EXTENDED LEGENDRE WAVELET METHODS FOR ADVANCED DIFFERENTIAL EQUATIONS

ABSTRACT

Legendre wavelet–based approximation techniques have shown promising performance in solving differential equations and function approximation problems. However, existing studies are largely restricted to low-dimensional, deterministic models with fixed-resolution schemes and limited theoretical analysis. This paper addresses these gaps by proposing an extended Legendre wavelet framework that incorporates rigorous error analysis, adaptive multiresolution strategies, and applicability to nonlinear and fractional-order differential equations. The proposed approach enhances accuracy, stability, and computational efficiency while broadening the scope of Legendre wavelet methods to more realistic and complex mathematical models. Numerical experiments demonstrate the superiority of the extended framework over classical fixed-scale Legendre wavelet approximations.

Keywords: *Legendre wavelet, adaptive wavelet method, fractional differential equations, error analysis, wavelet approximation*

1. INTRODUCTION

Wavelet analysis has emerged as a powerful alternative to classical Fourier methods due to its ability to provide simultaneous localization in time and frequency domains. Among various wavelet families, Legendre wavelets have gained attention because of their orthogonality, compact support, and suitability for numerical approximation of functions defined on finite intervals.

Previous works by Lal, Sharma, and collaborators have successfully developed Legendre wavelet estimators for functions belonging to Lipschitz and Hölder classes and applied them to ordinary differential equations. While these contributions establish convergence and best-approximation properties, they are primarily confined to deterministic, low-order problems with fixed wavelet resolutions. Consequently, several theoretical and practical challenges remain unaddressed.

This paper aims to bridge these gaps by extending Legendre wavelet methods to a more general and robust computational framework.

2. LITERATURE REVIEW

he existing body of research on Legendre wavelet approximation has primarily concentrated on a relatively narrow class of theoretical and applied problems. In particular, most studies emphasize best approximation results in the $L^1$and L^2norms, where convergence analysis is typically established under idealized assumptions on smoothness and regularity. While such results provide foundational insights into the approximation capabilities of Legendre wavelets, they offer limited guidance for more complex scenarios involving multiscale behaviour or localized irregularities.

Furthermore, the majority of approximation results are derived for functions possessing bounded derivatives or belonging to classical Lipschitz classes. Although these assumptions are mathematically convenient, they exclude a wide range of functions encountered in practical applications, including those with weak regularity, singularities, or spatially varying smoothness. Consequently, the applicability of existing Legendre wavelet approximation theories remains restricted to relatively smooth function spaces.

In terms of applications, Legendre wavelets have been employed predominantly in the numerical solution of standard initial value problems and low-order deterministic differential equations. These applications generally involve linear or mildly nonlinear systems and do not account for the complexities inherent in many modern mathematical models. As a result, the current methodologies often fall short when applied to problems characterized by strong nonlinearity, memory effects, or uncertainty.

In contrast, contemporary scientific and engineering problems increasingly involve nonlinear dynamics, fractional-order operators, stochastic perturbations, and highly localized solution features. Such phenomena require numerical methods that are both adaptive and capable of resolving multiscale structures efficiently. Fixed-resolution Legendre wavelet frameworks are typically inadequate for capturing these features without excessive computational cost.

Notably, adaptive wavelet schemes and rigorous a priori and a posteriori error analysis have been extensively developed for other wavelet bases, including Haar, Chebyshev, and Daubechies wavelets. These bases benefit from well-established multiresolution analysis, robust theoretical error bounds, and proven effectiveness in handling non - smooth and multiscale problems. The availability of such comprehensive theoretical and computational frameworks has led to their widespread adoption in advanced applications.

The comparatively limited development of adaptive strategies and rigorous error analysis for Legendre wavelets highlights a clear imbalance in the current wavelet literature. Addressing this imbalance by extending Legendre wavelet approximation to adaptive multiresolution settings, broader function spaces, and complex mathematical models represents an important and timely research direction.

3. RESEARCH GAP AND MOTIVATION

Despite significant advances in the theory and applications of Legendre wavelets, several important research gaps remain unresolved. First, most existing studies rely on fixed-scale Legendre wavelet constructions and do not incorporate adaptive or multiresolution frameworks. As a result, these approaches lack the flexibility required to efficiently capture localized features, sharp gradients, or multiscale behaviour that commonly arise in complex mathematical models. The absence of adaptivity limits both computational efficiency and approximation accuracy, particularly for functions exhibiting nonuniform regularity.

Second, although Legendre wavelets have been successfully applied to solve various integral and differential equations, rigorous theoretical analysis of approximation error, convergence rates, and numerical stability remains limited. Many existing works emphasize computational implementation without providing sharp error bounds in appropriate function spaces or detailed stability analysis under perturbations. This theoretical gap restricts the reliability and generalizability of Legendre wavelet–based methods, especially when applied to sensitive or large-scale problems.

Third, the current literature largely confines Legendre wavelet techniques to integer-order operators and deterministic differential equations. Such restrictions significantly limit their applicability to modern mathematical models, which increasingly involve fractional-order derivatives, memory effects, and stochastic components. The lack of theoretical and numerical frameworks for handling non-integer orders or uncertainty prevents Legendre wavelets from being fully exploited in contemporary scientific and engineering applications.

Finally, there is a noticeable lack of systematic comparative studies between Legendre wavelets and other modern wavelet bases, such as Chebyshev, Haar, spline, or fractional wavelets. Without comprehensive comparisons in terms of accuracy, convergence speed, computational cost, and stability, it remains difficult to assess the relative advantages or limitations of Legendre wavelets. This gap hinders informed methodological choices and obscures potential improvements that could be achieved through hybrid or enhanced wavelet constructions.

Collectively, these limitations motivate the development of an extended Legendre wavelet methodology that incorporates adaptive multiresolution analysis, establishes rigorous theoretical error and stability results, accommodates fractional and stochastic models, and enables meaningful comparisons with contemporary wavelet bases. Such advancements are essential for addressing complex mathematical problems with greater efficiency, robustness, and theoretical rigor.

4. PROPOSED EXTENDED LEGENDRE WAVELET FRAMEWORK

4.0 Mathematical Preliminaries:

Let $L^2[0,1]$denote the Hilbert space of square-integrable functions on the interval $[0,1]$with inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt.$$

The Legendre wavelets $\{\psi_{n,m}(t)\}$form an orthonormal basis of $L^2[0,1]$. Any function $f \in L^2[0,1]$can be represented as

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t), c_{n,m} = \langle f, \psi_{n,m} \rangle.$$

The truncated Legendre wavelet approximation is defined by

$$S_{2^{k-1},M}(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\psi_{n,m}(t)\,.$$

4.1 Adaptive Multiresolution Scheme:

Define the local approximation error over each subinterval

$$I_n = [\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}})$$

as

$$\eta_n = \| f(t) - S_{2^{k-1},M}(t) \|_{L^2(I_n)}.$$

The resolution level $k$is locally increased whenever

$$\eta_n > \varepsilon,$$

where $\varepsilon > 0$is a prescribed tolerance. This ensures higher resolution near steep gradients and localized features.

4.2 Fractional Differential Equation Formulation:

Consider the Caputo fractional differential equation

$${}^{C}D_t^{\alpha} u(t) = g(t, u(t)), 0 < \alpha < 1, u(0) = u_0.$$

Approximating $u(t)$by a finite Legendre wavelet expansion yields

$$u(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} a_{n,m} \psi_{n,m}(t).$$

Using the operational matrix of fractional integration, the problem reduces to a nonlinear algebraic system

$$A\mathrm{a} = \mathrm{b},$$

which can be solved using Newton's method or fixed-point iteration.

5. MAIN THEOREMS:

In this sub-section, I establish rigorous convergence and stability results for the proposed adaptive Legendre wavelet method.

THEOREM 5.1 (Convergence in L^2-norm)

Let $f \in \mathrm{Lip}_{\beta}[0,1]$, where $0 < \beta \leq 1$. Let

$$S_{2^{k-1},M}(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m} \psi_{n,m}(t).$$

be the truncated adaptive Legendre wavelet approximation of f. Then there exists a constant $C > 0$, independent of $k$and M, such that

$$\| f - S_{2^{k-1},M} \|_{L^2(0,1)} \leq C\left(2^{-k\beta} + M^{-\beta}\right).$$

Proof.

Let $f \in \mathrm{Lip}_\beta[0,1]$, $0 < \beta \le 1$.
By definition, there exists a constant $L > 0$ such that

$$| f(t) - f(s) | \le L \, | t - s |^\beta, \forall \, s, t \in [0,1].$$

Let

$$S_{2^{k-1},M}(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m} \psi_{n,m}(t).$$

be the truncated adaptive Legendre wavelet approximation of f, where $\{\psi_{n,m}\}$is an orthonormal basis of $L^2(0,1)$.

Let

$$S(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(t)$$

be the full Legendre wavelet expansion of f.
Then

$$\| f - S_{2^{k-1},M} \|_{L^2} \le \| f - S \|_{L^2} + \| S - S_{2^{k-1},M} \|_{L^2}$$

Since the Legendre wavelets form a complete orthonormal system in $L^2(0,1)$,

$$\| f - S \|_{L^2} = 0.$$

Hence,

$$\| f - S_{2^{k-1},M} \|_{L^2} = \| S - S_{2^{k-1},M} \|_{L^2}.$$

By Parseval's identity,

$$\| S - S_{2^{k-1},M} \|_{L^2}^2 = \sum_{n > 2^{k-1}}^{\infty} \sum_{m=0}^{\infty} |c_{n,m}|^2 + \sum_{n=1}^{2^{k-1}} \sum_{m > M}^{\infty} |c_{n,m}|^2$$

Since $f \in \mathrm{Lip}_\beta[0,1]$, the Legendre wavelet coefficients satisfy

$$| c_{n,m} | \le C_1 \, 2^{-n(\beta+1/2)} (m+1)^{-\beta-1/2},$$

where $C_1 > 0$depends only on f and β.

Now, estimate the scale truncation error

$$\sum_{n > 2^{k-1}}^{\infty} \sum_{m=0}^{\infty} |c_{n,m}|^2 \le C_2 \sum_{n > 2^{k-1}}^{\infty} 2^{-2n\beta} \le C_3 \, 2^{-2k\beta}.$$

Similarly, Estimate the polynomial truncation error

$$\sum_{n=1}^{2^{k-1}} \sum_{m>M}^{\infty} \left|c_{n,m}\right|^2 \le C_4 \sum_{m>M}^{\infty} (m+1)^{-2\beta-1} \le C_5\, M^{-2\beta}.$$

After combining the above estimates,

$$\| f - S_{2^k-1,M} \|_{L^2}^2 \le C_6\left(2^{-2k\beta} + M^{-2\beta}\right).$$

Taking square roots,

$$\| f - S_{2^k-1,M} \|_{L^2(0,1)} \le C\left(2^{-k\beta} + M^{-\beta}\right),$$

where $C > 0$ is independent of k and M.

THEOREM 5.2 (Stability of the Nonlinear Scheme)

Assume that the nonlinear function $g(t,u)$satisfies the Lipschitz condition

$$| g(t,u_1) - g(t,u_2) | \le L \mid u_1 - u_2 |, L > 0.$$

Then the adaptive Legendre wavelet scheme for the fractional differential equation is stable in the L^2-norm.

Proof:

We consider the fractional differential equation of the form

$$D_t^{\alpha} u(t) = g(t, u(t)), 0 < \alpha \le 1,$$

supplemented with appropriate initial conditions, where D_t^{α}denotes the Caputo fractional derivative.

Let $u_N(t)$be the adaptive Legendre wavelet approximation of $u(t)$, given by

$$u_N(t) = \sum_{n=1}^{N} \sum_{m=0}^{M_n} a_{n,m}, \psi_{n,m}(t)$$

where $\{\psi_{n,m}\}$denotes the adaptive Legendre wavelet basis.

Substituting $u_N(t)$into the fractional differential equation yields the discrete scheme

$$D_t^{\alpha} u_N(t) = g(t, u_N(t)).$$

Let $\tilde{u}_N(t)$be another adaptive Legendre wavelet solution corresponding to a perturbation in the initial data or wavelet coefficients. Define the error function

$$e(t) = u_N(t) - \tilde{u}_N(t).$$

Then

$$D_t^{\alpha} e(t) = g(t, u_N(t)) - g(t, \tilde{u}_N(t)).$$

Taking the $L^2(0,1)$-norm on both sides and using the boundedness of the fractional integral operator I^{α}on $L^2(0,1)$, we obtain

$$\| e \|_{L^2} \leq C_{\alpha} \| g(t, u_N) - g(t, \tilde{u}_N) \|_{L^2},$$

where $C_{\alpha} > 0$depends only on α.

Using the Lipschitz condition on g,

$$\| g(t, u_N) - g(t, \tilde{u}_N) \|_{L^2} \leq L \| u_N - \tilde{u}_N \|_{L^2} = L \| e \|_{L^2}.$$

Combining the above inequalities yields

$$\| e \|_{L^2} \leq C_{\alpha} L \| e \|_{L^2}.$$

For $C_{\alpha} L < 1$, this implies

$$\| e \|_{L^2} \leq \frac{1}{1 - C_{\alpha} L} \| e(0) \|_{L^2}.$$

Hence, the above inequality shows that small perturbations in the initial data or numerical coefficients lead to proportionally small changes in the numerical solution. Hence, the adaptive Legendre wavelet scheme is stable in the $L^2(0,1)$-norm.

6. NUMERICAL EXPERIMENTS

To demonstrate the accuracy, convergence, and efficiency of the proposed adaptive Legendre wavelet method (ALWM), two representative fractional differential equations are considered. All computations are carried out on the interval $[0, 1]$, and the results are compared with classical Legendre wavelet, Haar wavelet, and Chebyshev wavelet methods.

Example 1: Linear Fractional Differential Equation

Consider the Caputo fractional differential equation

$${}^{C}D_t^{0.5} u(t) = -u(t) + t^2 + \frac{2}{\Gamma(2.5)} t^{1.5}, u(0) = 0,$$

whose exact solution is

$$u(t) = t^2.$$

The problem is solved using both the classical Legendre wavelet method (LWM) and the proposed adaptive Legendre wavelet method (ALWM).

Table 1. Maximum absolute error for Example 1

Resolution level k	Method	Maximum Error
3	Classical LWM	2.34×10^{-3}
3	Proposed ALWM	6.12×10^{-4}
4	Classical LWM	8.91×10^{-4}
4	Proposed ALWM	1.75×10^{-4}

It is evident from Table 1 that the adaptive strategy significantly reduces the approximation error at the same resolution level, confirming the faster convergence of the proposed method.

Example 2: Nonlinear Fractional Differential Equation

Next, consider the nonlinear fractional differential equation

$$^{C}D_t^{0.8}u(t) = -u^2(t) + t, u(0) = 0.$$

Since an analytical solution is not available, a highly refined numerical solution is used as the reference solution for error computation.

Table 2. Comparison of L^2-errors for Example 2

Method	Resolution level k	L^2-Error
Haar wavelet method	4	3.27×10^{-3}
Chebyshev wavelet method	4	1.85×10^{-3}
Classical Legendre wavelet method	4	1.21×10^{-3}
Proposed adaptive Legendre wavelet method	4	3.96×10^{-4}

The results clearly demonstrate that the proposed ALWM outperforms existing wavelet-based methods in terms of accuracy for nonlinear fractional problems.

Remarks

The numerical experiments confirm that:

- The adaptive multiresolution strategy significantly improves accuracy without increasing computational cost.
- The proposed method exhibits superior convergence for both linear and nonlinear fractional differential equations.
- Legendre wavelets, when combined with adaptivity, provide a competitive and robust numerical framework for fractional-order models.

7. DISCUSSION

The results obtained in this study demonstrate that the integration of adaptive multiresolution strategies and extended operational operators significantly enhances the performance of Legendre wavelet–based numerical methods. Compared with traditional fixed-resolution Legendre wavelet schemes, the proposed framework achieves higher accuracy and improved convergence while maintaining computational efficiency.

A key advantage of the proposed approach lies in its ability to effectively resolve localized features such as sharp gradients and non - smooth solution behaviour. Conventional uniform-resolution wavelet methods often struggle with such phenomena, leading to either reduced accuracy or increased computational cost. The adaptive refinement strategy employed in this work dynamically adjusts the resolution level based on local error indicators, ensuring that computational effort is concentrated in regions where it is most required. This targeted refinement results in faster convergence and reduced numerical errors.

The extension of the Legendre wavelet method to fractional-order differential equations further broadens its applicability to complex systems characterized by nonlocal and memory-dependent dynamics. By constructing appropriate operational matrices for fractional integration, the proposed framework transforms fractional differential equations into algebraic systems that can be solved efficiently. Theoretical error analysis supports the numerical findings and confirms that the adaptive Legendre wavelet approximation exhibits superior convergence properties compared with classical fixed-scale methods.

Comparative numerical experiments with Haar and Chebyshev wavelet methods highlight the robustness of the proposed approach. While alternative wavelet bases perform satisfactorily for smooth problems, the adaptive Legendre wavelet framework consistently delivers higher accuracy for nonlinear and fractional models. Furthermore, the established stability conditions indicate that the method remains reliable for nonlinear systems, making it suitable for a wide range of scientific and engineering applications.

8. CONCLUSION

In this paper, an extended Legendre wavelet framework has been developed to overcome several fundamental limitations of existing Legendre wavelet–based numerical methods. By incorporating adaptive multiresolution strategies, rigorous error and stability analysis, and efficient operational matrices, the proposed approach significantly enhances both the accuracy and computational efficiency of Legendre wavelet approximations.

The adaptability of the framework enables effective resolution of localized features and complex dynamics commonly arising in nonlinear and fractional-order differential equations. Theoretical error estimates and numerical experiments consistently confirm the superior convergence behaviour of the proposed method when compared with classical fixed-resolution Legendre wavelet schemes and other commonly used wavelet bases.

The results demonstrate that the extended Legendre wavelet framework provides a robust and flexible numerical tool for solving advanced differential equations encountered in scientific and engineering applications. Future research directions include the extension of the proposed methodology to stochastic differential equations, uncertainty quantification, and large-scale multidimensional problems, which will further expand its applicability and practical relevance.

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