Original Research Article

On the Graph-Theoretic Properties of Compressed Zero Divisor Graphs of Product Rings over $\prod_{k=1}^{n} \mathbb{Z}_{p_k}$

Abstract

This article explores the structural and graph-theoretic properties of the compressed zero-divisor graph of product rings $R = \prod_{k=1}^{n} \mathbb{Z}_{p_k}$ where each p_k is a prime and $n \geq 2$. The study examines several key invariants of $\Gamma_C(R)$, including eccentricity, radius, diameter, girth, chromatic number, chromatic index, and clique number. It is shown that for $n \geq 3$, $\Gamma_C(R)$ has radius 2, diameter 3, and contains cycles c_l of every possible length l satisfying $3 \leq l \leq n$. Moreover, it is proved that $\omega(\Gamma_C(R)) = \chi(\Gamma_C(R)) = n$, and the chromatic index follows $\chi(\Gamma_C(R))$ = $2^{n-1} - 1$. These results provide deeper insights into the structural behaviour of compressed zero-divisor graphs associated with product rings over finite fields.

Keywords: Product Ring; Annihilator; Compressed Zero-Divisor Graph; Girth; Clique Number; Chromatic Number.

2010 Mathematics Subject Classification: 53C25; 83C05; 57N16

1 Introduction

The concept of zero-divisor graphs for commutative rings was first introduced by Beck [1,7] in 1988. In his work, Beck introduced properties such as the chromatic number χ and clique number ω of zero-divisor graphs. Subsequently, Anderson and Livingston [2] revised Beck's approach. Various authors studies have been explored in recent years. A zero-divisor graph of a ring is denoted by $\Gamma(R)$. The vertices of $\Gamma(R)$ are the nonzero zero-divisors of R and the edge is established between two distinct vertices a and b if and only if ab = 0. The Zero-divisor graphs were studied for their ability to relate algebraic properties of commutative rings and graph-theoretic properties. Many authors [1,7,8,9,10,11,12,13,14,15] explored the relationship between the ring-theoretical properties and graph-theoretic properties by eccentricity, radius, diameter, and girth. The chromatic number χ [1,7,8,9,14] represents the complexity of vertex coloring in zero-divisor graphs.

Throughout the paper, R shall denote

$$R = \prod_{k=1}^{n} \mathbb{Z}_{p_k},$$

where each p_k is a prime and $n \geq 2$, unless otherwise stated. The concept of a compressed zero-divisor graph was first introduced by S. Spiroff and C. Wickham, providing a refined graphical framework that captures the structure of zero-divisors in a more compact and computationally efficient form compared to the classical zero-divisor graph [17,18]. The compressed zero-divisor graph, denoted by $\Gamma_C(R)$, offers several advantages compared to the zero-divisor graph $\Gamma(R)$. Notably, in many instances, $\Gamma_C(R)$ remains finite even in cases where the corresponding $\Gamma(R)$ is infinite, making it a more efficient and compact representation of the zero-divisor structure of a ring.
The compressed zero-divisor graph, denoted $\Gamma_C(R)$, retains the distance relationships among zero-divisors in R. Hence, to determine the distance between any two zero-divisors $x, y \in R$, it is sufficient to evaluate the distance $d([x],[y])$ between their corresponding vertices in $\Gamma_C(R)$. This approach considerably simplifies computations, as the number of vertices in $\Gamma_C(R)$ is significantly smaller than the total number of zero-divisors in R. Consequently,

$$d([x],[y]) = d(x,y), \quad \mathrm{Rad}(\Gamma_C(R)) = \mathrm{Rad}(\Gamma(R)), \quad \mathrm{diam}(\Gamma_C(R)) = \mathrm{diam}(\Gamma(R)).$$

For example, consider the following graphs:

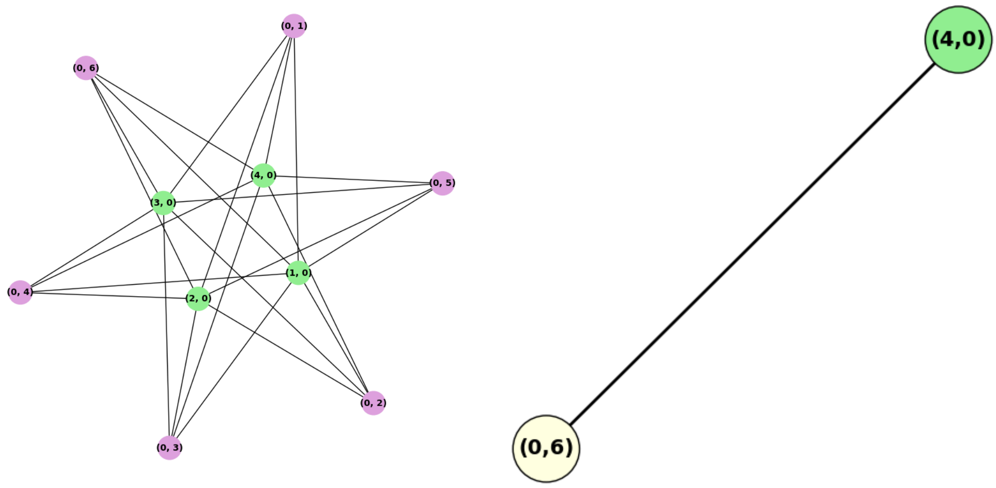

Figure 1: $\Gamma(\mathbb{Z}_5 \times \mathbb{Z}_7)$ and $\Gamma_C(\mathbb{Z}_5 \times \mathbb{Z}_7)$

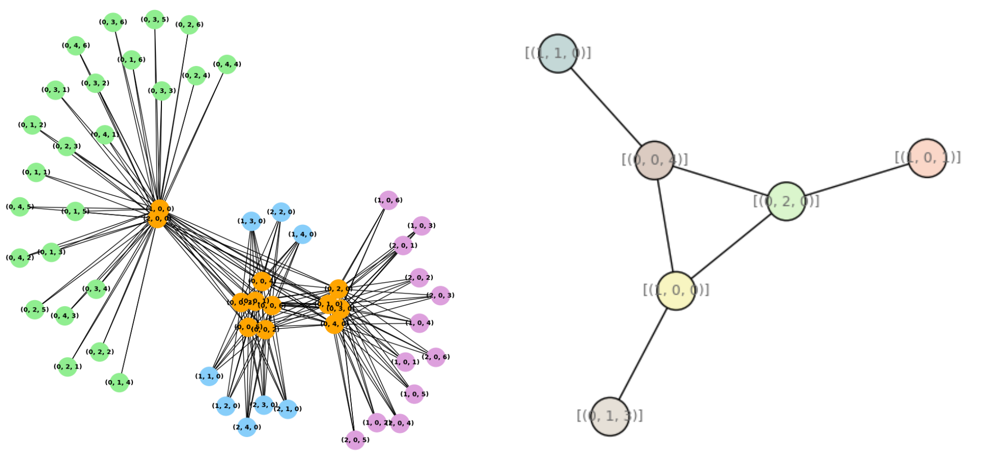

Figure 2: $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7)$ and $\Gamma_C(\mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7)$

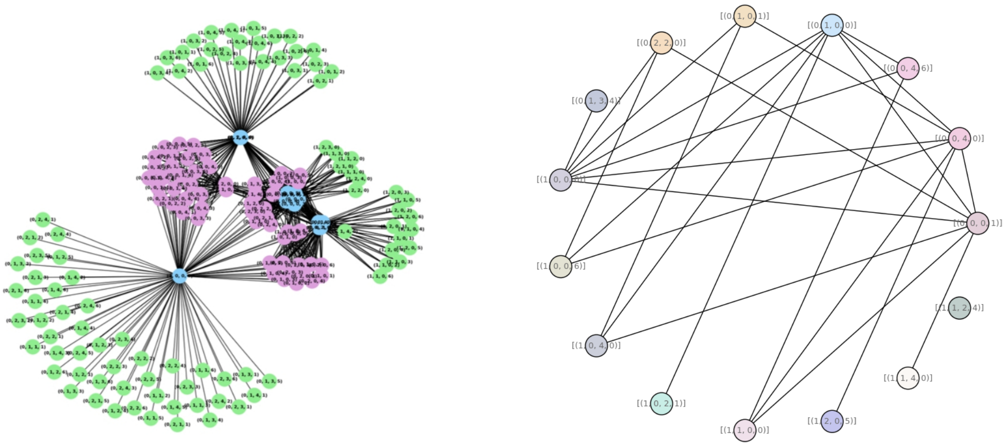

Figure 3: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7)$ and $\Gamma_c(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7)$

2 Preliminaries

2.1 Compressed Zero-Divisor Graph

Let R be a commutative ring with unity. The compressed zero-divisor graph of R $\Gamma_C(R)$ in which each vertex corresponds to an equivalence class of nonzero zero-divisors of R. Two elements $x, y \in R$ - $\{0\}$ belong to the same class if and only if $\mathrm{Ann}(x) = \mathrm{Ann}(y)$. $\mathrm{Ann}(x)$ = { $r \in R$|rx = 0 } is the annihilator of x. Two distinct classes [x] = [y] Two distinct classes $[x]$ and $[y]$ are adjacent in $\Gamma_C(R)$ if and only if $xy = 0$. Thus $\Gamma_C(R)$ provides a reduced representation of the zero-divisor structure by merging elements with identical annihilators into single vertices.

Graph-Theoretic Terminology

The Let $\Gamma_C(R)$ be a finite simple connected compressed zero-divisor graph of R

2.2 Distance and Eccentricity

For any two vertices $u, v \in V(\Gamma_C)$ the distance between u and v, denoted by $d(u, v)$.is the length of the shortest path connecting them. The eccentricity of a vertex v is defined as

$ecc(v) = \max_{u \in V(\Gamma_C(R))} d(u, v)$.

2.3 Radius of the compressed zero-divisor graph

The radius of a graph Γ_C is the minimum eccentricity among all vertices in Γ_C, defined as

$$\mathrm{rad}(\Gamma_C) = \min_{v \in V(\Gamma_C)} \mathrm{ecc}(v),$$

where $\mathrm{ecc}(v)$ is the maximum distance from v to any other vertex.

2.4 Diameter of the compressed zero-divisor graph

The diameter of a graph Γ_C, denoted $\mathrm{diam}(\Gamma_C)$, is the largest eccentricity among all the vertices of the graph Γ_C and is defined as

$$\mathrm{diam}(\Gamma_C) = \max_{v \in V(\Gamma_C)} \mathrm{ecc}(v).$$

2.5 Girth of the compressed zero-divisor graph

The girth of a graph Γ_C is the length of the shortest cycle in Γ_C. If Γ_C contains no cycles, its girth is defined to be infinite.

2.6 Chromatic Number of the compressed zero-divisor graph

The chromatic number of a graph Γ_C, denoted $\chi(\Gamma_C)$, is the smallest number of colors needed to properly color the vertices of Γ_C such that no two adjacent vertices share the same color.

2.7 Chromatic Index of the compressed zero-divisor graph

The chromatic index (also called the edge chromatic number) of a graph Γ_C, denoted by $\chi'(\Gamma_C)$, is the smallest number of colors needed to color all edges of Γ_C in such a way that no two adjacent edges share the same color.

2.8 Clique and Clique Number of the compressed zero-divisor graph

A clique in the zero-divisor graph Γ_C is a complete subgraph of Γ_C. The order of the largest clique in Γ_C is its clique number. It is denoted by $\omega(\Gamma_C)$.

2.9 Relation between R and $\Gamma_C(R)$:

The compressed zero-divisor graph $\Gamma_C(R)$ preserves the distances between zero-divisors. Therefore, to compute the distance between any two zero-divisors $x, y \in R$, it suffices to compute the distance $d([x],[y])$ between their corresponding vertices in $\Gamma_C(R)$. This reduces the complexity significantly, since the number of vertices in $\Gamma_C(R)$ is much smaller than the total number of zero-divisors in R.

Therefore,

$$d([x],[y]) = d(x,y), \quad \text{Rad}(\Gamma_C(R)) = \text{Rad}(\Gamma(R)), \quad \text{diam}(\Gamma_C(R)) = \text{diam}(\Gamma(R)).$$

3 Main Results

3.1 Theorem 1

Let $R = \prod_{k=1}^{n} \mathbb{Z}_{p_k}$ where each p_k is prime and $n \geq 2$. Then the compressed zero-divisor graph $\Gamma_C(R)$ satisfies the following:

$$\mathrm{diam}(\Gamma_C(R)) = \begin{cases} 1, & n = 2, \\ 3, & n \geq 3, \end{cases} \qquad \mathrm{Rad}(\Gamma_C(R)) = \begin{cases} 1, & n = 2, \\ 2, & n \geq 3. \end{cases}$$

Proof. Consider the product ring

$$R = \prod_{k=1}^{n} \mathbb{Z}_{p_k}, \quad \text{with each } p_k \text{ prime and } n \geq 2.$$

For any zero-divisor $r \in R$, its annihilator is given by the set

$$\text{Ann}_R(r) = [r] = \{x \in R : xr = 0\}.$$

Hence, the vertex set of the compressed zero-divisor graph $\Gamma_C(R)$ is

$$V(\Gamma_C(R)) = \{(p_1 - 1, 0, 0, \ldots, 0),\ (0, p_2 - 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, p_n - 1)\}.$$

Each vertex of $\Gamma_C(R)$ corresponds to a nonempty proper subset $U \subseteq \text{Ann}_R(r)$, representing the indices of nonzero coordinates of a nonzero zero-divisor. Two vertices U and V are adjacent if and only if they are disjoint, that is,

$$U \cap V = \emptyset.$$

□

Case if n=2,

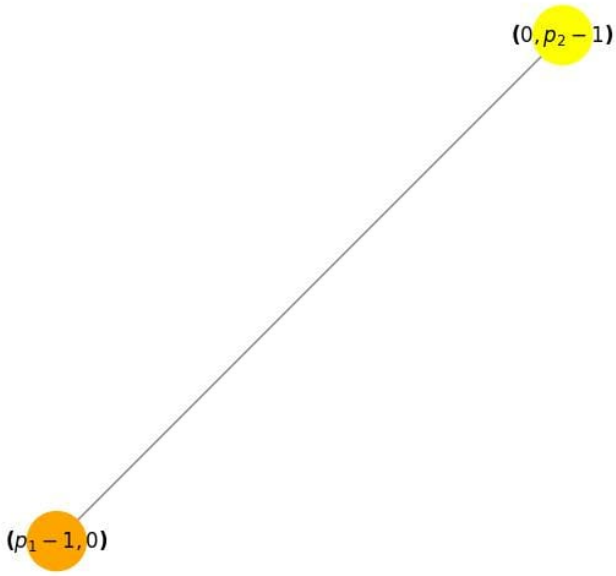

Figure 4: $\Gamma_C(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$

In the product ring

$$R = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}, \quad \text{where } p_1, p_2 \text{ are primes,}$$

the only nonempty proper subsets of $\text{Ann}_R(r)$ are $(p_1 - 1, 0)$ and $(0, p_2 - 1)$, which are disjoint.The graph $\Gamma_C(R)$ consists of exactly two distinct vertices corresponding to the two classes of annihilators of nonzero zero-divisors in R. Hence, $\Gamma_C(R)$ is isomorphic to K_2, and therefore $\Gamma_C(R)$ is

$$\text{Rad}(\Gamma_C(R)) = \text{diam}(\Gamma_C(R)) = 1.$$

Case if n ≥ 3,

The graph $\Gamma_C(R)$ is connected since every vertex U is adjacent to some

$$\{i\} = (0, 0, 0, \ldots, p_i - 1, \ldots, 0) \quad \text{with } i \notin U.$$

Consider arbitrary vertices U and V:

- If $U \cap V = \emptyset$, then $d(U, V) = 1$.
- If $U \cap V \neq \emptyset$ but $U \cup V \neq \text{Ann}_R(r)$, choose any index $j \notin U \cup V$. Then the vertex

$$\{j\} = (0, 0, 0, \ldots, p_j - 1, \ldots, 0)$$

is adjacent to both U and V, so $d(U, V) \leq 2$.

If $U \cap V \neq \emptyset$ and $U \cup V = \text{Ann}_R(r)$, choose the vertices $x \in U - V$ and $y \in V - U$. Then the path

$$U - \{y\} - \{x\} - V$$

has length 3, giving $d(U, V) \leq 3$. Then

$$\{j\} = (0, 0, 0, \ldots, p_j - 1, \ldots, 0)$$

is adjacent to both U and V, In this situation, there is no shorter path, so there exist pairs of vertices at distance exactly 3.

Hence,

$$\text{diam}(\Gamma_C(R)) \leq 3.$$

This implies that

$$\text{diam}(\Gamma_C(R)) = \begin{cases} 1, & n = 2 \\ 3, & n \geq 3 \end{cases}.$$

We observe that

$$[\text{ecc}(p_1 - 1, 0, 0, \ldots, 0)] = [\text{ecc}(0, p_2 - 1, 0, \ldots, 0)] = \cdots = [\text{ecc}(0, 0, 0, \ldots, p_n - 1)] = 2.$$

On the other hand, vertices of the form

$$[(p_1 - 1, p_2 - 1, 0, \ldots, 0)],\ [(0, p_2 - 1, p_3 - 1, 0, \ldots, 0]), \ldots, [(0, p_2 - 1, p_3 - 1, \ldots, p_n - 1)]$$

have an eccentricity equal to 3, i.e.,

$$\begin{aligned} [\text{ecc}(p_1 - 1, p_2 - 1, 0, \ldots, 0)] &= [\text{ecc}(0, p_2 - 1, p_3 - 1, 0, \ldots, 0)] \\ &= \cdots = \\ &= [\text{ecc}(0, p_2 - 1, p_3 - 1, \ldots, p_n - 1)] = 3. \end{aligned}$$

Since the graph is not complete when $n \geq 3$, no vertex has eccentricity 1, so radius ≥ 2.

Consider a vertex

$$\{i\} = (0, 0, 0, \ldots, p_i - 1, \ldots, 0).$$

If $i \notin V$, then $d(i, V) = 1$.

If $i \in V$, choose $j \notin V$, then the path is

$$\{i\} - \{j\} - V$$

that is,

$$[(0, 0, 0, \ldots, p_i - 1, \ldots, 0)] - [(0, 0, 0, \ldots, p_j - 1, \ldots, 0)] - V$$

has length 2, giving $d(\{j\}, V) \leq 2$. Therefore,

$$\text{ecc}(0, 0, 0, \ldots, p_i - 1, \ldots, 0) = 2.$$

This implies that

$$\text{Rad}(\Gamma_C(R)) = 2 \quad \text{for } n \geq 3.$$

Hence, we have

$$\text{Rad}(\Gamma_C(R)) = \begin{cases} 1, & n = 2 \\ 2, & n \geq 3 \end{cases}.$$

Example: For $n = 3$,

$$R = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3}, \quad \text{where } p_1, p_2, p_3 \text{ are primes.}$$

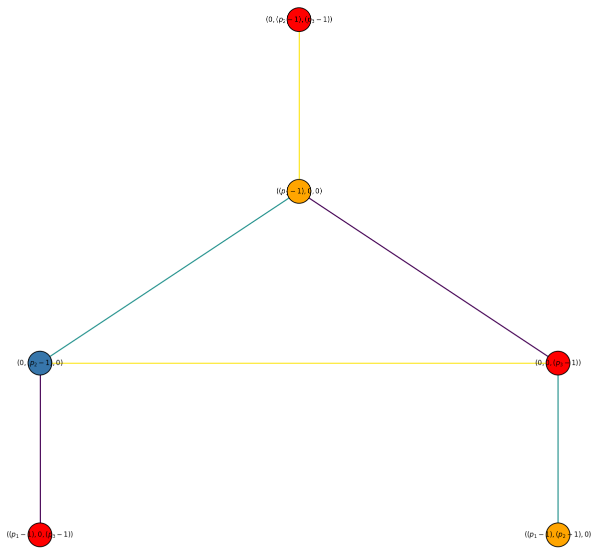

Figure 5: $\Gamma_C(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3})$

The collection of annihilators is

$$V(\Gamma_C(R)) = \{\, [(p_1 - 1, 0, 0)],\ [(0, p_2 - 1, 0)],\ [(0, 0, p_3 - 1)],\\ [(p_1 - 1, p_2 - 1, 0)],\ [(0, p_2 - 1, p_3 - 1)],\ [(p_1 - 1, 0, p_3 - 1)] \,\}.$$

We observe that

$$\text{ecc}[(p_1 - 1, 0, 0)] = \text{ecc}[(0, p_2 - 1, 0)] = \text{ecc}[(0, 0, p_3 - 1)] = 2.$$

On the other hand, vertices of the form

$$[(p_1 - 1, p_2 - 1, 0)],\ [(0, p_2 - 1, p_3 - 1)],\ [(p_1 - 1, 0, p_3 - 1)]$$

have an eccentricity equal to 3.

This implies that

$$\text{Rad}(\Gamma_C(R)) = \min_{v \in V(\Gamma_C(R))} \text{ecc}(v) = 2, \quad \text{diam}(\Gamma_C(R)) = \max_{v \in V(\Gamma_C(R))} \text{ecc}(v) = 3.$$

Therefore, for $n = 3$, the compressed zero-divisor Graph $\Gamma_C(R)$ has

$$\text{Rad}(\Gamma_C(R)) = 2, \quad \text{diam}(\Gamma_C(R)) = 3.$$

3.2 Theorem 2

Let

$$R = \prod_{k=1}^{n} \mathbb{Z}_{p_k}, \quad \text{where each } p_k \text{ is prime and } n \geq 2.$$

Then the clique number and chromatic number of the compressed zero-divisor graph satisfy

$$\omega(\Gamma_C(R)) = \chi(\Gamma_C(R)) = n.$$

Proof.

Let the collection

$$\Psi = \{U \subseteq \mathrm{Ann}_R(r) : U \neq \emptyset \text{ and } U \neq \mathrm{Ann}_R(r)\}.$$

Two vertices $U, V \in \Psi$ are adjacent if and only if $U \cap V = \emptyset$.

Consider the n vertices

$$[(p_1 - 1, 0, 0, \ldots, 0)],\ [(0, p_2 - 1, 0, \ldots, 0)],\ \ldots,\ [(0, 0, 0, \ldots, p_n - 1)],$$

which are pairwise disjoint. Hence, they all form a complete subgraph of $\Gamma_C(R)$ of size n. This implies that they form a clique of size n. Hence

$$\omega(\Gamma_C(R)) \geq n.$$

Any clique is a family of pairwise disjoint nonempty subsets of $\mathrm{Ann}_R(r)$.

Since the sum of the cardinality we cannot exceed n, This is such a family contains at most n elements. Therefore

$$\omega(\Gamma_C(R)) \leq n.$$

Combinedly we have

$$\omega(\Gamma_C(R)) = n.$$

Define a colouring map

$$c : \Psi \longrightarrow \{1, 2, 3, \ldots, n\}$$

defined by

$$c(U) = \min(U).$$

If $U \cap V = \emptyset$, then $(U) \neq (V)$, so adjacent vertices have distinct colours. Thus,

$$\chi(\Gamma_c) \leq n.$$

We know that the result

$$n = \omega(\Gamma_c(R)) = \chi(\Gamma_c(R)),$$

and this implies that

$$\chi(\Gamma_c(R)) \geq n.$$

Combinedly, we have

$$\chi(\Gamma_C(R)) = n.$$

Example 1.
For $n = 4$, consider

$$R = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4}, with \qquad p_1, p_2, p_3, p_4 \text{ primes.}$$

The collection of vertices $\Gamma_C(R))$ is

$$V(\Gamma_C(R)) = \{ (p_1-1,0,0,0),\ (0,p_2-1,0,0),\ (0,0,p_3-1,0),\ (0,0,0,p_4-1),\\ (p_1-1,p_2-1,0,0),\ (p_1-1,0,p_3-1,0),\ (p_1-1,0,0,p_4-1),\\ (0,p_2-1,p_3-1,0),\ (0,p_2-1,0,p_4-1),\ (0,0,p_3-1,p_4-1),\\ (p_1-1,p_2-1,p_3-1,0),\ (p_1-1,p_2-1,0,p_4-1),\\ (p_1-1,0,p_3-1,p_4-1),\ (0,p_2-1,p_3-1,p_4-1) \}.$$

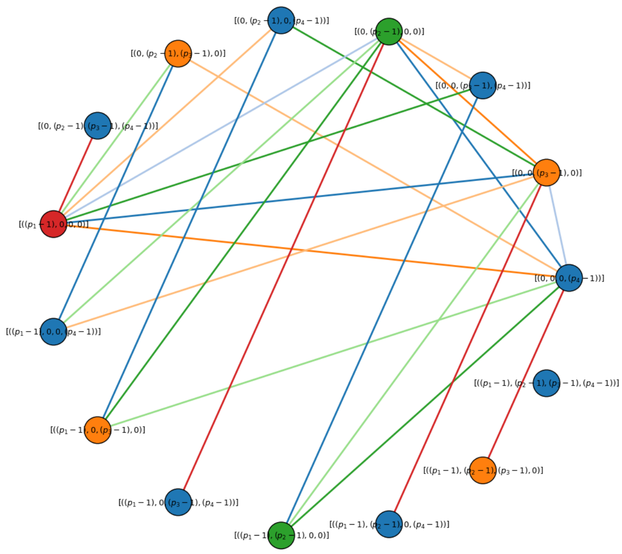

Figure 6: $\Gamma_C(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4})$

The vertices

$$(p_1-1,0,0,0),\ (0,p_2-1,0,0),\ (0,0,p_3-1,0),\ (0,0,0,p_4-1)$$

form the complete graph K_4, which is a clique of size 4, and it is the largest complete subgraph of $\Gamma_C(R)$. Therefore, the clique number is

$$\omega(\Gamma_C(R)) = 4.$$

Furthermore, since each of these vertices is mutually adjacent and the compressed zero-divisor graph contains a K_4, it requires exactly four colors for a proper vertex coloring. Hence, it follows that the chromatic number is

$$\chi(\Gamma_C(R)) = 4.$$

Example 2.
For $n = 5$, consider

$$R = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4} \times \mathbb{Z}_{p_5}, \qquad p_1, p_2, p_3, p_4, p_5 \text{ primes.}$$

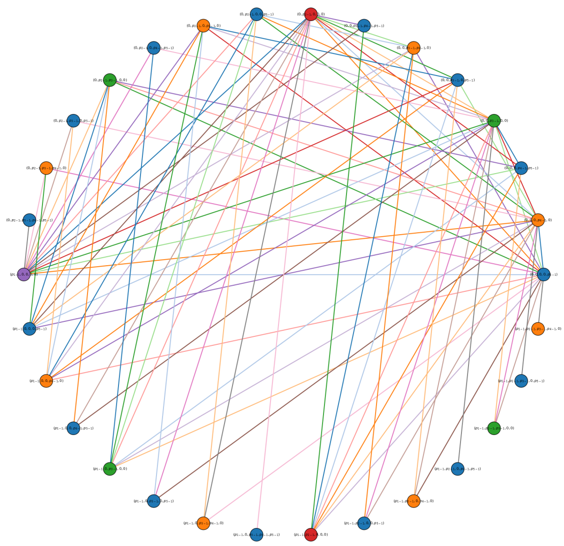

Figure 7: $\Gamma_C(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4} \times \mathbb{Z}_{p_5})$

The vertices

$$(p_1 - 1, 0, 0, 0, 0),\ (0, p_2 - 1, 0, 0, 0),\ (0, 0, p_3 - 1, 0, 0),\ (0, 0, 0, p_4 - 1, 0),\ (0, 0, 0, 0, p_5 - 1)$$

form the complete graph K_5, which is a clique of size 5, and it is the largest complete subgraph of $\Gamma_C(R)$. Therefore, the clique number is

$$\omega(\Gamma_C(R)) = 5.$$

Furthermore, since each of these vertices is mutually adjacent and the compressed zero-divisor graph contains a K_5, and it requires exactly five colors for a proper vertex coloring. it follows that the chromatic number

$$\chi(\Gamma_C(R)) = 5.$$

3.3 Theorem 3

Let

$$R = \prod_{k=1}^{n} \mathbb{Z}_{p_k},$$

where each p_k is a prime and $n \geq 2$. Then the edge chromatic number or chromatic index of the compressed zero-divisor graph satisfy

$$\chi'(\Gamma_C(R)) = \begin{cases} 1, & n = 2, \\ 2^{n-1} - 1, & n \geq 3. \end{cases}$$

Proof. Two vertices $U, V \in \Gamma_C(R)$ are adjacent if and only if

$$U \cap V = \varnothing.$$

Clearly, the degree of a vertex is the number of adjacent vertices
For $n \geq 3$, the n vertices

$$(p_1 - 1, 0, 0, \ldots, 0),\ (0, p_2 - 1, 0, \ldots, 0),\ \ldots,\ (0, 0, \ldots, p_n - 1)$$

have the largest $2^{\,n-1} - 1$ degree. That is,

$$\Delta(\Gamma_C(R)) = 2^{\,n-1} - 1.$$

For $n = 2$, the graph is the complete graph K_2. This implies that

$$\Delta(\Gamma_C(R)) = 1.$$

By Vizing's Theorem, for any simple graph the chromatic index is either Δ or $\Delta + 1$, where Δ is the maximum degree of any vertex in the graph. This implies that,

$$\Delta(\Gamma_C(R))\ \leq\ \chi'(\Gamma_C(R))\ \leq\ \Delta(\Gamma_C(R)) + 1.$$

For $n = 2$, the graph $\Gamma_C(R) = K_2$ is class 1. Therefore, we have

$$\Delta(c_R) = \chi'(\Gamma_C(R)) = 1.$$

For $n \geq 3$, $\Gamma_C(R)$ contains odd cycles and is not bipartite. Hence, it is class 2 and therefore, we have

$$\Delta(\Gamma_C(R)) = \chi'(\Gamma_C(R)) + 1 = 1 + 2^{n-1} - 1 = 2^{n-1} - 1.$$

3.4 Corollary

Let

$$R = \prod_{k=1}^{n} \mathbb{Z}_{p_k},$$

where each p_k is a prime and $n \geq 2$. Then the girth of the

$$\operatorname{girth}(\Gamma_C(R)) = \begin{cases} \infty, & n = 2, \\ 3, & n \geq 3. \end{cases}$$

Proof. The vertices of the graph $\Gamma_C(R)$ corresponds to nonempty proper subsets $U, V \subset \operatorname{Ann}_R(r)$, and two vertices U and V are adjacent if and only if

$$U \cap V = \varnothing.$$

Case 1: $n = 2$.
In the product ring

$$R = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2},$$

where p_1, p_2 are primes, the only nonempty proper subsets of $\operatorname{Ann}_R(r)$ are

$$\{[(p_1 - 1, 0)]\} \quad \text{and} \quad \{[(0, p_2 - 1)]\},$$

which are disjoint. These form a single edge; therefore, the girth is infinite:

$$\operatorname{girth}(\Gamma_C(R)) = \infty.$$

Case: $n \geq 3.$

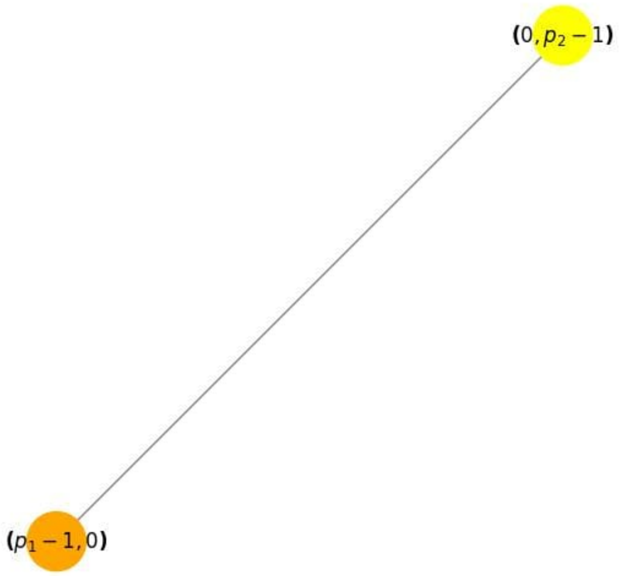

Figure 8: $\Gamma_C(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$

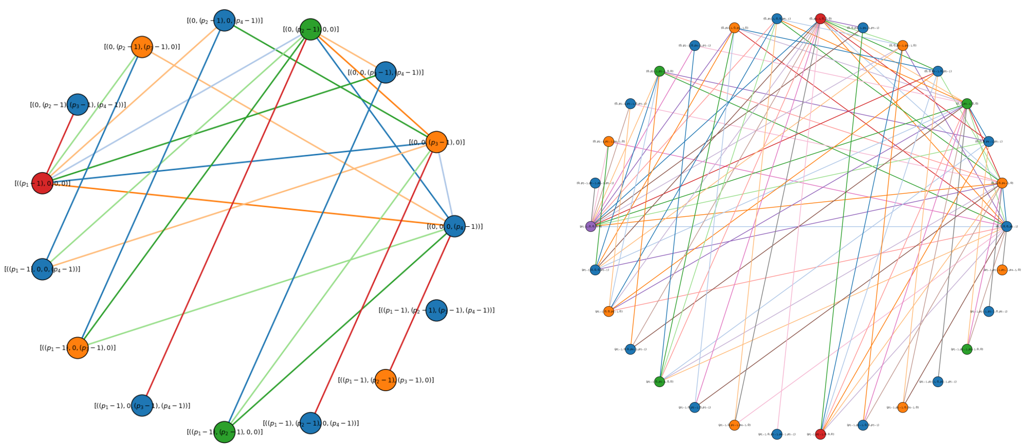

Figure 9: $\Gamma_c(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4})$ and $\Gamma_c(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3} \times \mathbb{Z}_{p_4} \times \mathbb{Z}_{p_5})$

Consider any three vertices

$$\{i\} = (0, 0, \ldots, p_i - 1, \ldots, 0),$$
$$\{j\} = (0, 0, \ldots, p_j - 1, \ldots, 0),$$
$$\{k\} = (0, 0, \ldots, p_k - 1, \ldots, 0),$$

with all indices (i, j, k) distinct. .

Then they form a 3-cycle

$$\{i\} - \{j\} - \{k\} - \{i\}.$$

Hence, this 3-cycle is the smallest cycle in $\Gamma_C(R)$, giving

$$\mathrm{girth}(\Gamma_C(R)) = 3.$$

3.5 Theorem.

Let

$$R = \prod_{k=1}^{n} \mathbb{Z}_{p_k},$$

where each p_k is a prime and $n \geq 3$. Then the compressed zero-divisor graph $\Gamma_C(R)$ contains cycles C_l of every possible length l, such that

$$3 \leq l \leq n.$$

Proof. For $n \geq 3$, consider the vertices

$$\begin{aligned} \{i\} &= (0, 0, \dots, p_i - 1, \dots, 0), \\ \{j\} &= (0, 0, \dots, p_j - 1, \dots, 0), \\ \{k\} &= (0, 0, \dots, p_k - 1, \dots, 0), \\ \dots&, \\ \{n\} &= (0, 0, \dots, p_n - 1) \end{aligned}$$

with all indices $i, j, k, ..., n$

These vertices together form a complete graph K_n. Since K_n is complete, it contains cycles C_l of every possible length l satisfying

$$3 \leq l \leq n.$$

4 Conclusion

In this paper, we have systematically investigated the structural and combinatorial properties of compressed zero-divisor graphs associated with finite commutative rings of the form

$$R = \prod_{k=1}^{n} \mathbb{Z}_{p_k},$$

where each p_k is a prime number. The study reveals a strong correspondence between the algebraic decomposition of the ring and the resulting graph-theoretic invariants of its compressed zero-divisor graph.

It has been shown that the diameter and radius are determined entirely by the number of prime components n. In particular, for $n = 2$, the graph attains minimal metric parameters with $\mathrm{diam} = 1$ and $\mathrm{Rad} = 1$, while for $n \geq 3$, the diameter increases to 3 and the radius to 2, reflecting the increased structural complexity induced by additional direct product components.

A key outcome of this work is the equality

$$\omega(\Gamma_c(R)) = \chi(\Gamma_c(R)) = n,$$

establishing that both the clique number and chromatic number are precisely equal to the number of prime factors in the ring. Furthermore, the edge chromatic number follows a predictable pattern: it equals 1 when $n = 2$ and takes the form $2^{n-1} - 1$ for $n \geq 3$, demonstrating the rapid growth in adjacency complexity as n increases.

The cyclic structure of the graph was also characterized. For $n = 2$, the graph is acyclic and hence has infinite girth. However, for $n \geq 3$, the girth equals 3, confirming the presence of triangular cycles. More generally, the graph contains cycles of all lengths l satisfying $3 \leq l \leq n$, indicating a rich cyclic structure.

Overall, these results show that the structural, chromatic, and cyclic properties of the compressed zero-divisor graph are directly governed by the multiplicative decomposition of the underlying finite ring, providing a foundation for further spectral and algebraic investigations.

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