Original Research Article

Automorphism of Zero-Divisor Graphs of Nilradicals of Semilocal Rings with Equal Prime Exponents

Abstract

A zero-divisor graph of a commutative ring R denoted as $\Gamma(R)$, is a graph whose vertices are the zero divisors of the ring. Any two distinct vertices of the graph are incident if and only if their product is zero. Zero-divisor graphs provide a powerful interface between commutative algebra and graph theory by encoding algebraic annihilation relations into combinatorial structures. While the graph-theoretic properties of $\Gamma(R)$ have been extensively studied, comparatively little is known about their automorphism groups, particularly for graphs arising from nilradicals of semilocal rings. In this paper, we investigate the automorphism groups of $\Gamma(R)$ associated with the nonzero nilradical of the semilocal ring $\mathbb{Z}_{p^n q^n}$, where $p \neq q$ are primes and $n \geq 2$. We show that the valuation structure induced by the prime-power decomposition yields a canonical partition of the vertex set into invariant layers. This stratification rigidly constrains graph automorphisms and forces the automorphism group to decompose as a direct product of symmetric groups indexed by valuation levels. Explicit formulas for these automorphism groups are obtained, thereby extending and unifying earlier results for local rings.

Keywords: Zero-divisor graph; nilradical; semilocal ring; automorphism group; valuation theory.

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1 Introduction

Zero-divisor graphs have played a central role in the interaction between commutative algebra and graph theory since the foundational work of Anderson and Livingston (2). By encoding the multiplicative behavior of zero divisors into adjacency relations, these graphs translate algebraic properties of rings into graph-theoretic invariants such as diameter, girth, chromatic number, and clique number. Subsequent studies have expanded this framework to include variants such as Beck's total graph (5) and the equivalence class graph $\Gamma_E(R)$ introduced by Mulay (11). For finite commutative rings, particularly local and semilocal rings, the structure of the nilradical plays a decisive role in shaping the associated zero-divisor graph (9). Numerous authors have investigated how nilpotency and prime decomposition influence graph properties, yielding sharp characterizations of completeness, multipartite structure, and connectivity (1; 4). However, despite these advances, the automorphism groups of zero-divisor graphs remain comparatively underexplored. Existing results on automorphisms of graphs have largely focused on reduced rings, direct products, or special constructions such as idealizations (7; 13). Even when zero-divisor graphs are finite and highly structured, explicit descriptions of $\mathrm{Aut}(\Gamma(R))$ are known only in limited cases. This gap is particularly evident for semilocal rings, where multiple maximal ideals introduce interacting valuation structures. Recent work by (8; 12) investigates automorphisms of $\Gamma(R)$ for certain classes of finite rings with bounded nilpotency conditions, particularly square and power-four radical zero completely primary rings. While their analysis does not address nilradicals explicitly, it highlights the strong influence of nilpotency depth and annihilator stratification on graph automorphism groups. Motivated by these developments, the present paper undertakes a systematic study of $\mathrm{Aut}(\Gamma(R))$ arising from nilradicals of semilocal rings with equal prime exponents. Specifically, we focus on rings of the form $\mathbb{Z}_{p^nq^n}$, where $p \neq q$ and $n \geq 2$. We show that the (p,q)-adic valuation induces a canonical decomposition of the vertex set into generator-based subsets, each of which is invariant under graph automorphisms. Even in the case of finite local rings, where $\Gamma(R)$ is finite and structurally constrained, the automorphism groups have not been systematically described in terms of intrinsic valuation-theoretic data.This layered structure completely determines the automorphism group, which decomposes as a direct product of symmetric groups.

2 Nilradical Structure of $\mathbb{Z}_{p^nq^n}$

Let $R = \mathbb{Z}_{p^nq^n}$, where $p \neq q$ and $n \geq 2$. R is Artinian and admits a complete primary decomposition determined by the prime factorization of its modulus. In any Artinian ring, the nilradical coincides with the Jacobson radical and is given by the intersection of all maximal ideals (10). Consequently, $\mathrm{Nil}(R) = (p) \cap (q)$. From a number-theoretic perspective, this intersection consists precisely of all residue classes modulo p^nq^n represented by integers divisible by both p and q, that is, by pq. Thus, nilpotent elements in R are exactly those elements whose prime-power valuations simultaneously involve both primes dividing the modulus. This observation allows the nilradical to be described explicitly in terms of p-adic and q-adic valuations, which will later serve as the foundation for a canonical decomposition of the vertex set $(V(\Gamma(R)))$.

Proposition 2.1. *An element $x \in \mathbb{Z}_{p^nq^n}$ is nilpotent if and only if it is divisible by pq. Moreover, every nonzero nilpotent element admits a unique representation $x = p^rq^sa, \quad 1 \leq r,s \leq n-1, \quad \gcd(a,pq) = 1.$*

Proof. Since $R = \mathbb{Z}_{p^nq^n}$ is finite, it is an Artinian ring. A standard result in commutative algebra asserts that, for any Artinian ring, the nilradical equals the intersection of all maximal ideals and consists precisely of the nilpotent elements (6). As (p) and (q) are the only maximal ideals of R, it follows that $\mathrm{Nil}(R) = (p) \cap (q)$. Hence, $x \in R$ is nilpotent if and only if it is divisible by both p and q, or equivalently, by their product pq. Let $x \neq 0$ be a nilpotent element. Viewing x as an integer representative modulo p^nq^n, we may write $x = p^rq^sa$, where $r, s \geq 1$ and $\gcd(a, pq) = 1$. The bounds $r, s \leq n-1$ follow from the fact that $x \not\equiv 0 \pmod{p^nq^n}$, so neither p^n nor q^n divides x. To verify nilpotency, consider powers of x: $x^m = p^{rm}q^{sm}a^m$. Since a is a unit modulo both p and q, the divisibility of x^m by p^nq^n depends entirely on the exponents of p and q. We have $x^m \equiv 0 \pmod{p^nq^n} \iff rm \geq n$ and $sm \geq n$. Such a positive integer m exists if and only if $r \geq 1$ and $s \geq 1$, which precisely characterizes the elements divisible by pq. Finally, the representation $x = p^rq^sa$ is unique. Indeed, the integers r and s are uniquely determined by the p-adic and q-adic valuations of x, while a is uniquely determined modulo $p^{n-r}q^{n-s}$ and is coprime to pq. □

3 Canonical Decomposition of the Vertex Set

The explicit valuation-theoretic description of nilpotent elements in $\mathbb{Z}_{p^nq^n}$ yields more than an algebraic classification: it induces a natural and highly structured partition of the vertex set of $\Gamma(R)$. Since adjacency in $\Gamma(\mathrm{Nil}^*(R))$ depends only on the p-adic and q-adic valuations of elements, vertices sharing the same pair of valuations are graph-theoretically indistinguishable. This observation leads to a canonical decomposition of the vertex set into valuation layers indexed by pairs (r, s). Such a decomposition plays a central role in the analysis of graph automorphisms. It isolates subsets of vertices whose neighborhoods are identical and whose cardinalities can be computed explicitly.

Proposition 3.1. *The vertex set of* $\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^nq^n}))$ *admits the disjoint decomposition*

$$V = \bigsqcup_{r=1}^{n-1} \bigsqcup_{s=1}^{n-1} S_{p^rq^s}, \qquad S_{p^rq^s} = \{\, p^rq^sa : \gcd(a, pq) = 1 \,\},$$

with

$$|S_{p^rq^s}| = p^{\,n-r-1}q^{\,n-s-1}(p-1)(q-1).$$

Proof. By Proposition 3.1, every nonzero nilpotent element of $\mathbb{Z}_{p^nq^n}$ admits a unique representation of the form $x = p^rq^sa, \quad 1 \leq r, s \leq n-1, \quad \gcd(a, pq) = 1$. This representation immediately associates to each vertex a unique valuation pair (r, s), and hence each vertex belongs to exactly one subset $S_{p^rq^s}$. Therefore, the union of all such subsets exhausts the vertex set $V(\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^nq^n})))$. Fix integers r and s with $1 \leq r, s \leq n-1$. Two elements p^rq^sa and p^rq^sa' represent the same vertex if and only if $a \equiv a' \pmod{p^{n-r}q^{n-s}}$. Moreover, the condition $\gcd(a, pq) = 1$ is equivalent to a being a unit modulo $p^{n-r}q^{n-s}$. Consequently, the set $S_{p^rq^s}$ is in bijection with the unit group $(\mathbb{Z}_{p^{n-r}q^{n-s}})^\times$. Since Euler's totient function is multiplicative and $\varphi(p^{n-r}) = p^{n-r-1}(p-1), \quad \varphi(q^{n-s}) = q^{n-s-1}(q-1)$, we obtain $|S_{p^rq^s}| = \varphi(p^{n-r}q^{n-s}) = p^{\,n-r-1}q^{\,n-s-1}(p-1)(q-1)$. Distinct valuation pairs $(r, s) \neq (r', s')$ correspond to distinct p-adic or q-adic orders and therefore yield disjoint subsets. Hence, the decomposition is both disjoint and exhaustive. □

4 Adjacency Relations and Degree Invariants

Once the vertex set has been decomposed into valuation-based subsets, the next step is to understand how these valuations control adjacency in $\Gamma(R)$. For rings of the form $\mathbb{Z}_{p^nq^n}$, adjacency is determined entirely by whether the combined p-adic and q-adic valuations of two elements are sufficient to annihilate the modulus. As a result, $\Gamma(R)$ exhibits a rigid layered structure in which vertices are connected according to valuation inequalities rather than specific representatives. This valuation dependence has two important consequences. First, it provides an explicit criterion for adjacency that is independent of unit coefficients. Second, it ensures that all vertices lying in the same valuation class have identical neighborhood structures, and hence identical degrees. These degree invariants will later play a crucial role in restricting graph automorphisms.

Proposition 4.1. *Let $x = p^r q^s a$ and $y = p^{r'} q^{s'} b$ be distinct vertices of $\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^n q^n}))$. Then x is incident to y if and only if $r + r' \geq n \quad$ and $\quad s + s' \geq n$. Consequently, all vertices in the same set $S_{p^r q^s}$ have identical degrees.*

Proof. By definition of the zero-divisor graph, two distinct vertices x and y are adjacent if and only if their product is zero modulo the ring modulus: $xy \equiv 0 \pmod{p^n q^n}$. Substituting the valuation representations yields $xy = (p^r q^s a)(p^{r'} q^{s'} b) = p^{r+r'} q^{s+s'} ab$. Since $\gcd(a, pq) = \gcd(b, pq) = 1$, both a and b are units modulo $p^n q^n$. Therefore, divisibility of xy by $p^n q^n$ depends exclusively on the powers of p and q appearing in the product. The congruence $p^{r+r'} q^{s+s'} ab \equiv 0 \pmod{p^n q^n}$ holds if and only if $p^n \mid p^{r+r'} \quad$ and $\quad q^n \mid q^{s+s'}$, which is equivalent to the simultaneous inequalities $r + r' \geq n \quad$ and $\quad s + s' \geq n$. This establishes the adjacency criterion. Now fix a valuation pair (r, s) and consider any vertex $x \in S_{p^r q^s}$. The neighbors of x are precisely those vertices whose valuation pairs (r', s') satisfy $r + r' \geq n$ and $s + s' \geq n$. Importantly, this condition depends only on (r, s) and not on the specific choice of the unit coefficient a. Hence every vertex in $S_{p^r q^s}$ has the same set of neighbors up to relabelling, and therefore the same degree. Thus, degree is constant on each valuation subset $S_{p^r q^s}$. □

5 Automorphism Group Decomposition

The valuation-based adjacency structure imposes strong constraints on possible graph automorphisms. Since automorphisms preserve adjacency, degrees, and neighborhood structures, they must respect the valuation layers identified earlier. In particular, vertices lying in different valuation classes cannot be permuted if their degrees or adjacency profiles differ. On the other hand, within a fixed valuation class, all vertices are graph-theoretically indistinguishable. This symmetry allows permutations to be unrestricted inside each layer, leading naturally to a decomposition of the automorphism group as a direct product of symmetric groups indexed by valuation pairs.

Proposition 5.1. *For $R = \mathbb{Z}_{p^n q^n}$ with $n \geq 3$, then,*

$$\mathrm{Aut}\big(\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^n q^n}))\big) \cong \prod_{r=1}^{n-1} \prod_{s=1}^{n-1} S_{p^{n-r-1} q^{n-s-1}(q-1)(p-1)}.$$

Proof. By proposition 4.1, the vertices in distinct valuation classes $S_{p^r q^s}$ and $S_{p^{r'} q^{s'}}$ have different adjacency conditions whenever $(r, s) \neq (r', s')$. In particular, their degree sequences or neighborhood structures are different. Since graph automorphisms preserve degrees and adjacency relations, no automorphism can map a vertex from one valuation class to another. Hence each subset $S_{p^r q^s}$ is invariant under $\mathrm{Aut}(\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^n q^n})))$. Fix a valuation pair (r, s). As shown earlier, all vertices in $S_{p^r q^s}$ have identical neighborhoods. Therefore, any permutation of the vertices within this subset preserves adjacency and extends to a graph automorphism. The group of all such permutations is precisely the symmetric group $S_{|S_{p^r q^s}|}$, where $|S_{p^r q^s}| = p^{n-r-1} q^{n-s-1}(p-1)(q-1)$. Since permutations acting on distinct valuation subsets operate on disjoint vertex sets, they commute and act independently. Consequently, the full automorphism group is the direct product of the symmetric groups associated with each valuation class, yielding the stated decomposition. □

Remark 5.1. It is important to note that the valuation-based decomposition described above assumes that the $\Gamma(R)$ is not complete. In the mixed-exponent setting, this distinction is sharp. Indeed, $\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^m q^n}))$ is complete if and only if $(m, n) = (2, 2)$. In this exceptional case, every product of two distinct nonzero nilpotent elements vanishes modulo $p^2 q^2$, and hence every pair of vertices is adjacent. As a consequence, all vertices have the same degree and identical neighborhood structure, so the valuation layers collapse graph-theoretically. Therefore, $\mathrm{Aut}\big(\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{p^2 q^2}))\big) \cong S_{|\mathrm{Nil}^*(\mathbb{Z}_{p^2 q^2})|}$. For $m, n > 2$, the graph remains connected but is no longer complete. In these nontrivial cases, vertices belonging to different valuation classes have distinct degree profiles, and the layered structure persists. This ensures that valuation subsets remain invariant under graph automorphisms, leading to the direct-product decompositions established above. Thus, completeness occurs only at the minimal exponent threshold and represents a degenerate case in the broader valuation-theoretic framework.

Example 5.1. *Let* $R = \mathbb{Z}_{3^2.5^2} = \mathbb{Z}_{225}$. *The nonzero nilradical of* R *is* $\mathrm{Nil}^*(\mathbb{Z}_{225}) = \{15, 30, 45, \ldots, 210\} = \{15k : 1 \leq k \leq 14\}$. *Hence,* $|\mathrm{Nil}^*(\mathbb{Z}_{225})| = 14$. *The zero-divisor graph* $\Gamma = \Gamma\big(\mathrm{Nil}^*(\mathbb{Z}_{225})\big)$ *has these 14 vertices, where two distinct vertices* x *and* y *are adjacent if and only if* $xy \equiv 0 \pmod{225}$. *Let* $x = 15k$ *and* $y = 15\ell$ *be two distinct vertices. Then* $xy = (15k)(15\ell) = 225\,k\ell \equiv 0 \pmod{225}$. *Thus, every pair of distinct vertices is adjacent. Therefore,* $\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{225})) \cong K_{14}$, *the complete graph on* 14 *vertices. The graph is connected and complete with* $\mathrm{diam}(\Gamma) = 1$ *and* $\mathrm{girth}(\Gamma) = 3$. *Since the graph is complete,* $\omega(\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{225}))) = \chi(\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{225}))) = 14$. *Since* $\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{225}))$ *is complete, every permutation of the vertices is an automorphism. Hence,* $\mathrm{Aut}(\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{225}))) \cong S_{14}, \qquad |\mathrm{Aut}(\Gamma(\mathrm{Nil}^*(\mathbb{Z}_{225})))| = 14!$.

6 Conclusion

This paper provides a complete description of the automorphism groups of $\Gamma(R)$ arising from the nonzero nilradical of semilocal rings of the form $\mathbb{Z}_{p^n q^n}$. By exploiting valuation-theoretic structure, we show that the vertex set admits a canonical invariant decomposition into valuation layers, each of which supports an independent symmetric action. This layered structure fully determines the graph automorphisms and extends the valuation-based phenomena observed in the local case. These results form a foundational step toward a unified theory of automorphism groups of zero-divisor graphs for general semilocal and mixed-exponent finite rings.

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