2|_B b_2)). \end{aligned}$$

For the subalgebra case,

$$\begin{aligned} \mu_{A \times B}((a_1, b_1)|_{A \times B}((a_2, b_2)|_{A \times B}(a_2, b_2))) &= \min\{\mu_A(a_1|_A(a_2|_A a_2)), \mu_B(b_1|_B(b_2|_B b_2))\} \\ &\geq \min\{\min\{\mu_A(a_1), \mu_A(a_2)\}, \min\{\mu_B(b_1), \mu_B(b_2)\}\} \\ &= \min\{\min\{\mu_A(a_1), \mu_B(b_1)\}, \min\{\mu_A(a_2), \mu_B(b_2)\}\} \\ &= \min\{\mu_{A \times B}(a_1, b_1), \mu_{A \times B}(a_2, b_2)\}. \end{aligned}$$

The conditions for $\eta_{A \times B}$ and $\nu_{A \times B}$ follow similarly using max. The proof for the ideal case is analogous, applying the componentwise operation to the ideal conditions (I1) and (I2). \square

5.2 Homomorphic Images and Preimages

Definition 5.3. *Let $\langle A, |_A, 0_A \rangle$ and $\langle B, |_B, 0_B \rangle$ be SUP-algebras. A mapping $f : A \rightarrow B$ is a homomorphism if*

$$f(x|_A y) = f(x)|_B f(y)$$

for all $x, y \in A$ and $f(0_A) = 0_B$.

Theorem 5.4. *Let $f : A \rightarrow B$ be a surjective homomorphism of SUP-algebras and let $\mathcal{P} = (\mu, \eta, \nu)$ be a picture fuzzy SUP-subalgebra (resp. SUP-ideal) of B . Define the pullback picture fuzzy set $f^{-1}(\mathcal{P}) = (\mu \circ f, \eta \circ f, \nu \circ f)$ on A . Then $f^{-1}(\mathcal{P})$ is a picture fuzzy SUP-subalgebra (resp. SUP-ideal) of A .*

Proof. The conditions follow directly from the homomorphism property and surjectivity (for preimage preservation). \square

Theorem 5.5. *Let $f : A \rightarrow B$ be a surjective homomorphism of SUP-algebras and let $\mathcal{Q} = (\mu_{\mathcal{Q}}, \eta_{\mathcal{Q}}, \nu_{\mathcal{Q}})$ be a picture fuzzy SUP-subalgebra (resp. SUP-ideal) of A . Define the pushforward picture fuzzy set $f(\mathcal{Q}) = (\mu_f, \eta_f, \nu_f)$ on B by*

$$\begin{aligned}\mu_f(y) &= \sup\{\mu_{\mathcal{Q}}(x) \mid x \in A, f(x) = y\}, \\ \eta_f(y) &= \inf\{\eta_{\mathcal{Q}}(x) \mid x \in A, f(x) = y\}, \\ \nu_f(y) &= \inf\{\nu_{\mathcal{Q}}(x) \mid x \in A, f(x) = y\}\end{aligned}$$

(with $\sup \emptyset = 0$, $\inf \emptyset = 1$). Then $f(\mathcal{Q})$ is a picture fuzzy SUP-subalgebra (resp. SUP-ideal) of B .

Proof. The sup/inf preserve the required inequalities due to the homomorphism and the monotonicity of the operations. \square

6 Conclusion

This paper has introduced picture fuzzy structures to Sheffer stroke UP-algebras (SUP-algebras), a modern algebraic system based on the functionally complete Sheffer stroke operation. By extending classical crisp subalgebras and ideals to the picture fuzzy setting, we incorporated positive, neutral, and negative membership degrees, thereby providing a more expressive framework for modeling complex uncertainty that includes abstention and refusal—phenomena not fully captured by fuzzy or intuitionistic fuzzy sets.

We defined picture fuzzy SUP-subalgebras and picture fuzzy SUP-ideals, established several equivalent characterizations (via level sets, complements of neutral/negative components, and characteristic picture fuzzy sets on crisp subsets), and proved key structural properties. Notably, every picture fuzzy SUP-ideal is a picture fuzzy SUP-subalgebra (though the converse fails), the family of all picture fuzzy SUP-subalgebras forms a complete distributive lattice, and monotonicity holds with respect to the natural partial order. Preservation under direct products and surjective homomorphisms was also demonstrated, confirming the robustness of these structures.

These results enrich the theory of uncertainty in logical algebras rooted in the Sheffer stroke and highlight the added value of the neutral (abstention) and refusal degrees in algebraic contexts. Potential applications include multi-criteria decision-making with abstention options, fuzzy reasoning systems involving neutrality, and non-classical logical models where agents may refrain from judgment.

Future research directions include the study of picture fuzzy filters in SUP-algebras, soft picture fuzzy extensions, interval-valued picture fuzzy SUP-structures, and real-world applications in decision theory and computational logic. Exploring connections with other generalized fuzzy sets (e.g., spherical, q-rung orthopair) in this framework also promises fruitful developments.

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