Dual Generalized Pandita Numbers

Abstract. In this paper, we introduce the generalized dual Pandita numbers defined over the bidimensional Clifford algebra of hyperbolic numbers. As special cases, we examine the dual Pandita and dual Pandita Lucas numbers. We derive Binet formulas, construct generating functions, and establish summation identities for these sequences. Furthermore, we present matrix representations associated with the proposed number sequences.

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1. Introduction

Dual numbers, first introduced by W.K. Clifford in 1873, represent a fascinating mathematical construct with a wide range of applications. They play a pivotal role in screw theory, the modeling of planar joints, and iterative techniques for displacement analysis in spatial mechanisms. Additionally, dual numbers are instrumental in the inertial force analysis of spatial systems and continue to find relevance in various branches of kinematics and robotics.Here are some general information about the applications of dual numbers.

- Engineering and Physics:

  Used in electrical engineering and control systems.

  Applied in wave analysis and signal processing.

  Utilized in mechanical engineering for vibration analysis, among other applications.

- Mathematics and Geometry:

  Alongside complex numbers, dual numbers contribute to the extension of mathematical structures.

  Employed in geometry to represent various transformations.

- Computer Science:

  Found in graphics and image processing.

  Used in robotics and control systems for modeling and analysis.

- Finance and Economics:

  Applied in risk analysis and financial engineering.

  Utilized in option pricing and portfolio management.

- Optimization Problems:

  Used for finding solutions in optimization problems.

  Acts as a tool in linear programming and decision-making models.

- Quantum Mechanics:

  Employed in quantum computers and quantum mechanics for mathematical representation.

Next, we give some information raleted to hypercomplex number system and then we give some properities about dual number. As discussed in [16], the hypercomplex numbers systems are extensions of real numbers. Some examples of hypercomplex number systems ,which is commutative , are complex numbers, hyperbolic numbers and dual numbers.

- Complex numbers are formed by extending the real number system with the imaginary unit, denoted as "i", which satisfies the equation $i^2 = -1$. Complex numbers is defined as follows,

$$C = \{z = a + ib : a, b \in R, i^2 = -1\}.$$

- As discussed in [30], hyperbolic numbers extend the real number system with the hyperbolic unit j, where $j^2 = 1$. Hyperbolic numbers is defined as follows,

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\}.$$

- As discussed in[11], dual numbers extend the real number system by introducing a new element ε, where $\varepsilon^2 = 0$. Dual numbers is defined as follows,

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

Let us now revisit the definition of generalized Pandita numbers.

A generalized Pandita sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$W_n = 2W_{n-1} - W_{n-2} + W_{n-3} - W_{n-4} \tag{1.1}$$

with the initial values W_0, W_1, W_2, W_3 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-1)} - W_{-(n-2)} + W_{-(n-3)} - W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

The initial values of the generalized Pandita numbers for both positive and negative subscripts are presented in Table 1.

Table 1. A few generalized Pandita numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$W_0 - W_1 + 2W_2 - W_3$
2	W_2	$W_1 + W_2 - W_3$
3	W_3	$W_0 + W_1 - W_2$
4	$W_1 - W_0 - W_2 + 2W_3$	$2W_0 - 2W_1 + 2W_2 - W_3$
5	$W_1 - 2W_0 - W_2 + 3W_3$	$3W_2 - 2W_3$
6	$W_1 - 3W_0 - 2W_2 + 5W_3$	$3W_1 - 2W_2$
7	$2W_1 - 5W_0 - 4W_2 + 8W_3$	$3W_0 - 2W_1$
8	$3W_1 - 8W_0 - 6W_2 + 12W_3$	$W_0 - 3W_1 + 6W_2 - 3W_3$
9	$4W_1 - 12W_0 - 9W_2 + 18W_3$	$5W_1 - 2W_0 - W_2 - W_3$
10	$6W_1 - 18W_0 - 14W_2 + 27W_3$	$3W_0 + W_1 - 5W_2 + 2W_3$
11	$9W_1 - 27W_0 - 21W_2 + 40W_3$	$4W_0 - 8W_1 + 8W_2 - 3W_3$
12	$13W_1 - 40W_0 - 31W_2 + 59W_3$	$4W_1 - 4W_0 + 5W_2 - 4W_3$
13	$19W_1 - 59W_0 - 46W_2 + 87W_3$	$9W_1 - 12W_2 + 4W_3$

If we set $W_0 = 0, W_1 = 1, W_2 = 2, W_3 = 3$ then $\{W_n\}$ is the well-known Pandita sequence and if we set $W_0 = 4, W_1 = 2, W_2 = 2, W_3 = 5$ then $\{W_n\}$ is the well-known Pandita -Lucas sequence. In other words, Pandita sequence $\{P_n\}_{n\geq 0}$ and Pandita -Lucas sequence $\{S_n\}_{n\geq 0}$ are defined by the second-order recurrence relations

$$P_n = 2P_{n-1} - P_{n-2} + P_{n-3} - P_{n-4}, \qquad P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 3, \qquad n \geq 4, \tag{1.2}$$

and

$$S_n = 2S_{n-1} - S_{n-2} + S_{n-3} - S_{n-4}, \qquad S_0 = 4, S_1 = 2, S_2 = 2, S_3 = 5, \qquad n \geq 4. \tag{1.3}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = P_{-(n-1)} - P_{-(n-2)} + 2P_{-(n-3)} - P_{-(n-4)}$$

and

$$S_{-n} = S_{-(n-1)} - S_{-(n-2)} + 2S_{-(n-3)} - S_{-(n-4)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.2) and (1.3) hold for all integer n.

We can list some important properties of generalized Pandita numbers that are needed.

- Binet formula of generalized Pandita sequence can be calculated using its characteristic equation which is given as

$$x^4 - 2x^3 + x^2 - x + 1 = (x^3 - x^2 - 1)(x - 1) = 0$$

The roots of characteristic equation are

$$\begin{aligned}
\alpha &= \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \\
\beta &= \frac{1}{3} + \omega\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \\
\gamma &= \frac{1}{3} + \omega^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \\
\delta &= 1,
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Using these roots and the recurrence relation, Binet formula can be given as

$$\begin{aligned}
W_n &= \frac{z_1\alpha^n}{3\alpha - 2} + \frac{z_2\beta^n}{3\beta - 2} + \frac{z_3\gamma^n}{3\gamma - 2} + z_4 \\
&= A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4,
\end{aligned}$$

where z_1, z_2 and z_3 are given below

$$\begin{aligned}
z_1 &= (\alpha W_3 - \alpha(2 - \alpha)W_2 + (-\alpha^2 + \alpha + 1)W_1 - W_0), \\
z_2 &= (\beta W_3 - \beta(2 - \beta)W_2 + (-\beta^2 + \beta + 1)W_1 - W_0), \\
z_3 &= (\gamma W_3 - \gamma(2 - \gamma)W_2 + (-\gamma^2 + \gamma + 1)W_1 - W_0), \\
z_4 &= -W_3 + W_2 + W_0.
\end{aligned}$$

and

$$\begin{aligned}
A_1 &= \frac{z_1}{3\alpha - 2}, \\
A_2 &= \frac{z_2}{3\beta - 2}, \\
A_3 &= \frac{z_3}{3\gamma - 2}, \\
A_4 &= z_4.
\end{aligned} \tag{1.4}$$

Binet formula of Pandita and Pandita Lucas sequences are

$$P_n = \frac{\alpha^{n+3}}{3\alpha-2} + \frac{\beta^{n+3}}{3\beta-2} + \frac{\gamma^{n+3}}{3\gamma-2} - 1,$$

and

$$S_n = \alpha^n + \beta^n + \gamma^n + 1,$$

respectively.

The generating function for generalized Pandita numbers is

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 2W_0)x + (W_2 - 2W_1 + W_0)x^2 + (W_3 - 2W_2 + W_1 - W_0)x^3}{1 - 2x + x^2 - x^3 + x^4}.$$

For more details about generalized Pandita numbers, see [36].

Next, we give the exponential generating function of $\sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n.

LEMMA 1. *[37, Lemma 1.4].Suppose that* $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ *is the exponential generating function of the generalized Pandita sequence* $\{W_n\}$.

Then $\sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \frac{(\alpha W_3 - \alpha(2-\alpha)W_2 + (-\alpha^2 + \alpha + 1)W_1 - W_0)}{3\alpha - 2} e^{\alpha x} \\ &+ \frac{(\beta W_3 - \beta(2-\beta)W_2 + (-\beta^2 + \beta + 1)W_1 - W_0)}{3\beta - 2} e^{\beta x} \\ &+ \frac{(\gamma W_3 - \gamma(2-\gamma)W_2 + (-\gamma^2 + \gamma + 1)W_1 - W_0)}{3\gamma - 2} e^{\gamma x} \\ &+ (-W_3 + W_2 + W_0)e^x.\end{aligned}$$

The previous Lemma 1 gives the following results as particular examples.

COROLLARY 2. *Exponential generating function of Pandita and Pandita-Lucas numbers*

a): $\sum_{n=0}^{\infty} P_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (\frac{\alpha^{n+3}}{3\alpha-2} + \frac{\beta^{n+3}}{3\beta-2} + \frac{\gamma^{n+3}}{3\gamma-2} - 1)\frac{x^n}{n!} = \frac{\alpha^3 e^{\alpha x}}{3\alpha-2} + \frac{\beta^3 e^{\beta x}}{3\beta-2} + \frac{\gamma^3 e^{\gamma x}}{3\gamma-2} - e^x.$

b): $\sum_{n=0}^{\infty} S_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (\alpha^n + \beta^n + \gamma^n + 1)\frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^{\gamma x} + e^x.$

Next, we provide an overview of selected publications in the literature that pertain to dual numbers.

- Göcen,Dikmen Kaya and Soykan [23] presented the dual generalized Fibonacci matrices as

$$DWn = \begin{pmatrix} W_{n+1} + \varepsilon W_{n+2} & W_n + \varepsilon W_{n+1} \\ W_n + \varepsilon W_{n+1} & W_{n-1} + \varepsilon W_n \end{pmatrix} = \begin{pmatrix} W_{n+1} + \varepsilon(Wn + 1 + Wn) & W_n + \varepsilon W_{n+1} \\ W_n + \varepsilon W_{n+1} & W_{n+1} - Wn + \varepsilon W_n \end{pmatrix}$$

with initial conditions $DW_0 = \begin{matrix} W_1 + \varepsilon(W_0 + W_1) & W_0 + \varepsilon W_1 \\ W_0 + \varepsilon W_1 & W_1 - W_0 + \varepsilon W_0 \end{matrix},$

$$DW_1 = \begin{matrix} W_0 + W_1 + \varepsilon(W_0 + 2W_1) & W_1 + \varepsilon(W_0 + W_1) \\ W_1 + \varepsilon(W_0 + W_1) & W_0 + \varepsilon W_1 \end{matrix} \quad \text{as } \varepsilon^2 = 0$$

.

- Halici [27] studied Dual Fibonacci Octonions as

$$p=\sum_{s=0}^{7}F_{n+s}e_s$$

where Fibonacci given by $F_n=F_{n-1}+F_{n-2},\ F_0=0,\ F_1=1.$

- Aydın [15] studied Dual Jacobsthal Quaternions as

$$QJ_{k;n}=J_{k;n}+i_1J_{k;n+1}+i_2J_{k;n+2}+i_3J_{k;n+3}$$

where $J_n=J_{n-1}+2J_{n-2},\ J_0=0,\ J_1=1.$

- Nurkan ,Guven, [10] studied Dual Fibonacci Quaternions as

$$\widetilde{Q}n=(F_n+F_{n+1})+i(F_{n+1}+F_{n+2})+j(F_{n+2}+F_{n+3})+k(F_{n+3}+F_{n+4})$$

where Fibonacci given by $F_n=F_{n-1}+F_{n-2},\ F_0=0,\ F_1=1.$

- Gürses, Şentürk, Yüce [26] studied dual-generalized complex Fibonacci and Lucas numbers, respectively, as

$$\begin{aligned}\widetilde{\mathcal{F}}_n &= F_n+jF_{n+1}+\varepsilon F_{n+2}+j\varepsilon F_{n+3},\\ \widetilde{\mathcal{L}}_n &= L_n+jL_{n+1}+\varepsilon L_{n+2}+j\varepsilon L_{n+3},\end{aligned}$$

where Fibonacci and Lucas numbers, respectively, given by $F_n=F_{n-1}+F_{n-2},\ F_0=0,\ F_1=1,\ L_n=L_{n-1}+L_{n-2},\ L_0=2,\ L_1=1.$

- Yılmaz and Soykan , [28] studied dual generalized Guglielmo numbers given by

$$\widetilde{W}_n=W_n+\varepsilon W_{n+1}$$

where generalized Guglielmo numbers are $W_n=3W_{n-1}-3W_{n-2}+W_{n-3}$ with the initial conditation $W_0,W_1,W_2\quad(n\geq 2).$

- Ayrılma and Soykan , [9] introduced On Dual Edouard Numbers are

$$DE_n=7DE_{n-1}-7DE_{n-2}+DE_{n-3}$$

where generalized Edouard numbers are $E_n=7E_{n-1}-7E_{n-2}+E_{n-3}$ with the initial conditation $E_0=0,E_1=1,E_2=7$

Following this, we provide details on dual hyperbolic sequences as they are presented in literature.

- Demirci and Soykan, [7] studied hyperbolic generalized Adrien numbers given by

$$HA_n = 3HA_{n-1} - HA_{n-2} - HA_{n-4}$$

where generalized Adrien numbers are $A_n = 3A_{n-1} - A_{n-2} + A_{n-4}$ with the initial conditation $A_0 = 0, A_1 = 1, A_2 = 3, A_3 = 8, \ \ n \geq 4$.

- Kalca and Soykan , [17] studied dual hyperbolic generalized Pandita numbers given by

$$\hat{P}_n = 2\hat{P}_{n-1} - \hat{P}_{n-2} + \hat{P}_{n-3} - \hat{P}_{n-4}$$

where generalized Pandita numbers are $P_n = 2P_{n-1} - P_{n-2} + P_{n-3} - P_{n-4}$ with the initial conditation $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 3, \ \ n \geq 4$.

In this paper, we define the dual generalized Pandita numbers in the next section and give some properties of them.

2. Dual Generalized Pandita Numbers and their Generating Functions and Binet's Formulas

In this section, we introduce the dual generalized Pandita numbers and derive their corresponding generating functions and Binet formulas. We now define the dual generalized Pandita numbers over the algebra $\mathbb{H}_{\mathbb{D}}$of dual dual numbers.The nth dual generalized Pandita number is

$$DW_n = W_n + \varepsilon W_{n+1}. \tag{2.1}$$

The sequence $\{DW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$DW_{-n} = W_{-n} + \varepsilon W_{-n+1},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.2) holds for all integer n.

Note that

$$\begin{aligned}
DW_0 &= W_0 + \varepsilon W_1, \\
DW_1 &= W_1 + \varepsilon W_2, \\
DW_2 &= W_2 + \varepsilon W_3, \\
DW_3 &= W_3 + \varepsilon W_4 = W_3 + \varepsilon (W_1 - W_0 - W_2 + 2W_3).
\end{aligned}$$

It can be easily shown that

$$DW_n = 2DW_{n-1} - DW_{n-2} + DW_{n-3} - DW_{n-4} \tag{2.2}$$

and

$$DW_{-n} = DW_{-(n-1)} - DW_{-(n-2)} + 2DW_{-(n-3)} - DW_{-(n-4)}$$

The initial values of the dual generalized Pandita numbers for both positive and negative subscripts are listed in Table 2.

table 2-A few dual generalized Pandita numbers

n	DW_n	DW_{-n}
0	DW_0	DW_0
1	DW_1	$DW_0 - DW_1 + 2DW_2 - DW_3$
2	DW_2	$DW_1 + DW_2 - DW_3$
3	DW_3	$DW_0 + DW_1 - DW_2$
4	$DW_1 - DW_0 - DW_2 + 2DW_3$	$2DW_0 - 2DW_1 + 2DW_2 - DW_3$
5	$DW_1 - 2DW_0 - DW_2 + 3DW_3$	$3DW_2 - 2DW_3$
6	$DW_1 - 3DW_0 - 2DW_2 + 5DW_3$	$3DW_1 - 2DW_2$
7	$2DW_1 - 5DW_0 - 4DW_2 + 8DW_3$	$3DW_0 - 2DW_1$
8	$3DW_1 - 8DW_0 - 6DW_2 + 12DW_3$	$DW_0 - 3DW_1 + 6DW_2 - 3DW_3$
9	$4DW_1 - 12DW_0 - 9DW_2 + 18DW_3$	$5DW_1 - 2DW_0 - DW_2 - DW_3$
10	$6DW_1 - 18DW_0 - 14DW_2 + 27DW_3$	$3DW_0 + DW_1 - 5DW_2 + 2DW_3$
11	$9DW_1 - 27DW_0 - 21DW_2 + 40DW_3$	$4DW_0 - 8DW_1 + 8DW_2 - 3DW_3$
12	$13DW_1 - 40DW_0 - 31DW_2 + 59DW_3$	$4DW_1 - 4DW_0 + 5DW_2 - 4DW_3$
13	$19DW_1 - 59DW_0 - 46DW_2 + 87DW_3$	$9DW_1 - 12DW_2 + 4DW_3$

As special cases, the nth dual dual Pandita numbers and the nth dual dual Pandita Lucas numbers are given as

$$DP_n = P_n + \varepsilon P_{n+1} \tag{2.3}$$

and

$$DS_n = S_n + \varepsilon S_{n+1} \tag{2.4}$$

respectively. The sequences $\{DP_n\}_{n\geq 0}$ and $\{DS_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$DP_{-n} = P_{-(n-1)} - P_{-(n-2)} + 2P_{-(n-3)} - P_{-(n-4)}$$

and

$$DS_{-n} = S_{-(n-1)} - S_{-(n-2)} + 2S_{-(n-3)} - S_{-(n-4)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.3) and (2.4) holds for all integer n.

For dual Pandita numbers (taking $W_n = P_n$, $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 3,$) we get

$$\begin{aligned} DP_0 &= \varepsilon, \\ DP_1 &= 2\varepsilon + 1, \\ DP_2 &= 3\varepsilon + 2 \end{aligned}$$

and for dual Pandita Lucas numbers (taking $W_n = S_n,\ \ S_0 = 4, S_1 = 2, S_2 = 2, S_3 = 5,$) we get

$$\begin{aligned} DS_0 &= 2\varepsilon + 4, \\ DS_1 &= 2\varepsilon + 2 \\ DS_2 &= 5\varepsilon + 2. \end{aligned}$$

Selected values of the dual Pandita numbers and dual Pandita Lucas numbers for both positive and negative subscripts are presented in Table 3 and Table 4, respectively.

Table 3. Dual Pandita numbers

n	DP_n	DP_{-n}
0	ε	ε
1	$2\varepsilon + 1$	0
2	$3\varepsilon + 2$	0
3	$5\varepsilon + 3$	-1
4	$8\varepsilon + 5$	$-\varepsilon - 1$
5	$12\varepsilon + 8$	$-\varepsilon$

Table 4. Dual Pandita- Lucas numbers

n	DS_n	DS_{-n}
0	$2\varepsilon + 4$	$2\varepsilon + 4$
1	$2\varepsilon + 2$	$4\varepsilon + 1$
2	$5\varepsilon + 2$	$\varepsilon - 1$
3	$6\varepsilon + 5$	$-\varepsilon + 4$
4	$7\varepsilon + 6$	$4\varepsilon + 3$
5	$11\varepsilon + 7$	$3\varepsilon - 4$

We now present the Binet formula for the dual generalized Pandita numbers, and for the remainder of the paper, we adopt the following notational conventions.

$$\widehat{\alpha} = 1 + \varepsilon\alpha, \tag{2.5}$$

$$\widehat{\beta} = 1 + \varepsilon\beta, \tag{2.6}$$

$$\widehat{\gamma} = 1 + \varepsilon\gamma \tag{2.7}$$

$$\widehat{\delta} = \widehat{1} = 1 + \varepsilon, \tag{2.8}$$

Note that we have the following identities:

$$\begin{aligned}
\widehat{\alpha}^2 &= 1+2\alpha\varepsilon, \\
\widehat{\beta}^2 &= 1+2\varepsilon\beta, \\
\widehat{\alpha}\widehat{\beta} &= 1+(\alpha+\beta)\varepsilon, \\
\widehat{\gamma}^2 &= 1+\gamma^2+2\varepsilon\gamma, \\
\widehat{\delta}^2 &= \widehat{1}^2 = 2+2\varepsilon, \\
\widehat{\gamma}\widehat{\delta} &= 1+\varepsilon(1+\gamma).
\end{aligned}$$

THEOREM 3. *(Binet's Formula) For any integer n, the nth dual generalized Pandita number is*

$$DW_n = A_1\alpha^n\widehat{\alpha} + A_2\beta^n\widehat{\beta} + A_3\gamma^n\widehat{\gamma} + \widehat{1}A_4. \tag{2.9}$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}, \widehat{\delta}$ are given as (2.5)-(2.8)

Proof. Using Binet's formula of the generalized Pandita numbers given below

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4.$$

where A_1, A_2, A_3, A_4 are given (1.4) we get

$$\begin{aligned}
DW_n &= W_n + \varepsilon W_{n+1}, \\
&= A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4 + \varepsilon(A_1\alpha^{n+1} + A_2\beta^{n+1} + A_3\gamma^{n+1} + A_4) \\
&= A_1\alpha^n(1+\varepsilon\alpha) + A_2\beta^n(1+\varepsilon\beta) + A_3\gamma^n(1+\varepsilon\gamma) + A_4(1+\varepsilon) \\
&= A_1\alpha^n\widehat{\alpha} + A_2\beta^n\widehat{\beta} + A_3\gamma^n\widehat{\gamma} + \widehat{1}A_4.
\end{aligned}$$

This proves (2.9).

As special cases, for any integer n, the Binet's Formula of nth dual Pandita number is

$$DP_n = \frac{\alpha^{n+3}\widehat{\alpha}}{3\alpha-2} + \frac{\beta^{n+3}\widehat{\beta}}{3\beta-2} + \frac{\gamma^{n+3}\widehat{\gamma}}{3\gamma-2} - \widehat{1} \tag{2.10}$$

and the Binet's Formula of nth dual Pandita Lucas number is

$$DS_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n + \widehat{\gamma}\gamma^n + \widehat{1}, \tag{2.11}$$

Next, we present generating function.

THEOREM 4. *The generating function for the dual generalized Pandita numbers is*

$$f_{DW_n}(x) = \sum_{n=0}^{\infty} DW_n x^n = \frac{DW_0 + (DW_1 - 2DW_0)x + (DW_2 - 2DW_1 + DW_0)x^2 + (DW_3 - 2DW_2 + DW_1 - DW_0)x^3}{1-2x+x^2-x^3+x^4}.$$

Proof. Using the definition of dual dual Pandita numbers, and substracting $xf(x)$, $x^2f(x)$ and $x^3f(x)$ from $f(x)$ we obtain $(1-2x+x^2-x^3+x^4)f_{DW_n}(x)$

$$
\begin{aligned}
&(1-2x+x^2-x^3+x^4)f_{DW_n}(x)\\
=\;&\sum_{n=0}^{\infty}DW_nx^n-2x\sum_{n=0}^{\infty}DW_nx^n+x^2\sum_{n=0}^{\infty}DW_nx^n-x^3\sum_{n=0}^{\infty}DW_nx^n+x^4\sum_{n=0}^{\infty}DW_nx^n,\\
=\;&\sum_{n=0}^{\infty}DW_nx^n-2\sum_{n=0}^{\infty}DW_nx^{n+1}+\sum_{n=0}^{\infty}DW_nx^{n+2}-\sum_{n=0}^{\infty}DW_nx^{n+3}+\sum_{n=0}^{\infty}DW_nx^{n+4},\\
=\;&\sum_{n=0}^{\infty}DW_nx^n-2\sum_{n=1}^{\infty}DW_{(n-1)}x^n+\sum_{n=2}^{\infty}DW_{(n-2)}x^n-\sum_{n=3}^{\infty}DW_{(n-3)}x^n+\sum_{n=4}^{\infty}DW_{(n-4)}x^n,\\
=\;&(DW_0+DW_1x+DW_2x^2+DW_3x^3)-2(DW_0x+DW_1x^2+DW_2x^3)+(DW_0x^2+DW_1x^3)-DW_0x^3\\
&+\sum_{n=4}^{\infty}(DW_n-2DW_{n-1}-DW_{n-2}-DW_{n-3}+DW_{n-4})x^n,\\
=\;&DW_0+(DW_1-2DW_0)x+(DW_2-2DW_1+DW_0)x^2+(DW_3-2DW_2+DW_1-DW_0)x^3.
\end{aligned}
$$

And rearranging above equation, we get (4). □

The following results are immediate consequences of the preceding Theorem.

COROLLARY 5. *For all integers n, we have following identities:*

a): $\sum_{n=0}^{\infty}DP_nx^n=\dfrac{x+\varepsilon}{1-2x+x^2-x^3+x^4}$.

b): $\sum_{n=0}^{\infty}DS_nx^n=\dfrac{(-4\varepsilon-1)x^3+(3\varepsilon+2)x^2+(-2\varepsilon-6)x+2\varepsilon+4}{1-2x+x^2-x^3+x^4}$.

Theorem (4) gives the following results as special cases,

$(1-2x+x^2-x^3+x^4)f_{DP_n}(x)=DP_0+(DP_1-2DP_0)x+(DP_2-2DP_1+DP_0)x^2+(DP_3-2DP_2+DP_1-DP_0)x^3=x+\varepsilon$,

$(1-2x+x^2-x^3+x^4)f_{DS_n}(x)=DS_0+(DS_1-2DS_0)x+(DS_2-2DS_1+DS_0)x^2+(DS_3-2DS_2+DS_1-DS_0)x^3=(-4\varepsilon-1)x^3+(3\varepsilon+2)x^2+(-2\varepsilon-6)x+2\varepsilon+4$.

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty}DW_n\frac{x^n}{n!}$ of the sequence DW_n.

LEMMA 6. *Suppose that $f_{DW_n}(x)=\sum\limits_{n=0}^{\infty}DW_n\frac{x^n}{n!}$ is the exponential dual generating function of the generalized Pandita sequence $\{DW_n\}$.*

Then $\sum\limits_{n=0}^{\infty}DW_n\frac{x^n}{n!}$ is given by

$$\sum_{n=0}^{\infty}DW_n\frac{x^n}{n!}==A_1e^{\alpha x}\widehat{\alpha}+A_2e^{\beta x}\widehat{\beta}+A_3e^{\gamma x}\widehat{\gamma}+A_4e^{x}\widehat{1}.$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}$, $\widehat{\delta}$ are given as (2.5)-(2.8)

Proof. Using Binet's formula

$$W_n=A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4.$$

where A_1, A_2, A_3, A_4 are given in (1.4) we get

$$\begin{aligned}
\sum_{n=0}^{\infty} DW_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty} W_n \frac{x^n}{n!} + \varepsilon \sum_{n=0}^{\infty} W_{n+1} \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} (A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4)\frac{x^n}{n!} + \varepsilon \sum_{n=0}^{\infty} (A_1\alpha^{n+1} + A_2\beta^{n+1} + A_3\gamma^{n+1} + A_4)\frac{x^n}{n!} \\
&= (A_1e^{\alpha x} + A_2e^{\beta x} + A_3e^{\gamma x} + A_4e^{x}) + \varepsilon(A_1\alpha e^{\alpha x} + A_2\beta e^{\beta x} + A_3\gamma e^{\gamma x} + A_4e^{x}) \\
&= A_1e^{\alpha x}(1+\varepsilon\alpha) + A_2e^{\beta x}(1+\varepsilon\beta) + A_3e^{\gamma x}(1+\varepsilon\gamma) + A_4e^{x}(1+\varepsilon) \\
&= A_1e^{\alpha x}\widehat{\alpha} + A_2e^{\beta x}\widehat{\beta} + A_3e^{\gamma x}\widehat{\gamma} + A_4e^{x}\widehat{1}
\end{aligned}$$

This proves (6). □

The previous Lemma 6 gives the following results as particular examples.

COROLLARY 7. *Exponential dual generating function of Pandita and Pandita-Lucas numbers are*

a): $\sum_{n=0}^{\infty} DP_n \frac{x^n}{n!} = \frac{\alpha^3 e^{\alpha x}\widehat{\alpha}}{3\alpha-2} + \frac{\beta^3 e^{\beta x}\widehat{\beta}}{3\beta-2} + \frac{\gamma^3 e^{\gamma x}\widehat{\gamma}}{3\gamma-2} - e^{x}\widehat{1}.$

b): $\sum_{n=0}^{\infty} DS_n \frac{x^n}{n!} = e^{\alpha x}\widehat{\alpha} + e^{\beta x}\widehat{\beta} + e^{\gamma x}\widehat{\gamma} + e^{x}\widehat{1}.$

3. Obtaining Binet Formula From Generating Function

We next find Binet's formula generalized dual Pandita number $\{DW_n\}$ by the use of generating function for DW_n.

THEOREM 8. *Binet's formula of generalized dual Pandita numbers:*

$$DW_n = \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}. \tag{3.1}$$

where

$$\begin{aligned}
q_1 &= DW_0\alpha^3 + (DW_1 - 2DW_0)\,\alpha^2 + (DW_0 - 2DW_1 + DW_2)\,\alpha - DW_0 + DW_1 - 2DW_2 + DW_3, \\
q_2 &= DW_0\beta^3 + (DW_1 - 2DW_0)\,\beta^2 + (DW_0 - 2DW_1 + DW_2)\,\beta - DW_0 + DW_1 - 2DW_2 + DW_3, \\
q_3 &= DW_0\gamma^3 + (DW_1 - 2DW_0)\,\gamma^2 + (DW_0 - 2DW_1 + DW_2)\,\gamma - DW_0 + DW_1 - 2DW_2 + DW_3, \\
q_4 &= DW_0\delta^3 + (DW_1 - 2DW_0)\,\delta^2 + (DW_0 - 2DW_1 + DW_2)\,\delta - DW_0 + DW_1 - 2DW_2 + DW_3.
\end{aligned}$$

Proof. Let

$$h(x) = x^4 - x^3 + x^2 - 2x + 1.$$

Then for some α, β, γ and δ we write

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x).$$

i.e.,

$$x^4 - x^3 + x^2 - 2x + 1 = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x). \tag{3.2}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$ are the roots of $h(x)$. This gives α, β, γ and δ as the roots of

$$h(\frac{1}{x}) = \frac{1}{x^2} - \frac{2}{x} - \frac{1}{x^3} + \frac{1}{x^4} + 1 = 0.$$

This implies $x^4 - x^3 + x^2 - 2x + 1 = 0$. Now, by it follows that

$$\sum_{n=0}^{\infty} DW_n x^n = \frac{(DW_1 - DW_0 - 2DW_2 + DW_3)\, x^3 + (DW_0 - 2DW_1 + DW_2)\, x^2 + (DW_1 - 2DW_0)\, x + DW_0}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)}.$$

Then we write

$$\frac{(DW_1 - DW_0 - 2DW_2 + DW_3)\, x^3 + (DW_0 - 2DW_1 + DW_2)\, x^2 + (DW_1 - 2DW_0)\, x + DW_0}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)} \tag{3.3}$$

$$= \frac{B_1}{(1 - \alpha x)} + \frac{B_2}{(1 - \beta x)} + \frac{B_3}{(1 - \gamma x)} + \frac{B_4}{(1 - \delta x)}. \tag{3.4}$$

So

$$\begin{aligned}
& (DW_1 - DW_0 - 2DW_2 + DW_3)\, x^3 + (DW_0 - 2DW_1 + DW_2)\, x^2 + (DW_1 - 2DW_0)\, x + DW_0 \\
= \; & B_1(1 - \beta x)(1 - \gamma x)(1 - \delta x) + B_2(1 - \alpha x)(1 - \gamma x)(1 - \delta x) \\
& + B_3(1 - \alpha x)(1 - \beta x)(1 - \delta x) + B_3(1 - \alpha x)(1 - \beta x)(1 - \gamma x).
\end{aligned}$$

If we consider $x = \frac{1}{\alpha}$, we get $DW_0 + \frac{1}{\alpha^2}(DW_0 - 2DW_1 + DW_2) - \frac{1}{\alpha^3}(DW_0 - DW_1 + 2DW_2 - DW_3) + \frac{1}{\alpha}(DW_1 - 2DW_0) = -B_1\left(\frac{1}{\alpha}\beta - 1\right)\left(\frac{1}{\alpha}\gamma - 1\right)\left(\frac{1}{\alpha}\delta - 1\right).$

This gives

$$\begin{aligned}
B_1 &= \alpha^3 (DW_0 + \frac{1}{\alpha^2}(DW_0 - 2DW_1 + DW_2) + \frac{1}{\alpha^3}(DW_1 - 5DW_0 - 4DW_2 + DW_3) + \frac{1}{\alpha}(DW_1 - 2DW_0)) \\
&= \frac{DW_0 \alpha^3 + (DW_1 - 2DW_0)\, \alpha^2 + (DW_0 - 2DW_1 + DW_2)\, \alpha - DW_0 + DW_1 - 2DW_2 + DW_3}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
B_2 &= \frac{DW_0 \beta^3 + (DW_1 - 2DW_0)\, \beta^2 + (DW_0 - 2DW_1 + DW_2)\, \beta - DW_0 + DW_1 - 2DW_2 + DW_3}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)}, \\
B_3 &= \frac{DW_0 \gamma^3 + (DW_1 - 2DW_0)\, \gamma^2 + (DW_0 - 2DW_1 + DW_2)\, \gamma - DW_0 + DW_1 - 2DW_2 + DW_3}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)}, \\
B_4 &= \frac{DW_0 \delta^3 + (DW_1 - 2DW_0)\, \delta^2 + (DW_0 - 2DW_1 + DW_2)\, \delta - DW_0 + DW_1 - 2DW_2 + DW_3}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}.
\end{aligned}$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty} DW_n x^n = B_1(1 - \alpha x)^{-1} + B_2(1 - \beta x)^{-1} + B_3(1 - \gamma x)^{-1} + B_4(1 - \delta x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} DW_n x^n = B_1 \sum_{n=0}^{\infty} \alpha^n x^n + B_2 \sum_{n=0}^{\infty} \beta^n x^n + B_3 \sum_{n=0}^{\infty} \gamma^n x^n + B_4 \sum_{n=0}^{\infty} \delta^n x^n = \sum_{n=0}^{\infty} (B_1 \alpha^n + B_2 \beta^n + B_3 \gamma^n + B_4 \delta^n) x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$DW_n = B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n.$$

The following identity reveals a connection between the dual Pandita numbers and the Pandita–Lucas numbers.

COROLLARY 9. *For all integers m, n the following identities holds:*

$$DW_{m+n} = P_{m-2}DW_{n+3} + (P_{m-4} - P_{m-3} - P_{m-5})DW_{n+2} + (P_{m-3} - P_{m-4})DW_{n+1} - DW_nP_{m-3}.$$

Proof. First we assume that $m, n \geq 0$.The Theorem (9) can be proved by mathematical induction on m. If $m = 0$ we get

$$DW_n = P_{-2}DW_{n+3} + (P_{-4} - P_{-3} - P_{-5})DW_{n+2} + (P_{-3} - P_{-4})DW_{n+1} - DW_nP_{-3}.$$

which is true since $P_{-2} = 0, P = -1, P_{-4} = -1, P_{-5} = 0$. Assume that the equality holds for $m \leq k$. For $m = k + 1$, we get

$$\begin{aligned} DW_{k+1+n} &= 2DW_{n+k} - DW_{n+k-1} + DW_{n+k-2} - DW_{n+k-3}, \\ & 2(P_{m-2}DW_{n+3} + (P_{m-4} - P_{m-3} - P_{m-5})DW_{n+2} + (P_{m-3} - P_{m-4})DW_{n+1} - DW_nP_{m-3}) \\ & -(P_{m-3}DW_{n+3} + (P_{m-5} - P_{m-4} - P_{m-6})DW_{n+2} + (P_{m-4} - P_{m-5})DW_{n+1} - DW_nP_{m-4}) \\ & +(P_{m-4}DW_{n+3} + (P_{m-6} - P_{m-5} - P_{m-7})DW_{n+2} + (P_{m-5} - P_{m-6})DW_{n+1} - DW_nP_{m-5}) \\ & -(P_{m-5}DW_{n+3} + (P_{m-7} - P_{m-6} - P_{m-8})DW_{n+2} + (P_{m-6} - P_{m-7})DW_{n+1} - DW_nP_{m-6}). \end{aligned}$$

Consequently, by mathematical induction on m, this proves Theorem 9.

The other cases of m, n can be proved smilarly for all integers m, n. □

Taking $DW_n = DP_n$ or $DW_n = DS_n$ in above Theorem, respectively, we get:

COROLLARY 10.

$$\begin{aligned} DP_{m+n} &= P_{m-2}DP_{n+3} + (P_{m-4} - P_{m-3} - P_{m-5})DP_{n+2} + (P_{m-3} - P_{m-4})DP_{n+1} - DP_nP_{m-3}, \\ DS_{m+n} &= P_{m-2}DS_{n+3} + (P_{m-4} - P_{m-3} - P_{m-5})DS_{n+2} + (P_{m-3} - P_{m-4})DS_{n+1} - DS_nP_{m-3}. \end{aligned}$$

4. SIMSON'S FORMULA

In this section, we present Simpson's formula for the dual generalized Pandita numbers . This is a special case of [35, Theorem 4.1].

THEOREM 11. *(Simpson's formula for dual generalized Pandita numbers) For all integers n we have,*

$$\begin{vmatrix} DW_{n+3} & DW_{n+2} & DW_{n+1} & DW_n \\ DW_{n+2} & DW_{n+1} & DW_n & DW_{n-1} \\ DW_{n+1} & DW_n & DW_{n-1} & DW_{n-2} \\ DW_n & DW_{n-1} & DW_{n-2} & DW_{n-3} \end{vmatrix} = \begin{vmatrix} DW_3 & DW_2 & DW_1 & DW_0 \\ DW_2 & DW_1 & DW_0 & DW_{-1} \\ DW_1 & DW_0 & DW_{-1} & DW_{-2} \\ DW_0 & DW_{-1} & DW_{-2} & DW_{-3} \end{vmatrix}$$

$= (DW_0 + DW_2 - DW_3)(-DW_3^3 + 3DW_2^3 - DW_1^3 + DW_0^3 + (5DW_2 - 2DW_1)DW_3^2 + (4DW_0 - 5DW_1 - 8DW_3)DW_2^2 + (4DW_0 + 4DW_2 - 5DW_3)DW_1^2$

$+(DW_2 - 3DW_1 - DW_3)DW_0^2 + 9DW_1DW_2DW_3 - 3DW_0DW_2DW_3 + 5DW_0DW_1DW_3 - 7DW_0DW_1DW_2)$

.

Proof. Using Theorem 3 it can be proved by using induction use [35, Theorem 4.1]. □

From the Theorem 11, we get the following Corollary.

Corollary 12. *For all integers n, the Simson's formulas of dual Pandita numbers and dual Pandita Lucas numbers are given as,*

a):

$$n = 0$$

$$\begin{vmatrix} DP_{n+3} & DP_{n+2} & DP_{n+1} & DP_n \\ DP_{n+2} & DP_{n+1} & DP_n & DP_{n-1} \\ DP_{n+1} & DP_n & DP_{n-1} & DP_{n-2} \\ DP_n & DP_{n-1} & DP_{n-2} & DP_{n-3} \end{vmatrix} = \begin{vmatrix} DP_3 & DP_2 & DP_1 & DP_0 \\ DP_2 & DP_1 & DP_0 & DP_{-1} \\ DP_1 & DP_0 & DP_{-1} & DP_{-2} \\ DP_0 & DP_{-1} & DP_{-2} & DP_{-3} \end{vmatrix}$$

$$= \begin{vmatrix} 5\varepsilon+3 & 3\varepsilon+2 & 2\varepsilon+1 & \varepsilon \\ 3\varepsilon+2 & 2\varepsilon+1 & \varepsilon & 0 \\ 2\varepsilon+1 & \varepsilon & 0 & 0 \\ \varepsilon & 0 & 0 & -1 \end{vmatrix} = \varepsilon^4 + \varepsilon^3 + \varepsilon^2 + 2\varepsilon + 1$$

$$= 2\varepsilon + 1$$

b):

$$
\begin{aligned}
n &= 0 \\
\begin{vmatrix} DS_{n+3} & DS_{n+2} & DS_{n+1} & DS_{n} \\ DS_{n+2} & DS_{n+1} & DS_{n} & DS_{n-1} \\ DS_{n+1} & DS_{n} & DS_{n-1} & DS_{n-2} \\ DS_{n} & DS_{n-1} & DS_{n-2} & DS_{n-3} \end{vmatrix} &= \begin{vmatrix} DS_{3} & DS_{2} & DS_{1} & DS_{0} \\ DS_{2} & DS_{1} & DS_{0} & DS_{-1} \\ DS_{1} & DS_{0} & DS_{-1} & DS_{-2} \\ DS_{0} & DS_{-1} & DS_{-2} & DS_{-3} \end{vmatrix} \\
&= \begin{vmatrix} 6\varepsilon+5 & 5\varepsilon+2 & 2\varepsilon+2 & 2\varepsilon+4 \\ 5\varepsilon+2 & 2\varepsilon+2 & 2\varepsilon+4 & 4\varepsilon+1 \\ 2\varepsilon+2 & 2\varepsilon+4 & 4\varepsilon+1 & \varepsilon-1 \\ 2\varepsilon+4 & 4\varepsilon+1 & \varepsilon-1 & -\varepsilon+4 \end{vmatrix} \\
&= -31\varepsilon^4 - 31\varepsilon^3 - 31\varepsilon^2 - 62\varepsilon - 31 \\
&= -62\varepsilon - 31.
\end{aligned}
$$

respectively.

5. Linear Sums

: In this section, we give the summation formulas of the dual generalized Pandita numbers with positive and negatif subscripts.

Now, we present the summation formulas of the generalized Pandita numbers.

THEOREM 13. *For the generalized Pandita numbers, we have the following formulas:*

(a): $\sum\limits_{k=0}^{n} W_k = -(n+3)W_{n+3} + (n+4)W_{n+2} + (n+4)W_n + 3W_3 - 4W_2 - 3W_0.$

(b): $\sum\limits_{k=0}^{n} W_{2k} = \frac{1}{3}(-3(n+2)W_{2n+2} + (3n+8)W_{2n+1} + 2W_{2n} + (3n+7)W_{2n-1} + 7W_3 - 8W_2 - W_1 - 6W_0).$

(c): $\sum\limits_{k=0}^{n} W_{2k+1} = \frac{1}{3}(-(3n+4)W_{2n+2} + (3n+8)W_{2n+1} + W_{2n} + 3(n+2)W_{2n-1} + 6W_3 - 8W_2 + W_1 - 7W_0).$

Proof. For the proof, see Soykan [33, Theorem 3.12]. □

THEOREM 14. *For the dual Pandita numbers, we have the following formulas:*

(a): $\sum\limits_{k=0}^{n} DW_k = -(n+3)DW_{n+3} + (n+4)DW_{n+2} + (n+4)DW_n + 3DW_3 - 4DW_2 - 3DW_0.$

(b): $\sum\limits_{k=0}^{n} DW_{2k} = \frac{1}{3}(-3(n+2)DW_{2n+2} + (3n+8)DW_{2n+1} + 2DW_{2n} + (3n+7)DW_{2n-1} + 7DW_3 - 8DW_2 - DW_1 - 6DW_0).$

(c): $\sum\limits_{k=0}^{n} DW_{2k+1} = \frac{1}{3}(-(3n+4)DW_{2n+2} + (3n+8)DW_{2n+1} + DW_{2n} + 3(n+2)DW_{2n-1} + 6DW_3 - 8DW_2 + DW_1 - 7DW_0).$

Proof. Use Theorem 13 and the definition of DW_n. □

As a special case of the theorem 14, we present the following Corollary.

Corollary 15. *For $n \geq 0$, dual Pandita numbers have the following properties:*

(a): $\sum\limits_{k=0}^{n} DW_k = -(n+3)DW_{n+3} + (n+4)DW_{n+2} + (n+4)DW_n + 1.$

(b): $\sum\limits_{k=0}^{n} DW_{2k} = \frac{1}{3}(-3(n+2)DW_{2n+2} + (3n+8)DW_{2n+1} + 2DW_{2n} + (3n+7)DW_{2n-1} + 3\varepsilon 4).$

(c): $\sum\limits_{k=0}^{n} DW_{2k+1} = \frac{1}{3}(-(3n+4)DW_{2n+2} + (3n+8)DW_{2n+1} + DW_{2n} + 3(n+2)DW_{2n-1} + \varepsilon + 3).$

Corollary 16. *For $n \geq 0$, dual Pandita Lucas numbers have the following properties.*

(a): $\sum\limits_{k=0}^{n} DS_k = -(n+3)DS_{n+3} + (n+4)DS_{n+2} + (n+4)DS_n - 8\varepsilon - 5.$

(b): $\sum\limits_{k=0}^{n} DS_{2k} = \frac{1}{3}(-3(n+2)DS_{2n+2} + (3n+8)DS_{2n+1} + 2DS_{2n} + (3n+7)DS_{2n-1} + -12\varepsilon - 7).$

(c): $\sum\limits_{k=0}^{n} DS_{2k+1} = \frac{1}{3}(-(3n+4)DS_{2n+2} + (3n+8)DS_{2n+1} + DS_{2n} + 3(n+2)DS_{2n-1} + -16\varepsilon - 12).$

Next, we give the ordinary generating functions of some special cases of dual generalized Pandita numbers.

Theorem 17. *The ordinary generating functions of the sequences DW_{2n}, DW_{2n+1} are given as follows:*

(a): $\sum_{n=0}^{\infty} DW_{2n}x^n = \dfrac{DW_2\left(x^3+3x^2-x\right) + DW_0\left(2x^2+2x-1\right) - DW_1\left(x^2-x^3\right) - DW_3\left(x^3+2x^2\right)}{-x^4-x^3+x^2+2x-1}.$

(b): $\sum_{n=0}^{\infty} DW_{2n+1}x^n = \dfrac{DW_0\left(x^3+2x^2\right) - DW_3\left(x^3+x^2+x\right) - DW_1\left(x^3-2x+1\right) + DW_2\left(2x^3+x^2\right)}{-x^4-x^3+x^2+2x-1}.$

Proof. Similary, the proof can be constructed as in [4, Theorem 4].

From the last Theorem, we have the following Corollary which gives sum formula of dual Pandita numbers (Take $DW_n = DP_n$ whit $DP_0 = \varepsilon$,$DP_1 = 2\varepsilon+1, DP_2 = 3\varepsilon+2, DP_3 = 5\varepsilon+3$)

Corollary 18. *For $n \geq 0$ dual Pandita numbers have the following properties.*

(a): $\sum_{n=0}^{\infty} DP_{2n}x^n = \dfrac{-x^2(1+\varepsilon) - x(2+\varepsilon) - \varepsilon}{1-2x+x^2-x^3+x^4},$

(b): $\sum_{n=0}^{\infty} DW_{2n+1}x^n = \dfrac{-x(1+\varepsilon) - x^2 - 2\varepsilon - 1}{1-2x+x^2-x^3+x^4}.$

6. Matrices related with dual Generalized Pandita Numbers

In this sectiion, using dual dual Pandita numbers, we give some matrices related to dual dual Pandita numbers.

We define the square matrix A of order 4 as

$$A = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

uch that $detA = 1$. Note that

$$A^n = \begin{pmatrix} P_{n+1} & -P_n + P_{n-1} - P_{n-2} & P_n - P_{n-1} & -P_n \\ P_n & -P_{n-1} + P_{n-2} - P_{n-3} & P_{n-1} - P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-2} + P_{n-3} - P_{n-4} & P_{n-2} - P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-3} + P_{n-4} - P_{n-5} & P_{n-3} - P_{n-4} & -P_{n-3} \end{pmatrix}$$

for the proof see [34].

Then we give the following lemma.

LEMMA 19. *For $n \geq 0$ the following identitiy is true:*

$$\begin{pmatrix} DW_{n+3} \\ DW_{n+2} \\ DW_{n+1} \\ DW_n \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}.$$

Proof. The identitiy(19) can be proved by mathematical induction on n. If $n = 0$ we obtain

$$\begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$. Thus the following identitiy is true

$$\begin{pmatrix} DW_{k+3} \\ DW_{k+2} \\ DW_{k+1} \\ DW_k \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}.$$

For $n = k+1$, we get

$$\begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} DW_{k+3} \\ DW_{k+2} \\ DW_{k+1} \\ DW_k \end{pmatrix}$$
$$= \begin{pmatrix} DW_{k+4} \\ DW_{k+3} \\ DW_{k+2} \\ DW_{k+1} \end{pmatrix}.$$

Consequently, by mathematical induction on n, the proof completed. □

We define

$$N_{DW} = \begin{pmatrix} DW_3 & DW_2 & DW_1 & DW_0 \\ DW_2 & DW_1 & DW_0 & DW_{-1} \\ DW_1 & DW_0 & DW_{-1} & DW_{-2} \\ DW_0 & DW_{-1} & DW_{-2} & DW_{-3} \end{pmatrix}, \tag{6.1}$$

$$E_{DW} = \begin{pmatrix} DW_{n+3} & DW_{n+2} & DW_{n+1} & DW_n \\ DW_{n+2} & DW_{n+1} & DW_n & DW_{n-1} \\ DW_{n+1} & DW_n & DW_{n-1} & DW_{n-2} \\ DW_n & DW_{n-1} & DW_{n-2} & DW_{n-3} \end{pmatrix}. \tag{6.2}$$

Now, we have the following theorem with N_{DW} and E_{DW}

Theorem 20. *Using N_{DW} and E_{DW} , we get*

$$A^n N_{DW} = E_{DW}.$$

Proof. Note that we get

$$A^nN_{DW} = \begin{pmatrix} P_{n+1} & -P_n+P_{n-1}-P_{n-2} & P_n-P_{n-1} & -P_n \\ P_n & -P_{n-1}+P_{n-2}-P_{n-3} & P_{n-1}-P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-2}+P_{n-3}-P_{n-4} & P_{n-2}-P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-3}+P_{n-4}-P_{n-5} & P_{n-3}-P_{n-4} & -P_{n-3} \end{pmatrix} \begin{pmatrix} DW_3 & DW_2 & DW_1 & DW_0 \\ DW_2 & DW_1 & DW_0 & DW_{-1} \\ DW_1 & DW_0 & DW_{-1} & DW_{-2} \\ DW_0 & DW_{-1} & DW_{-2} & DW_{-3} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

where

$$\begin{aligned}
a_{11} &= DW_1(P_n-P_{n-1})-DW_2(P_n-P_{n-1}+P_{n-2})-DW_0P_n+W_3P_{n+1}=DW_{n+3}, \\
a_{12} &= DW_0(P_n-P_{n-1})-DW_1(P_n-P_{n-1}+P_{n-2})-P_nDW_{-1}+DW_2P_{n+1}=DW_{n+2}, \\
a_{13} &= DW_{-1}(P_n-P_{n-1})-DW_0(P_n-P_{n-1}+P_{n-2})-P_nDW_{-2}+DW_1P_{n+1}=DW_{n+1}, \\
a_{14} &= DW_{-2}(P_n-P_{n-1})-DW_{-1}(P_n-P_{n-1}+P_{n-2})-P_nDW_{-3}+DW_0P_{n+1}=DW_n, \\
a_{21} &= DW_3P_n-DW_2(P_{n-1}-P_{n-2}+P_{n-3})+DW(P_{n-1}-P_{n-2})-DW_0P_{n-1}=DW_{n+2}, \\
a_{22} &= DW_2P_n-DW_{-1}P_{n-1}-DW_1(P_{n-1}-P_{n-2}+P_{n-3})+DW(P_{n-1}-P_{n-2})=DW_{n+1}, \\
a_{23} &= DW_{-1}(P_{n-1}-P_{n-2})-DW_{-2}P_{n-1}+DW_1P_n-DW_0(P_{n-1}-P_{n-2}+P_{n-3})=DW_n, \\
a_{24} &= DW_{-2}(P_{n-1}-P_{n-2})-DW_{-3}P_{n-1}+DW_0P_n-DW_{-1}(P_{n-1}-P_{n-2}+P_{n-3})=DW_{n-1}, \\
a_{31} &= DW_1(P_{n-2}-P_{n-3})-DW_2(P_{n-2}-P_{n-3}+P_{n-4})-DW_0P_{n-2}+DW_3P_{n-1}=DW_{n+1}, \\
a_{32} &= DW_0(P_{n-2}-P_{n-3})-DW_1(P_{n-2}-P_{n-3}+P_{n-4})-DW_{-1}P_{n-2}+DW_2P_{n-1}=DW_n, \\
a_{33} &= DW_{-1}(P_{n-2}-P_{n-3})-DW_{-2}P_{n-2}-DW_0(P_{n-2}-P_{n-3}+P_{n-4})+DW_1P_{n-1}=DW_{n-1}, \\
a_{34} &= DW_{-2}(P_{n-2}-P_{n-3})-DW_{-3}P_{n-2}-DW_{-1}(P_{n-2}-P_{n-3}+P_{n-4})+DW_0P_{n-1}=DW_{n-2}, \\
a_{41} &= DW_1(P_{n-3}-P_{n-4})-DW_2(P_{n-3}-P_{n-4}+P_{n-5})-DW_0P_{n-3}+DW_3P_{n-2}=DW_n, \\
a_{42} &= DW_0(P_{n-3}-P_{n-4})-DW_1(P_{n-3}-P_{n-4}+P_{n-5})-DW_{-1}P_{n-3}+DW_2P_{n-2}=DW_{n-1}, \\
a_{43} &= DW_{-1}(P_{n-3}-P_{n-4})-DW_{-2}P_{n-3}-DW_0(P_{n-3}-P_{n-4}+P_{n-5})+DW_1P_{n-2}=DW_{n-2}, \\
a_{44} &= DW_{-2}(P_{n-3}-P_{n-4})-DW_{-3}P_{n-3}-DW_{-1}(P_{n-3}-P_{n-4}+P_{n-5})+DW_0P_{n-2}=DW_{n-3}.
\end{aligned}$$

Using the theorem (9) the proof is done. □

By taking $DW_n = DP_n$ with DP_0, DP_1, DP_2, DP_3 in (6.1) and (6.2)

$DW_n = S_n$ with DS_0, DS_1, DS_2, DS_3 in (6.1) and (6.2) respectively, we get:

$$N_{DP} = \begin{pmatrix} 5j+8\varepsilon+12j\varepsilon+3 & 3j+5\varepsilon+8j\varepsilon+2 & 2j+3\varepsilon+5j\varepsilon+1 & j+2\varepsilon+3j\varepsilon \\ 3j+5\varepsilon+8j\varepsilon+2 & 2j+3\varepsilon+5j\varepsilon+1 & j+2\varepsilon+3j\varepsilon & \varepsilon+2j\varepsilon \\ 2j+3\varepsilon+5j\varepsilon+1 & j+2\varepsilon+3j\varepsilon & \varepsilon+2j\varepsilon & -j\varepsilon \\ j+2\varepsilon+3j\varepsilon & \varepsilon+2j\varepsilon & -j\varepsilon & -1 \end{pmatrix},$$

$$E_{DP} = \begin{pmatrix} DP_{n+3} & DP_{n+2} & DP_{n+1} & DP_{n} \\ DP_{n+2} & DP_{n+1} & DP_{n} & DP_{n-1} \\ DP_{n+1} & DP_{n} & DP_{n-1} & DP_{n-2} \\ DP_{n} & DP_{n-1} & DP_{n-2} & DP_{n-3} \end{pmatrix},$$

$$N_{DS} = \begin{pmatrix} 6j+7\varepsilon+11j\varepsilon+5 & 5j+6\varepsilon+7j\varepsilon+2 & 2j+5\varepsilon+6j\varepsilon+2 & 2j+2\varepsilon+5j\varepsilon+4 \\ 5j+6\varepsilon+7j\varepsilon+2 & 2j+5\varepsilon+6j\varepsilon+2 & 2j+2\varepsilon+5j\varepsilon+4 & 4j+2\varepsilon+2j\varepsilon+1 \\ 2j+5\varepsilon+6j\varepsilon+2 & 2j+2\varepsilon+5j\varepsilon+4 & -4j+2\varepsilon+2j\varepsilon+1 & j+4\varepsilon+2j\varepsilon-1 \\ 2j+2\varepsilon+5j\varepsilon+4 & -4j+2\varepsilon+2j\varepsilon+1 & j+4\varepsilon+2j\varepsilon-1 & \varepsilon-j+4j\varepsilon+4 \end{pmatrix},$$

$$E_{DS} = \begin{pmatrix} DS_{n+3} & DS_{n+2} & DS_{n+1} & DS_{n} \\ DS_{n+2} & DS_{n+1} & DS_{n} & DS_{n-1} \\ DS_{n+1} & S_{n} & DS_{n-1} & DS_{n-2} \\ DS_{n} & DS_{n-1} & DS_{n-2} & DS_{n-3} \end{pmatrix}.$$

From Theorem [20], we can write the following corollary.

Corollary 21. *The following identities are hold:*

a): $A^n N_{DP} = E_{DP}$.

b): $A^n N_{DS} = E_{DS}$.

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