Energy of Knight's Hypergraph

Abstract

Hypergraphs are generalization of graphs, introduced by Berge [1]. In an ordinary graph, an edge connects exactly two vertices, where as in hypergraphs a hyperedge can join any number of vertices. Hypergraphs have applications in the field of Computer Science, Machine learning, Neural networks etc [2]. In this paper, we focus on Knight's Hypergraph in which the squares of a chessboard are taken as vertices and each hyperedge include a vertex and all the vertices which are reachable by a knight in one move.We find the adjacency matrix ,Laplacian matrix, their eigen values and corresponding energies of these graphs with the help of python programming.

Keywords: Hypergraph, Adjacency matrix, Eigen values, Energy, Laplacian matrix

1 Introduction

Hypergraphs are systems of finite sets and form the most general concept in Discrete mathematics.It was only in 1960s that Hypergraph become an independent theory.It was mostly in Hungary and France under the leadership of mathematicians like Paul Erdos, Laszlo Lovasz, Paul Turan and C Berge.This branch of mathematics has developed very rapidly in the later part of twentieth century, influenced by the advent of computer science.Knight's graphs are graphs in which squares of a chess board are taken as vertices and two vertices are adjacent if they are reachable by a knight in one move.Lot of works are done in Knights graphs and many solutions are for knight tour problem which traces all the vertices by a knight exactly once.Knight's Hypergraphs are generelisation of knights graph where one vertex can be adjacent to many vertices.More precisely a Hyper edge include a vertex and all vertices which are reachable from that vertex by a knight in one move. In this paper we find out $64 * 64$ adjacency matrix of knight's hypergraph and there after the energy of Knight's hypergraph.

2 Preliminaries

Definition 1 [3]:- A hypergraph H is a pair (V, E), where $V = \{v_1, v_2, ..., v_n\}$ is a finite set of vertices and $E = \{E_1, E_2, ..., E_m\}$ is a finite set of non empty subsets of V called Hyperedges of H, such that $\cup_{j=1}^{m} E_j = V$. The number of vertices is called the order of H and the number of hyperedges is called the size of H. In a hypergraph,the vertices in a hyper edge are said to be adjacent. Degree of a vertex v_i of a hypergraph H is the number of hyper edges which contains the vertex v_i and is denoted by $d(v_i)$.

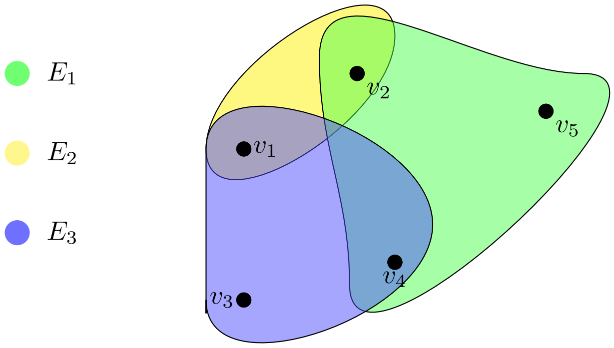


Fig. 1

$$\begin{aligned}
V &= \{v_1, v_2, v_3, v_4, v_5\} \\
E &= \{E_1, E_2, E_3\} \\
E_1 &= \{v_2, v_4, v_5\} \\
E_2 &= \{v_1, v_2\} \\
E_3 &= \{v_1, v_3, v_4\}
\end{aligned}$$

Definition 2 [3]:- Adjacency matrix of a hypergraph H is denoted by $A(H) = [a_{ij}]$ where

$$a_{ij} = \begin{cases} |E_k \in E : \{v_i, v_j\} \subseteq E_k| & if \quad i \neq j \\ 0 & Otherwise \end{cases}$$

Eigen values of $A(H)$ are called the eigen values of hypergraph H. Since $A(H)$ is a real symmetric matrix, all the eigen values are real. The spectrum of H is the set of all eigen values of $A(H)$ together with their multiplicities.
If $\lambda_1, \lambda_2, ..., \lambda_s$ are distinct eigen values of H with multiplicities $m_1, m_2, ..., m_s$, then

$$spec(H) = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_s \\ m_1 & m_2 & \dots & m_s \end{pmatrix}$$

Definition 3 [4]:- Laplacian degree of a vertex $v_i \in V(H)$ is $\delta_l(v_i) = \sum_{j=1}^{m} a_{ij}$. The Laplacian matrix of a hypergraph H is denoted by $L = L(H)$ and is defined as $L = D - A$ where $D = diag(\delta_l(v_1), \delta_l(v_2), ..., \delta_l(v_n))$. The matrix L is symmetric and positive definite. So all eigen values are real and non-negativewith smallest eigen value zero. These eigen values with their multiplicitiies collectively called the Laplacian Spectrum.

3 Knight's Hypergraph

Definition 4 (Knight's Graph) :- Knights graph is the graph in which 64 squares of chess board are taken as vertices and two squares or vertices are connected in the graph if they are reachable by a knight in one move.

Definition 5 (Knight's Hypergraph) :- Knight's Hypergraph is the hyper graph in which 64 squares in a chess board are taken as vertices and every hyper edge consists of a vertex and all the vertices which are reachable by a knight in one move.
This graph contains 64 vertices and 64 hyperedges. Hence pictorial representation is not so easy . So for analysing the graph and to obtain adjacency mtrix, we follow certain notations.These are the notations used for the 64 squares in a chess game.
In a chessboard,we call the first column as A file and 8 squares in A file as a_1,a_2,..a_8 , second column as B file and squares in B file as b_1,b_2...b_8 and so on up to H file and last square h_8.
Figure 2 shows all the possibilities that a knight can move from the square d_4 and they are b_3, b_5, c_2, c_6, e_2, e_6, $f_3$and f_5. So let us call this hyper edge as hyperedge with centre at d_4 denoted by E_{d_4}.
$E_{d_4} = \{d_4, b_3, b_5, c_2, c_6, e_2, e_6, f_3, f_5\}$

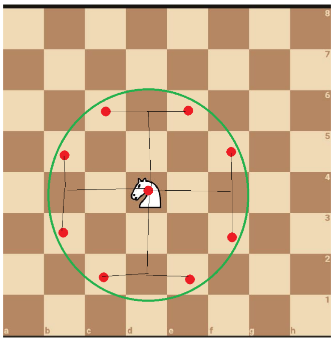


Fig. 2

Hence the Knight's hypergraph is $H = (V, E)$

Where

$$\begin{aligned} V &= \{a_8, b_8, c_8, d_8, e_8, f_8, g_8, h_8, \\ . \quad & \quad a_7, b_7, c_7, d_7, e_7, f_7, g_7, h_7, \\ & \quad \ldots \\ & \quad \ldots \\ & \quad a_1, b_1, c_1, d_1, e_1, f_1, g_1, h_1\} \end{aligned}$$

And

$$\begin{aligned} E &= \{E_{a_8}, E_{b_8}, E_{c8}, E_{d8}, E_{e_8}, E_{f_8}, E_{g_8}, E_{h_8}, \\ & \quad E_{a_7}, E_{b_7}, E_{c_7}, E_{d_7}, E_{e_7}, E_{f_7}, E_{g_7}, E_{h_7}, \\ & \quad \ldots \\ & \quad \ldots \\ & \quad E_{a_1}, E_{b_1}, E_{c_1}, E_{d_1}, E_{e_1}, E_{f_1}, E_{g_1}, E_{h_1}\} \end{aligned}$$

Where

$$\begin{aligned} E_{a_i} &= \{a_i, b_{(i-2)}, b_{(i+2)}, c_{(i-1)}, c_{(i+1)} : \\ & 0 \leq i-2, i-1, i+1, i+2 \leq 8\}, \\ & i = 1, 2, ..., 8 \end{aligned}$$

$$\begin{aligned} E_{b_i} &= \{b_i, a_{(i-2)}, a_{(i+2)}, c_{(i-2)}, c_{(i+2)}, d_{(i-1)}, d_{(i+1)} : \\ & 0 \leq i-2, i-1, i+1, i+2 \leq 8\}, \\ & i = 1, 2, ..., 8 \end{aligned}$$

$$\begin{aligned} E_{c_i} &= \{c_i, a_{(i-1)}, a_{(i+1)}, b_{(i-2)}, b_{(i+2)}, d_{(i-2)}, d_{(i+2)}, e_{(i-1)}, e_{(i+1)} : \\ & 0 \leq i-2, i-1, i+1, i+2 \leq 8\}, \\ & i = 1, 2, ..., 8 \end{aligned}$$

$$\begin{aligned} E_{d_i} &= \{d_i, b_{(i-1)}, b_{(i+1)}, c_{(i-2)}, c_{(i+2)}, e_{(i-2)}, e_{(i+2)}, f_{(i-1)}, f_{(i+1)} : \\ & 0 \leq i-2, i-1, i+1, i+2 \leq 8\}, \\ & i = 1, 2, ..., 8 \end{aligned}$$

$$\begin{aligned} E_{e_i} &= \{e_i, c_{(i-1)}, c_{(i+1)}, d_{(i-2)}, d_{(i+2)}, f_{(i-2)}, f_{(i+2)}, g_{(i-1)}, g_{(i+1)} : \\ & 0 \leq i-2, i-1, i+1, i+2 \leq 8\}, \\ & i = 1, 2, ..., 8 \end{aligned}$$

$$\begin{aligned} E_{f_i} &= \{f_i, d_{(i-1)}, d_{(i+1)}, e_{(i-2)}, e_{(i+2)}, g_{(i-2)}, g_{(i+2)}, h_{(i-1)}, h_{(i+1)} : \\ & 0 \leq i-2, i-1, i+1, i+2 \leq 8\}, \\ & i = 1, 2, ..., 8 \end{aligned}$$

$$\begin{aligned} E_{g_i} &= \{g_i, e_{(i-1)}, e_{(i+1)}, f_{(i-2)}, f_{(i+2)}, h_{(i-2)}, h_{(i+2)} : \\ & 0 \leq i-2, i-1, i+1, i+2 \leq 8\}, \\ & i = 1, 2, ..., 8 \end{aligned}$$

$$\begin{aligned} E_{h_i} &= \{h_i, f_{(i-1)}, f_{(i+1)}, g_{(i-2)}, g_{(i+2)} : \\ & 0 \leq i-2, i-1, i+1, i+2 \leq 8\}, \\ & i = 1, 2, ..., 8 \end{aligned}$$

Definition 6 (Adajcency matrix of Knight's hypergraph) :- In a Hypergraph , two vertices are adjacent if they are common in one hyper edge and the corresponding entry in the adjacency matrix is the number of hyper edges containig both vertices.
In Knights hyper graph, two vertices are adjacent if they are reachable by a knight in one move or in two moves and the corresponding entry in the adjacency matrix is the number of possibilities by which these two vertices are reachable by a knight in one or in two moves.

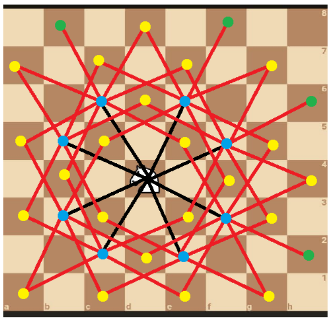


Fig. 3

Hence adjacency matrix of Knight's Hypergraph is a square matrix of order 64.
Figure 3 shows how d_4 vertex is adjacent to other vertices .
The vertex d_4 is adjacent to blue, green labelled vertices in one way.
The vertex d_4 is adjacent to yellow labelled vertices in two ways and all other vertices in zero ways.
Using figure 3 we find out the adjacency possibilities of 64 vertices with each other,And we observe that

1. This matrix is symmetric wmatrix with all diagonal entries zeros
2. 64∗64 matrix can be treated as a combination of 64 block matrices of order 8∗8.And these block matrices are the adjacency matrces of A file with A file, A file with B file, ...H file with H file.
3. Due to summetry of the chess board, many of these block matrices are same and are given the same notations and we get the following different blocks
   (a) Adjacency matrix of A file with A file (say [AA]), which is same as the adjacency matrix of H file with H file

$$[AA] = \begin{bmatrix} 0&0&1&0&1&0&0&0 \\ 0&0&0&1&0&1&0&0 \\ 1&0&0&0&1&0&1&0 \\ 0&1&0&0&0&1&0&1 \\ 1&0&1&0&0&0&1&0 \\ 0&1&0&1&0&0&0&1 \\ 0&0&1&0&1&0&0&0 \\ 0&0&0&1&0&1&0&0 \end{bmatrix}$$

(b) Adajacency matrix of A file with B file (say [AB]),which is same as adjacency matrix of H file with G file.
Due to symmetry, adjacency matrices of B file with A file and G file with H file are transpose of [AB].

$$[AB] = \begin{bmatrix} 0&0&1&1&0&0&0&0 \\ 1&0&1&1&1&0&0&0 \\ 1&1&0&1&1&1&0&0 \\ 1&1&1&0&1&1&1&0 \\ 0&1&1&1&0&1&1&1 \\ 0&0&1&1&1&0&1&1 \\ 0&0&0&1&1&1&0&1 \\ 0&0&0&0&1&1&0&0 \end{bmatrix}$$

(c) Adjacency matrix of B file with Bfile (say [BB]) ,which is same as the adjacency matrix of G file with G file.

$$[BB] = \begin{bmatrix} 0&0&1&0&2&0&0&0 \\ 0&0&0&1&0&2&0&0 \\ 1&0&0&0&1&0&2&0 \\ 0&1&0&0&0&1&0&2 \\ 2&0&1&0&0&0&1&0 \\ 0&2&0&1&0&0&0&1 \\ 0&0&2&0&1&0&0&0 \\ 0&0&0&2&0&1&0&0 \end{bmatrix}$$

(d) Adjacency matrix of A file with C file [AC] = adjacency matrix of Bfile with D file [BD]=adjacency matrix of C file with E file [CE]=adjacency matrix of Dfile with F file [DF]=adjacency matrix of E file with G file [EG]=adjacency matrix of Ffile with H file [FH].SImilarly due to symmetry we get these matrices are same as [CA],[DB],[EC],FD,[GE],[HF].

We call these block matrices as [Q]

$$[Q] = \begin{bmatrix} 1&1&0&0&1&0&0&0 \\ 1&1&1&0&0&1&0&0 \\ 0&1&2&1&0&0&1&0 \\ 0&0&1&2&1&0&0&1 \\ 1&0&0&1&2&1&0&0 \\ 0&1&0&0&1&2&1&0 \\ 0&0&1&0&0&1&1&1 \\ 0&0&0&1&0&0&1&1 \end{bmatrix}$$

(e) . [BC]=[CD]=[DE]=[EF]=[FG]=[CB]=[DC]=[ED]=[FE]=[GF].

We call these block matrices as [P].

$$[P]=\begin{bmatrix}0&1&1&2&0&0&0&0\\1&0&2&1&2&0&0&0\\1&2&0&2&1&2&0&0\\2&1&2&0&2&1&2&0\\0&2&1&2&0&2&1&2\\0&0&2&1&2&0&2&1\\0&0&0&2&1&2&0&1\\0&0&0&0&2&1&1&0\end{bmatrix}$$

(f) . [AD]=[BE]=[CF]=[DG]=[EH]=[DA]=[EB]=[FC]=[GD]=[HE].

We call all these block matrices as [R].

$$[R]=\begin{bmatrix}0&1&0&2&0&0&0&0\\1&0&2&0&2&0&0&0\\0&2&0&2&0&2&0&0\\2&0&2&0&2&0&2&0\\0&2&0&2&0&2&0&2\\0&0&2&0&2&0&2&0\\0&0&0&2&0&2&0&1\\0&0&0&0&2&0&1&0\end{bmatrix}$$

(g) . [AE]=[BF]=[CG]=[DH]=[EA]=[FB]=[GC]=[HD].

We call these block matrices as [S].

$$[S]=\begin{bmatrix}1&0&1&0&0&0&0&0\\0&2&0&1&0&0&0&0\\1&0&2&0&1&0&0&0\\0&1&0&2&0&1&0&0\\0&0&1&0&2&0&1&0\\0&0&0&1&0&2&0&1\\0&0&0&0&1&0&2&0\\0&0&0&0&0&1&0&1\end{bmatrix}$$

(h) . [CC]=[DD]=[EE]=[FF].

We call these block matrices as [D].

$$[D] = \begin{bmatrix} 0&0&2&0&2&0&0&0\\ 0&0&0&2&0&2&0&0\\ 2&0&0&0&2&0&2&0\\ 0&2&0&0&0&2&0&2\\ 2&0&2&0&0&0&2&0\\ 0&2&0&2&0&0&0&2\\ 0&0&2&0&2&0&0&0\\ 0&0&0&2&0&2&0&0 \end{bmatrix}$$

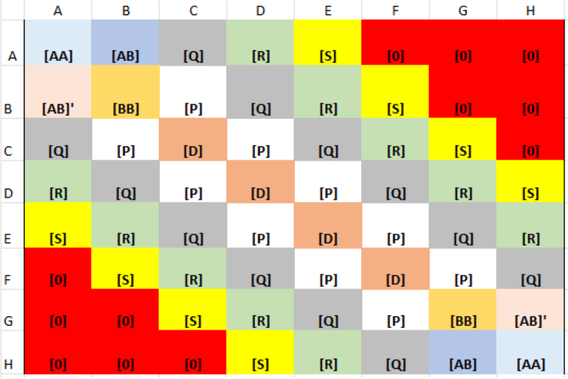
(i) .

$$[AF] = [AG] = [AH] = [BG] = [BH] = [CH] = [FA] = [GA] = [GB] = [HA] = [HB] = [HC].$$

All these block matrices are zero block matrix denoted by [0].

Combining all these we get the adjacency matrix as an $8*8$ matrix with entries are $8*8$ block matrices mentioned above.

And the Adjacency matrix is as follows.



	A	B	C	D	E	F	G	H
A	[AA]	[AB]	[Q]	[R]	[S]	[0]	[0]	[0]
B	[AB]'	[BB]	[P]	[Q]	[R]	[S]	[0]	[0]
C	[Q]	[P]	[D]	[P]	[Q]	[R]	[S]	[0]
D	[R]	[Q]	[P]	[D]	[P]	[Q]	[R]	[S]
E	[S]	[R]	[Q]	[P]	[D]	[P]	[Q]	[R]
F	[0]	[S]	[R]	[Q]	[P]	[D]	[P]	[Q]
G	[0]	[0]	[S]	[R]	[Q]	[P]	[BB]	[AB]'
H	[0]	[0]	[0]	[S]	[R]	[Q]	[AB]	[AA]

Fig 4 :

4 Energy of knight's hypergraph

Definition 7 Energy of a graph/hypergraph:-
Energy of a graph/hypergraph is the sum of absolute values of eigen values of its adjacency matrix.
For knight's hypergraph,the adjacency matrix is a $64*64$ matrix. So we use Python programming to find its eigen values and hence graph energy.Since the adjacency matrix is a symmetric matrix of order 64,we get 64 real numbers as its eigen values .The collection of eigen values or the eigen spectrum obtained using Python is listed below.

The eigen values of Knight's hypergraph are

$$\left\{\begin{array}{llll}
-6.76629207, & -6.76629207, & -6.0536517, & -5.96629206, \\
-5.96629206, & -5.83056446, & -5.80505071, & -5.52267497, \\
-5.3983357, & -5.3983357, & -5.31093628, & -5.13028815, \\
-4.93057489, & -4.78337845, & -4.73877083, & -4.73877083, \\
-4.60585049, & -4.38763183, & -4.19935141, & -4.19935141, \\
-3.8189587, & -3.8189587, & -3.73713403, & -3.56269421, \\
-3.29045932, & -3.29045932, & -2.82285377, & -2.77046619, \\
-2.6664803, & -2.34367463, & -2.34367463, & -2.25830655, \\
-1.57235606, & -1.33264629, & -0.97495884, & -0.95073415, \\
-0.72000298, & -0.60433008, & -0.60433008, & -0.35385596, \\
-0.35385596, & -0.30121246, & 0.24745974, & 0.42622305, \\
0.42622305, & 1.53815809, & 1.53815809, & 2.11331664, \\
2.66257058, & 2.67917741, & 2.84503095, & 3.28174244, \\
3.98504752, & 3.98504752, & 4.56654951, & 4.56654951, \\
7.91874659, & 7.91874659, & 9.46778072, & 9.54778863, \\
11.04559601, & 11.04559601, & 23.59996913, & 35.5856115,
\end{array}\right\}$$

Energy of Hypergraph is the sum of absolute values of all eigen values.

Energy= $\sum \mid \lambda_i \mid =$ **301.9821785666297**

Definition 8 (Laplacian energy of Knight's Hypergraph) Laplacian degree of a vertex $v_i \in V(H)$ is $\delta_l(v_i) = \sum_{j=1}^{m} a_{ij}$. The Laplacian matrix of a hypergraph H is denoted by $L = L(H)$ and is defined as $L = D - A$ where $D = diag(\delta_l(v_1), \delta_l(v_2), ..., \delta_l(v_n))$.
-9 Here, from the adjacency matrix (figure 4)

We get

$$\begin{aligned}
\delta_l(v_i) &= \textstyle\sum_{j=1}^{m} a_{ij} \\
&= \left\{\begin{array}{l}
12,\ 18,\ 23,\ 26,\ 26,\ 23,\ 18,\ 12, \\
18,\ 24,\ 32,\ 37,\ 37,\ 32,\ 24,\ 18, \\
23,\ 32,\ 42,\ 48,\ 48,\ 42,\ 32,\ 23, \\
26,\ 37,\ 48,\ 56,\ 56,\ 48,\ 37,\ 26, \\
26,\ 37,\ 48,\ 56,\ 56,\ 48,\ 37,\ 26, \\
23,\ 32,\ 42,\ 48,\ 48,\ 42,\ 32,\ 23, \\
18,\ 24,\ 32,\ 37,\ 37,\ 32,\ 24,\ 18, \\
12,\ 18,\ 23,\ 26,\ 26,\ 23,\ 18,\ 12
\end{array}\right\}
\end{aligned}$$

Now,
$L = D - A$ where $D = diag(\delta_l(v_1), \delta_l(v_2), ..., \delta_l(v_n))$.

$$L = \begin{bmatrix} 12 & 0 & -1 & \dots & 0 & 0 & 0 \\ 0 & 18 & 0 & \dots & 0 & 0 & 0 \\ -1 & 0 & 23 & \dots & 0 & 0 & 0 \\ . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & \dots & 23 & 0 & -1 \\ 0 & 0 & 0 & \dots & 0 & 18 & 0 \\ 0 & 0 & 0 & \dots & -1 & 0 & 12 \end{bmatrix}$$

Now using python programming , we find the Laplacian eigen values and hence Laplacian energy . Laplacian eigen values are

$$\left\{ \begin{array}{llll} 0, & 9.16738846, & 10.9649665, & 10.9649665. \\ 12.4932678, & 12.8641821, & 14.6345557, & 14.6345557, \\ 17.0886644, & 18.0450611, & 18.0450611, & 18.4121099, \\ 18.7205134, & 18.8615491, & 20.0355417, & 20.0355417, \\ 22.4650656, & 22.4650656, & 22.5866526, & 22.8944773, \\ 22.9819833, & 23.3428841, & 23.3428841, & 24.6519214, \\ 25.6647137, & 25.7132628, & 25.7132628, & 26.7049696, \\ 26.7482473, & 27.4884118, & 28.3998092, & 28.5706526, \\ 28.5706526, & 29.0547667, & 30.5561609, & 30.5561609, \\ 33.7916809, & 33.8355291, & 33.9952090, & 33.9952090, \\ 36.4063523, & 36.4063523, & 37.3230872, & 38.0299265, \\ 39.5749735, & 39.7243451, & 39.7243451, & 40.4252155, \\ 42.7317928, & 44.1613902, & 44.8193037, & 44.8193037, \\ 48.0742861, & 48.7552547, & 48.9079869, & 48.9079869, \\ 50.3525104, & 50.5428064, & 52.7263458, & 52.7263458, \\ 57.1899290, & 58.4279898, & 59.0923061, & 59.0923061 \end{array} \right\}$$

$$\begin{aligned} \text{Laplacian energy} &= \text{Sum of Laplacian eigen values} \\ &= \mathbf{2008.0000000000016} \end{aligned}$$

5 Conclusion

As we all know knight is one of the most dangerous pieces in a chess board as it can jump over the pieces and due to its 'L' shape pattern of movements.And the same pattern makes interesting and challenging while analysing its graph theoretical aspects.Throughout this paper we focused on knight's movements as a hypergraph and finally successful in analysisng and evaluating its eigen spectrum and energy.

6 Open Problems

1. Compare the eigen spectra and connectedness of Knight's graph and Knight's Hypergraph.

2. Obtain the eigen spectra of hypergraphs of movements of Bishop, Rook, Queen in a chessboard and compare their connectiveness with thsat og Knight's hypergraph.

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