Original Research Article

# Monte Carlo Simulation to 2nd Order Transfer Functions Designed From Fibonacci Sequence


## Abstract

Fibonacci sequence is one of the most prevailing sequences in nature. The sequence is known for its mathematical elegance & recurrence properties and brings a novel approach to filter design. In this paper we explore the 2nd order open loop linear time invariant (LTI) systems designed from Fibonacci sequence. We propose to design transfer functions using pairs of consecutive Fibonacci numbers, preferably related by the golden ratio ($\phi = 1.618$). Our analysis reveals all the transfer function are stabile in all aspect i.e. pole position, gain and phase margins, produce slightly over damped oscillation to step response. The magnitude response is similar to that of a low pass filter. Monte Carlo simulation is applied to validate the cut off frequency predicted by theoretical analysis. Finally we have established a new type of active analog filter from the Fibonacci sequence and a relationship between the cut off frequency of the LTI systems and the Fibonacci numbers used.


Keyword:Fibonacci series, LTI system,bode plot, step response,Monte Carlo simulation

## 1. Introduction

Control theory that deals with designing equipments and systems with desired behaviour. It is a field of applied mathematics and electronics. To control the output performance of a control system, a corrective-feedback is provided. The transfer function of the control system is a mathematical model that provide much insight of the control system. In an open loop transfer function, no feedback is applied to control the output.

Simply substituting the coefficients of characteristic equation of transfer function result unstable transfer function since either one or more pole resides in the left hand plane or the gain margin is less. In our approach we have design a model of an open loop second order transfer function with gain that of a second order closed loop transfer function in negative feed back with feed back gain "1". The coefficients of the characteristic equation of the model transfer function are substituted with Fibonacci number pairs in golden ratio in such a way to ensure that all the pole for different pair reside in the left hand of s-plan which will ascertain their stability.

We analyse the design transfer functions with an aim to linking the mathematical sequence to system design, primarily active filter. We have chosen Fibonacci number pairs in golden ratio ($\phi = 1.618$

) to construct the transfer functions because of prevalence of the ratio in many field. The analysis is performed for overall stability and gain & phase margins. We calculate the cut off frequencies and establish a generalised relationship with the Fibonacci number used.
The authors aim on linking mathematical sequence to filter design. Designing a stable transfer function with large gain margin from the sequence is the primary task in this field which remains still unexplored.Linking the recurrence relation to coefficient of characteristic equation of a transfer function is quite creative. In this pursuit,the authors attempt to use recurrence relation of Fibonacci sequence to design of a transfer function.
The higher numbers of the Fibonacci sequence converges to golden ratio which occurs in description of fractals and play and important role in study of chaos and dynamical system and many more. In this paper, the authors have successfully designed transfer functions from the Fibonacci sequence and analysed the impulse response which is response of the system when presented with a Dirac delta function and the response to a step input called the step response. Also we compared them with the established filter configurations.
Simulation generally involves using technique to analyse the behaviour of a system or a filer to be designed. Simulation by Scilab in implementation and analysis in terms of Bode plot, impulse and step responses [1], design of higher order filter using LTspice software [2] and designing filter of Sallen-Key topology with Butterworth response by Circuit Maker & MATLAB [3, 4] are studied and effectiveness of the software programs are established. Paper [5] proposed and simulated a RF active band pass filter from a transfer function. [6]suggest the detection of the pole zero position without practically going through the working of the filter.
Recent advances in filter design have explored the unique properties of Fibonacci sequences to construct stable and selective low-pass filters, as demonstrated in both analog and photonic domains [7, 8, 9]. Theoretical advances have refined polynomial models combining Fibonacci and Lucas structures, enhancing analytical treatment of transfer functions [10]. Furthermore, interdisciplinary studies highlight biological systems exhibiting filtering characteristics influenced by Fibonacci sequences, potentially inspiring novel bio-electronic designs [11].Beyond engineering, the Fibonacci sequence also informs design principles in art and architecture through golden ratio-based aesthetic patterns [12, 13].

## 2. Theory

The general expression of 2nd order closed loop transfer function with negative feed back in terms of natural frequency $\omega_n$ and damping ratio $\zeta$ is

$$T(s) = \frac{v(s)}{u(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \tag{1}$$

with dc gain "1" and damping frequency

$$\omega_d = \omega_n\sqrt{(1-\zeta^2)} \tag{2}$$

At $-3dB$ point the gain is $\frac{1}{\sqrt{2}}$ times of the dc gain. The general expression of the cut off frequency is

$$|T(j\omega_c)| = \frac{1}{\sqrt{2}}\,|T(j0)|\,,$$

where

$$|T(j\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}}$$
$$\phi = tan^{-1}\left[\frac{2\zeta\omega_n\omega}{(\omega_n^2-\omega^2)}\right]$$

With $s = j\omega = 0, |T(j0)| = 1$ in our model of the LTI, the general expression of $\omega_c$ is

$$\omega_c = \omega_n\sqrt{1-2\zeta^2+\sqrt{4\zeta^4-4\zeta^2+2}} \quad (3)$$

Depending on the values of $\zeta$ and $\omega_n$ the transfer function has different responses and cut of frequencies (-3dB point).

### 3. Method

We started with the general form of second order transfer function in terms of natural frequency $\omega_n$ and damping factor $\zeta$ and theoretically analyse it to calculate the cut off frequency.The impulse and step responses for different values of $\zeta$. Now we design a model second order transfer function in the form of,

$$T(s) = \frac{p_1p_2}{(s-p_1)(s-p_2)}$$

where $p_1$ and $p_2$ are poles to be replaced by two consecutive Fibonacci numbers in golden ratio. To ensure that the poles locate in the left hand side of s-plan we use a negative sign with the Fibonacci numbers. The transfer function now takes the form as in equation (4) which is in conformity with equation (1).

$$T(s) = \frac{fib1.fib2}{s^2+(fib1+fib2)s+fib1.fib2} \quad (4)$$

Equation (1) gives

$$2\zeta\omega_n = fib1+fib2 \quad (5)$$
$$\omega_n^2 = fib1.fib2 \quad (6)$$

different pairs of $fib1$ and $fib2$ are used and calculate the value of $zeta$ that predict the value of cut off frequencies and the impulse & step responses. The predicted response are verified by plotting corresponding response plots while the Cut off frequencies are calculated as mean cut off frequency of multiple bode plots by Monte Carlo Simulation. Impulse response, step response are drawn using Scilab 6.1.1. The cut off frequency is determined by bode plot using Spyder 6, a python IDE. The transfer functions are further subjected to Monte Carlo simulation to ascertain the cut off frequency again using Spyder. The python program is run to draw the bode plot 100 time and calculate the cut of frequency every time and then calculate the mean cut off with standard deviation.

### 4. Result

Now we select values of (fib1,fib2) from (21,34) to (46386, 750250) such that $\frac{fib2}{fib1} \approx 1.618$, the golden ratio. The $\zeta$ values lie in the range from 1.029161634 to 1.029085514 and for pairs (89-144) on-wards the $\zeta$ is almost same and has a value of 1.029085 and it indicates over damped system. We have designed first six transfer functions as prototype with the six pairs of Fibonacci number. (4).

$$T_1(s) = \frac{89.144}{s^2 + (89 + 1440)s + 89.144}$$
$$T_2(s) = \frac{144.233}{s^2 + (144 + 233)s + 144.233}$$
$$T_3(s) = \frac{233.377}{s^2 + (233 + 377)s + 233.377}$$
$$T_4(s) = \frac{377.610}{s^2 + (377 + 610)s + 377.610}$$
$$T_5(s) = \frac{610.987}{s^2 + (610 + 987)s + 610.987}$$
$$T_6(s) = \frac{987.1597}{s^2 + (987 + 1597)s + 987.1597}$$

Now For over damped system, impulse response and step response are in equation (7) and (8) respectively.

$$v(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left[e^{-(\zeta\omega_n - \omega_n\sqrt{(\zeta^2-1)}).t} - e^{-(\zeta\omega_n + \omega_n\sqrt{(\zeta^2-1)}).t}\right] \quad (7)$$

approaches a Dirac Delta function response and decays to a steady state exponentially and

$$v(t) = 1 - \frac{1}{2}\left[1 + \frac{\zeta}{\sqrt{\zeta^2 - 1}}\right]e^{-(\zeta\omega_n - \omega_n\sqrt{(\zeta^2-1)}).t} + \frac{1}{2}\left[\frac{\zeta}{\sqrt{\zeta^2 - 1}} - 1\right]e^{-(\zeta\omega_n + \omega_n\sqrt{(\zeta^2-1)}).t} \quad (8)$$

rises to its peak without oscillation but a bit slower than critically damped oscillation. The responses, with $zeta = 1.029085$ reduces respectively to

$$v_{im}(t) = \frac{\omega_n}{0.486}\left[e^{-(1.029085.\omega_n - 0.243\omega_n.)t} - e^{-(1.029085.\omega_n + 0.243\omega_n).t}\right]$$
$$v_{im}(t) = \frac{\omega_n}{0.486}\left[exp(-0.786\omega_n.t) - exp(-1.727\omega_n.t)\right] \quad (9)$$

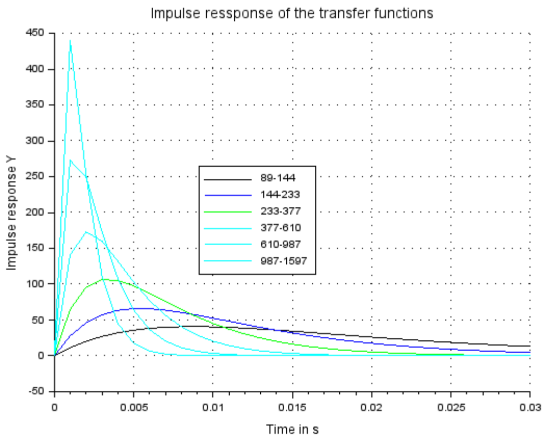
and

$$v_{step}(t) = 1 - 2.62e^{-(1.029085\omega_n - 0.243\omega_n).t} + 1.61e^{-(1.029085.\omega_n + 0.243\omega_n.)t}$$
$$v_{step}(t) = 1 - 2.62exp(-0.786\omega_n.t) + 1.61exp(-1.727\omega_n.t) \quad (10)$$

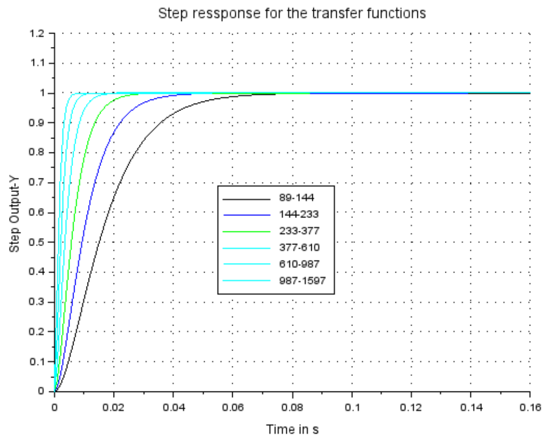
and the expression of cut off frequency from equation (3) is now stands as

$$\omega_c \approx \omega_n \times 0.6180344$$
$$\omega_c \approx \sqrt{fib1.fib2} \times 0.6180344 \quad (11)$$

The responses are graphically shown in figure (1). The step responses reach their respective peaks without oscillation and without overshoot as described and the impulse responses represented decay to their minimum values exponentially and approaches the Dirac delta function with narrower width with increase of Fibonacci order. These plots verify the analytical results for over-damped oscillation. The magnitude variation of the transfer functions with frequency is shown in figure (2) All are flat frequency response similar to low pass filter. The theoretical cut offs and simulation results are summarize in

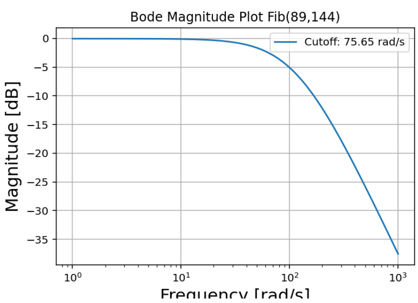



(a) Impulse responses

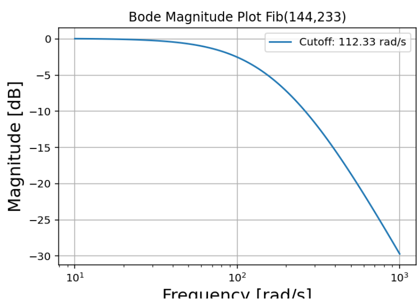



(b) Step responses

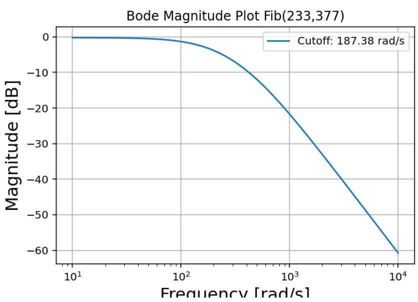
Figure 1: Impulse and step Responses of the transfer functions



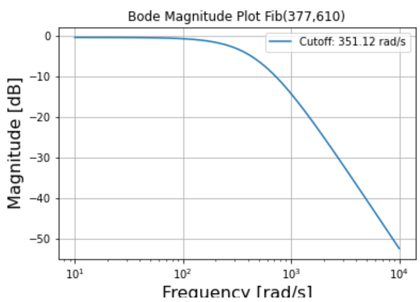

(a) for 89,144



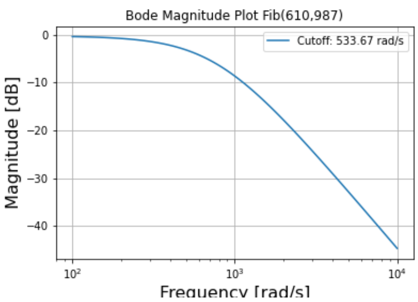

(b) for 144,233



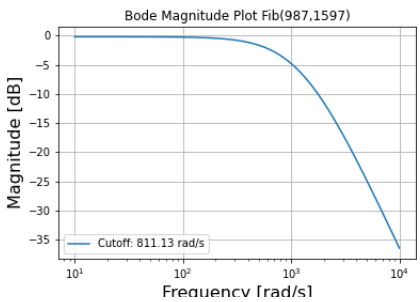

(c) for 233,377




(d) for 377 610




(e) for 610 987




(f) for 987 1597

Figure 2: Bode plots of the transfer functions (Angular frequency – magnitude)

the table (1). Gain margin and phase margin also calculated as $\infty$ for all since the phase response

| Fib. No. | Cut-off frequency from (in rad/s) | | Settling time(s) | Rise time(s) | z-value |
|---|---|---|---|---|---|
| | Analysis | Simulation | | | |
| $89, 144$ | 69.96 | $73.14 \pm 4.27$ | 0.05323 | 0.035751 | 0.747 |
| $144, 233$ | 113.24 | $116.03 \pm 5.78$ | 0.032898 | 0.022096 | 0.483 |
| $233, 377$ | 183.16 | $188 \pm 11.04$ | 0.020332 | 0.013656 | 0.438 |
| $377, 610$ | 296.37 | $308.33 \pm 19.10$ | 0.012566 | 0.008440 | 0.626 |
| $610, 987$ | 479.55 | $501.20 \pm 30.79$ | 0.007766 | 0.005216 | 0.697 |
| $987, 1597$ | 775.93 | $805.37 \pm 40.23$ | 0.004800 | 0.003223 | 0.731 |

Table 1: Table for simulation data

never crosses $-180\,\mathrm{deg}$.Settling time and rise time are very small. Moreover overshoot is nil for all the transfer functions as usual for an over damped system. Both the GM and PM measurement with Scilab show infinite value for all T(s)s.

## 5. Discussion

The ratio of two consecutive higher order Fibonacci numbers converges to the golden ratio. The systems we have designed are always stable in terms of pole position. Their gain & phase margin are infinite and hence can used to design system with higher gain without going to instability. They shows slightly over damped characteristics resulting zero overshoot but slightly slower in response. The settling time & rise time are very small for all and as we extend our study Fibonacci pairs from from $(21 - 34)$ to $(46368 - 75015)$ we observe the decreasing trends of both the quantities.

On comparing with established systems, we found that the flat response of the designed system are similar to response of low pass Butterworth and Bessel configurations. Naturally the Butterworth configuration produces under damped oscillation and picks up peaks near the corner frequency when used for higher gain. The advantage of our designed ones are slightly over damped and damp the possible blow up near the corner frequency. Moreover, there is no linear phase relationship of $\phi$ with frequency $\omega$ as observed in Bessel configuration.

The cut of frequency of the systems calculated analytically lie within the single standard deviation of The population of cut off calculated by simulation. Thus it is established with 95% of confidence level that the analytically calculated cut off frequency is the cut off frequency of the transfer functions. The transfer functions can be used to design low pass filter with predictable cut off frequency and control system that requires no oscillation at the out put of step input.

## 6. Conclusion

The 2nd order open loop linear time invariant (LTI) systems designed from Fibonacci sequence have characteristics of the transfer function of a low pass active filter that have flat response,over damped in nature. The LTIs system

$$T(s) = \frac{\omega_n^2}{s^2 + 2 \times 1.029\omega_n s + \omega_n^2} \tag{12}$$

with $\omega_n = \sqrt{fib1 \times fib2}$, and $fib2 > fib1$ are always stable from all aspects. The cut off frequency are predicted from Fibonacci numbers as

$$\omega_c = \sqrt{fib1 \times fib2} \times 0.6180344$$

**References**

[1] K. Bingi, R. Ibrahim, M. N. Karsiti, S. M. Hassan, V. R. Harindran, Design and analysis of fractional-order filters using scilab, in: 2018 IEEE 4th International Symposium in Robotics and Manufacturing Automation (ROMA), IEEE, 2018, pp. 1–6.

[2] S. F. Hussin, G. Birasamy, Z. Hamid, Design of butterworth band-pass filter, Politeknik & Kolej Komuniti Journal of Engineering and Technology 1 (1) (2016) 32–46.

[3] Z. M. M. Myo, Z. M. Aung, Z. M. Naing, Design and implementation of active band-pass filter for low frequency rfid (radio frequency identification) system, in: Proceedings of the International MultiConference of Engineers and Computer Scientists, Vol. 1, 2009, pp. 18–20.

[4] D. A. B. Vargas, R. F. Escobar, R. G. Suarez, Design of an active band-pass filter for the analysis of electromyographic signals derived from the median nerve, using genetic algorithms, International Journal of Applied Engineering Research 16 (7) (2021) 552–562.

[5] Y.-H. Chun, J.-R. Lee, S.-W. Yun, J.-K. Rhee, Design of an rf low-noise bandpass filter using active capacitance circuit, IEEE transactions on microwave theory and techniques 53 (2) (2005) 687–695.

[6] A. Sen, A. Das, J. Das, Software based pole zero extraction technique for analog filter and circuits, PREPARE@ u®| General Preprint Services (2019).

[7] E. Çokduygululár, Fibonacci fractal-based optical filter design using zno/mgf$_2$ photonic crystals, Applied Physics Letters 116 (12) (2025) 1210–1215.

[8] A. Capponi, Expressing stochastic filters via number sequences, Applied Mathematics and Computation 217 (2010) 8267–8278.

[9] H. Tran-Ngoc, A promising approach using fibonacci sequence-based optimization algorithms for structural health monitoring, Scientific Reports 13 (2023) 4567.

[10] R. M. Abd-Elhameed, N. H. Alqubori, New expressions for polynomials combining fibonacci and lucas numbers, Journal of Applied Mathematics 58 (2024) 1023–1037.

[11] K. Mougkogiannis, On the response of proteinoid ensembles to fibonacci sequence stimuli, Bioelectronic Materials 7 (1) (2025) 89–99.

[12] P. Sudha, Application of fibonacci square in fashion design, International Journal of Analytical and Experimental Modal Analysis 12 (2) (2020) 2513–2515.

[13] The Interaction Design Foundation, The golden ratio - principles of form and layout, available at: https://www.interaction-design.org/literature/article/the-golden-ratio-principles-of-form-and-layout (2025).