Discretization and Neimark-Sacker Bifurcation Analysis of a Fractional Order COVID - 19 Model Using Conformable Derivatives

Abstract

In order to illustrate the discretization process and bifurcation analysis, a fractional order COVID-19 mathematical model that discusses the healthy population class and the diseased population class is revisited in this study. The discretization has been carried out of an autonomous nonlinear model. A fractional order discrete system is created from a continuous biological model. Discretization is accomplished using the conformable fractional operator for the piecewise constant approximation. Two fixed points namely the trivial fixed point and the coexisted one have been calculated. Through the stability of two fixed points and bifurcation analysis, the dynamics of the model are examined. By specifying the bifurcation parameter σ, the Neimark Sacker bifurcation's existence is studied and the result has been stated. By assigning the parameters numbers, numerical variations are explored for different discrete parameter h and for different fractional orders. Additionally, graphical interpretations are provided for various fractional orders and discrete parameters.

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1 Introduction

Mathematical models are effective research tools for a variety of phenomena and practical issues. We can learn about the spread of an infectious disease within a population, its fatality rates, and effective management measures by utilizing mathematical models to describe infectious diseases [1, 4]. A significant outbreak caused by the Corona virus, also known as COVID-19, occurred in Hubei Province of China at the end of 2019. There are numerous hypotheses regarding the virus's origin, and some experts have looked into the possibility that it was spread from bats to humans through improper animal handling in a Wuhan seafood market. Researchers confirmed that person-to-person contact is what causes the disease to spread so widely. The COVID-19 outbreak has captured the interest of numerous experts from across the branch. For all mathematicians working in the field of mathematical modeling, it is the most popular topic. The first publication may be examined in April of 2020, and following that, several mathematical models and their analyses may be examined in this field [25, 26, 27, 28, 29]. Qualitative analysis given by Kamal Shah et. al. approach is revisited here with conformable fractional order derivative [25].

$$\begin{aligned} T_\alpha S(t) &= \sigma S(t) - \beta S(t)I(t) + \gamma I(t), \\ T_\alpha I(t) &= \beta S(t)I(t) + \theta I(t) - \mu I(t) - \gamma I(t). \end{aligned} \tag{1}$$

The above model is described by a system of fractional order system of equation containing two non overlapping classes, namely, S (Healthy Populations Class) and I (Infected Populations Class). Where the parameter β denotes the infection rate, σ indicates the rate of conversion of susceptible individuals to infected ones, θ designates the rate at which infection displaces, μ is the death rate and γ stands for the cure rate.

Instead of using the more common ordinary derivatives for analysis, fractional derivatives produce more important conclusions that are more beneficial for comprehending biological phenomena. A fractional derivative is a way to expand the classical one to any number of orders. The conformable sense fractional derivative was first introduced by Khalil et. al. [10] and it is further improved by Abdeljawad [11]. Fractional theory has evolved as a theory in a conformable sense in addition to being based on formulas. On the stability of fractional differential equations, Cauchy issues of fractional systems, and existence theory in a conformable sense, there is still much to be done. Using conformable derivatives, we address the idea of discretization in this study. We will need some conceptual foundations for this. We first check some definition of the Conformable fractional order derivative. For details [10, 11] can be referred. The following definitions can be checked from [11].

Let f be a real valued function defined on $[a, \infty)$. The left conformable derivative starting from a of order $0 < \alpha < 1$ is defined as

$$T_\alpha^a f(t) = \lim_{\vartheta \to 0} \frac{f(t + \vartheta(t-a)^{1-\alpha} - f(t)}{\vartheta}, \tag{2}$$

Similarly, the right conformable derivative terminating at b of order $0 < \alpha < 1$ is defined as;

$$T_\alpha^b f(t) = -\lim_{\vartheta \to 0} \frac{f(t + \vartheta(b-t)^{1-\alpha} - f(t)}{\vartheta}, \tag{3}$$

Where α derivative of the function f exists everywhere.
Let f be a real valued function defined on $[t_0, \infty)$. The left conformable integral starting from t_0 of order $0 < \alpha < 1$ is defined as:

$$I_\alpha^a f(t) = \int_a^t (\tau - t_0)^{\alpha-1} f(\tau) d\tau. \tag{4}$$

In this case, the integral is the standard improper Riemann integral.
Lemma:[11] Let the derivative of order $0 < \alpha < 1$ of the function f exists,and $t_0 > 0$. The the left conformable derivative hold the following relation;

$$T_\alpha^a f(t) = (t-a)^{1-\alpha} \frac{df(t)}{dt}. \tag{5}$$

2 Discretization of the Model

Difference equations are a wide and active discipline that emerged in practically all biologically applied fields of study. The technique of discretization involves creating a difference equation that closely resembles the differential equation, the equation with no derivative and only variations in function values. We already have studied the different approaches of discretizing fractional order systems given by Ravi P. Agarwal et. al. in 2013 in which Caputo fractional operator has been used [9], C.N. Angstmann et. al work in 2017 [15], S. Kartal and F. Gurcan in 2018 [18]. We have been through both the approaches of discretization of fractional differential equations using Conformable sense and Caputo sense. Here we take the α derivative in conformable sense. We rewrite above systems using discretization parameter h. Discretization process is done by using the piecewise constant approximation and the left conformable fractional derivative. Let $t \in [nh, (n+1)h), n = 0, 1, 2, - - -$

$$\begin{aligned} T_\alpha S(t) &= \sigma S(t) - \beta S(t) I\left(\left[\frac{t}{h}\right] h\right) + \gamma I\left(\left[\frac{t}{h}\right] h\right), \\ T_\alpha I(t) &= \beta S\left(\left[\frac{t}{h}\right] h\right) I(t) + \theta I(t) - \mu I(t) - \gamma I(t). \end{aligned} \tag{6}$$

With $S(0) = 0$ and $I(0) = 0$, where $[.]$ denotes the integer part of t, $0 \le t < \infty$. Solving very first equation we get,

$$(t-nh)^{1-\alpha} \frac{dS(t)}{dt} = (\sigma - \beta I(nh)) S(t) + \gamma I(nh),$$

Solving by converting the above equation into first order ODE,

$$\frac{dS(t)}{dt} + (\beta I(nh) - \sigma)(t-nh)^{\alpha-1} S(t) = \gamma I(nh)(t-nh)^{\alpha-1},$$

Multiply both side by $e^{(\beta I(nh)-\sigma)\frac{(t-nh)^\alpha}{\alpha}}$ the equation gets converted into;

$$\frac{d}{dt}\left(S(t)e^{(\beta I(nh)-\sigma)\frac{(t-nh)^\alpha}{\alpha}}\right) = \gamma I(nh)e^{(\beta I(nh)-\sigma)\frac{(t-nh)^\alpha}{\alpha}}(t-nh)^{\alpha-1},$$

Integrating above equation with respect to t between nh to t.

$$S(t)e^{(\beta I(nh)-\sigma)\frac{(t-nh)^\alpha}{\alpha}} - S(nh) = \gamma I(nh)\int_{nh}^{t} e^{(\beta I(nh)-\sigma)\frac{(t-nh)^\alpha}{\alpha}}(t-nh)^{\alpha-1}dt,$$
$$S(t)e^{(\beta I(nh)-\sigma)\frac{(t-nh)^\alpha}{\alpha}} = S(nh) + \frac{\gamma I(nh)}{(\beta I(nh)-\sigma)}\left[e^{(\beta I(nh)-\sigma)\frac{(t-nh)^\alpha}{\alpha}} - 1\right],$$

$$S(t) = S(nh)e^{(\sigma-\beta I(nh))\frac{(t-nh)^\alpha}{\alpha}} + \frac{\gamma I(nh)}{(\beta I(nh)-\sigma)}\left[1 - e^{(\beta I(nh)-\sigma)\frac{(t-nh)^\alpha}{\alpha}}\right]. \quad (7)$$

Now repeating the procedure for second equation of the system;

$$(t-nh)^{1-\alpha}\frac{dI(t)}{dt} = I(t)(\beta S(nh)+\theta-\mu-\gamma),$$
$$\frac{dI(t)}{dt} = (t-nh)^{\alpha-1}(\beta S(nh)+\theta-\mu-\gamma),$$

Integrating between nh to t we get;

$$I(t) = I(nh)e^{(\beta S(nh)+\theta-\mu-\gamma)\frac{(t-nh)^\alpha}{\alpha}}. \quad (8)$$

Replacing in 7 and 8 as $t \longrightarrow (n+1)h$ we get the discretized model of the proposed system;

$$\begin{aligned} S_{n+1} &= S_n e^{(\sigma-\beta I_n)\frac{h^\alpha}{\alpha}} + \frac{\gamma I_n}{(\beta I_n - \sigma)}\left(1 - e^{(\sigma-\beta I_n)\frac{h^\alpha}{\alpha}}\right), \\ I_{n+1} &= I_n e^{(\beta S_n+\theta-\mu-\gamma)\frac{h^\alpha}{\alpha}}. \end{aligned} \quad (9)$$

Equation 9 is the final discretized model of proposed system.

3 Stability of Equilibrium Points

Here we got two equilibrium points of the proposed system denoted as $E_0 = E(0,0)$ and $E_1 = E\left(\frac{\mu+\gamma-\theta}{\beta}, \frac{\sigma(\mu+\gamma-\theta)}{\beta(\mu-\theta)}\right)$. We linearize the system and get the jacobian matrix as follows;

$$J(E) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad (10)$$

$$a_{11} = e^{(\sigma-\beta I_n)\frac{h^\alpha}{\alpha}},$$
$$a_{12} = \left(\frac{-\beta h^\alpha}{\alpha}\right) S_n e^{(\sigma-\beta I_n)\frac{h^\alpha}{\alpha}} + \frac{\gamma I_n \beta h^\alpha}{\alpha(\beta I_n - \sigma)} e^{(\sigma-\beta I_n)\frac{h^\alpha}{\alpha}} - \frac{\gamma\sigma(1-e^{(\sigma-\beta I_n)\frac{h^\alpha}{\alpha}})}{(\beta I_n - \sigma)^2},$$
$$a_{21} = I_n e^{(\beta S_n+\theta-\mu-\gamma)} \frac{\beta h^\alpha}{\alpha},$$
$$a_{22} = e^{(\beta S_n+\theta-\mu-\gamma)} \frac{\beta h^\alpha}{\alpha}.$$

The following theorems can be proved for discussing the stability of the fixed points E_0 and E_1.

Theorem 3.1 *The fixed point E_0 is saddle point if $\frac{\theta}{\mu+\gamma} < 1$ and is source if $\frac{\theta}{\mu+\gamma} > 1$.*

Proof By linearising the system 9, the jacobian matrix of the system at E_0 is,

$$J(E_0) = \begin{bmatrix} e^{\frac{\sigma h^\alpha}{\alpha}} & -\frac{\gamma}{\sigma}(1-e^{\frac{\sigma h^\alpha}{\alpha}}) \\ 0 & e^{(\theta-\mu-\gamma)\frac{h^\alpha}{\alpha}} \end{bmatrix}$$

From the eigenvalues of the jacobian matrix it is quite straightforward that the system is unstable in both the cases.

Theorem 3.2 *The fixed point E_1 is local asymptotically stable if $\sigma < \frac{\alpha}{(\mu+\gamma-\theta)(\mu-\theta)h^\alpha}$.*

Proof: The jacobian matrix calculated at fixed point E_1 is,

$$J(E_1) = \begin{bmatrix} e^{\frac{-\sigma\gamma h^\alpha}{(\mu-\theta)\alpha}} & (\mu-\theta)^2(e^{\frac{-\sigma\gamma h^\alpha}{(\mu-\theta)\alpha}}-1) \\ \frac{\sigma(\mu+\gamma-\theta)h^\alpha}{(\mu-\theta)\alpha} & 1 \end{bmatrix}$$

Here we use the Jury's necessary and sufficient condition to prove the desired result. The characteristics equation is given as;

$$\lambda^2 + q_1\lambda + q_0 = 0,$$

Where

$$q_1 = -1 - e^{\frac{-\sigma\gamma h^\alpha}{(\mu-\theta)\alpha}},$$
$$q_0 = e^{\frac{-\sigma\gamma h^\alpha}{(\mu-\theta)\alpha}} - \sigma(\mu+\gamma-\theta)(\mu-\theta)\frac{h^\alpha}{\alpha}(e^{\frac{-\sigma\gamma h^\alpha}{(\mu-\theta)\alpha}}-1).$$

Necessary condition of Jury test: Condition I:

$$1 + q_1 + q_0 > 0,$$
$$\sigma(\mu+\gamma-\theta)(\mu-\theta)\frac{h^\alpha}{\alpha}(1-e^{\frac{-\sigma\gamma h^\alpha}{(\mu-\theta)\alpha}}) > 0.$$

Condition II:

$$1 - q_1 + q_0 > 0,$$

$$\frac{2\alpha(1 - e^{\frac{-\sigma\gamma h^\alpha}{(\mu-\theta)\alpha}}) + \sigma(\mu + \gamma - \theta)(\mu - \theta)h^\alpha(1 - e^{\frac{-\sigma\gamma h^\alpha}{(\mu-\theta)\alpha}})}{\alpha} > 0.$$

Sufficient Condition of Jury test:

$$1 - q_0 > 0,$$

$$\frac{(1 - e^{\frac{-\sigma\gamma h^\alpha}{(\mu-\theta)\alpha}})(\alpha + \sigma(\mu + \gamma - \theta)(\theta - \mu)h^\alpha)}{\alpha} > 0.$$

Hence the proof is completed.

4 Neimark Sacker Bifurcation

The modeling of experimental data is frequently appropriate for discrete dynamical systems. Bifurcations are crucial in many real-world systems as a switching mechanism because often a small change in parameter values results in a dramatic, qualitative change in the system's behavior. When parameters are changed in a dynamical system, this is referred to as bifurcation. A system of differential equations can bifurcate if there are non-hyperbolic fixed points and the roots of the polynomial that characterizes the linearization of those fixed points lie on the unit circle. Bifurcation is a crucial nonlinear feature that can signal a qualitative shift in the characteristics of a system as a system parameter changes. The existence of an eigen value close to 1 is associated with saddle-node, transcritical, or pitchfork bifurcation, while the existence of an eigen value close to -1 is associated with flip or period doubling bifurcation. The existence of a pair of complex conjugate eigen values with a modulus close to 1 is associated with a Neimark Scaker bifurcation. Closed invariant curves produced by Neimark Sacker bifurcations exhibit more intriguing complicated behavior. Many more work can be checked on this from the studies [5, 7, 19, 22, 24]. Here we are working on the COVID - 19 mathematical model. The characteristics polynomial associated with 10 is calculated as;

$$\lambda^2 + a_1\lambda + a_0 = 0, \tag{11}$$

Where

$$a_1 = e^{\frac{h^\alpha(\sigma-\beta I_n)}{\alpha}} \left(e^{\frac{h^\alpha(I_n\beta+\beta(S_n\beta-\gamma+\theta-\mu)-\sigma)}{\alpha}} + 1 \right),$$

and

$$a_0 = \frac{e^{\frac{4h^{\alpha}(\sigma-\beta I_n)}{\alpha}}\left(e^{\frac{h^{\alpha}(3I_n\beta+\beta(S_n\beta-\gamma+\theta-\mu)-3\sigma)}{\alpha}}+1\right)}{\alpha(I_n\beta-\sigma)^2}a_2,$$

where

$$a_2 = (I_n(I_nh^{\alpha}\beta\gamma(I_n\beta-\sigma)-S_nh^{\alpha}\beta(I_n\beta-\sigma)^2-\alpha\gamma\sigma(e^{\frac{h^{\alpha}(I_n\beta-\sigma)}{\alpha}}-1))+\alpha(I_n\beta-\sigma)^2).$$

The characteristics equation calculated at $S_n = \frac{\mu+\gamma-\theta}{\beta}$ and $I_n = \frac{\sigma(\mu+\gamma-\theta)}{\beta(\mu-\theta)}$ with $\sigma^* = \frac{\alpha}{(\mu+\gamma-\theta)(\mu-\theta)h^{\alpha}}$ the matrix 10 is get converted as;

$$J(E_1(\sigma^*)) = \begin{bmatrix} e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}} & (\mu-\theta)^2(e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}-1) \\ \frac{1}{(\mu-\theta)^2} & 1 \end{bmatrix}$$

Eigen values $J(E_1(\sigma^*))$ are

$$\lambda_{1,2}(\sigma^*) = \frac{-p(\sigma^*) \pm \sqrt{p^2(\sigma^*)-4q(\sigma^*)}}{2},$$

$$p(\sigma^*) = -1 - e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}, \; q(\sigma^*) = 1,$$

$$\lambda_{1,2}(\sigma^*) = \frac{1+e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}} \pm \sqrt{(1+e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}})^2-4}}{2},$$

$$\lambda_{1,2}(\sigma^*) = \frac{1+e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}}{2} \pm \frac{ie^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}\sqrt{(-1+e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}})(1+3e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}})}}{2} = \lambda_1 + i\lambda_2.$$

Here $|\lambda_{1,2}(\sigma^*)| = \sqrt{q(\sigma^*)}$ and

$$\left.\frac{d|\lambda_{1,2}(\sigma)|}{d\sigma}\right|_{\sigma=\sigma^*} = (\mu+\gamma-\theta)(\mu-\theta)\frac{h^{\alpha}}{\alpha}\left(e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}-1\right) \neq 0.$$

Using the transformation $u_n = S_n - S^*$ and $v_n = I_n - I^*$ where $(S^*, I^*) = \left(\frac{\mu+\gamma-\theta}{\beta}, \frac{\sigma(\mu+\gamma-\theta)}{\beta(\mu-\theta)}\right)$ the system 9 is transformed as;

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = J(E_1(\sigma^*)) \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} f(u_n, v_n) \\ g(u_n, v_n) \end{bmatrix}$$

where
$f(u_n, v_n) = \xi_{13}u_n^2 + \xi_{14}u_nv_n + \xi_{15}v_n^2 + \xi_{16}u_n^3 + \xi_{17}u_n^2v_n + \xi_{18}u_nv_n^2 + \xi_{19}v_n^3 + o((|S|+$

$|I|)^4)$,

$g(u_n,v_n) = \xi_{23}u_n^2 + \xi_{24}u_nv_n + \xi_{25}v_n^2 + \xi_{26}u_n^3 + \xi_{27}u_n^2v_n + \xi_{28}u_nv_n^2 + \xi_{29}v_n^3 + o((|S|+|I|)^4)$,

Where

$\xi_{13} = 0,\ \xi_{14} = \frac{-\beta h^\alpha}{\alpha} e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}},\ \xi_{16} = 0,\ \xi_{17} = 0,\ \xi_{18} = \frac{\beta^2 h^{2\alpha}}{\alpha^2} e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}$

$$\xi_{15} = \beta\left(\frac{2a\beta(1-e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}})}{b^3} - \frac{2\gamma(1-e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}})}{b^2} - \frac{h^{2\alpha}a\beta e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}}{b\alpha^2} + \frac{(\mu+\gamma-\theta)h^{2\alpha}e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}}{\alpha^2} - \frac{2e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}h^{\alpha a\beta}}{b^2\alpha} + \frac{2e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}h^\alpha\gamma}{b\alpha}\right),$$

$$\xi_{19} = \beta^2\left(\frac{6a\beta(1-e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}})}{b^4} + \frac{6\gamma(1-e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}})}{b^3} + \frac{3h^{2\alpha}a\beta e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}}{b^2\alpha^2} - \frac{(\mu+\gamma-\theta)h^{3\alpha}e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}}{\alpha^3} + \frac{3h^{2\alpha}a\beta e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}}{b^2\alpha^2} - \frac{3h^{2\alpha}\gamma e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}}{b\alpha^2} + \frac{6h^\alpha a\beta e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}}{b^3\alpha} - \frac{6h^\alpha\gamma e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}}{b^2\alpha}\right),$$

$\xi_{23} = \frac{h^{\alpha\beta}}{\alpha(\theta-\mu)^2},\ \xi_{24} = \frac{h^{\alpha\beta}}{\alpha},\ \xi_{25} = 0,\ \xi_{26} = \frac{h^{2\alpha\beta^2}}{\alpha^2(\theta-\mu)^2},\ \xi_{27} = \frac{h^{2\alpha\beta^2}}{\alpha^2},\ \xi_{28} = 0,\ \xi_{29} = 0.$
On finding the eigen vectors associated with the eigen values of $J(E_1(\sigma^*))$ we construct the invertible matrix denoted as T, where T is,

$$T = \begin{bmatrix} 0 & 1 \\ P & Q \end{bmatrix}$$

Where $P = \frac{e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}\sqrt{(-1+e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}})(1+3e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}})}}{2(\mu-\theta)^2(e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}-1)}$, and $Q = \frac{-1}{2(\mu-\theta)^2}$.
Using the transformation

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ P & Q \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}$$

The proposed system can be exhibited as;

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} F(X,Y) \\ G(X,Y) \end{bmatrix}$$

$$F(X,Y) = \frac{-Qf(u_n,v_n)}{P} + \frac{g(u_n,v_n)}{P},$$

$$G(X,Y) = f(u_n, v_n).$$

$F(X,Y) = \varsigma_{13}X^2 + \varsigma_{14}XY + \varsigma_{15}Y^2 + \varsigma_{16}X^3 + \varsigma_{17}X^2Y + \varsigma_{18}XY^2 + \varsigma_{19}Y^3 + o((|X|+|Y|)^4)$,

$G(X,Y) = \varsigma_{13}X^2 + \varsigma_{14}XY + \varsigma_{15}Y^2 + \varsigma_{16}X^3 + \varsigma_{17}X^2Y + \varsigma_{18}XY^2 + \varsigma_{19}Y^3 + o((|X|+|Y|)^4)$,

Where

$$\varsigma_{13} = \frac{-2h^\alpha\beta\sqrt{(-1+e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}})}}{\alpha\sqrt{(1+3e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}})}},$$

$$\varsigma_{14} = \frac{e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}h^\alpha\beta\sqrt{e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}-1}(-2e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}(\mu-\theta)^2+1)}{\alpha\sqrt{3e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}+1}},$$

$$\varsigma_{15} = \frac{e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}\beta\sqrt{e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}-1}}{b^3\alpha^2\sqrt{(1+3e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}})}}A_1,$$

$$\varsigma_{16} = \frac{-2h^{2\alpha}\beta^2\sqrt{e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}-1}}{\alpha^2\sqrt{(1+3e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}})}},$$

$$\varsigma_{17} = \frac{-2h^{2\alpha}\beta^2\sqrt{e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}-1}(\mu-\theta)^2}{\alpha^2\sqrt{(1+3e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}})}},$$

$$\varsigma_{18} = \frac{-e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}h^{2\alpha}\beta^2\sqrt{e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}-1}}{\alpha^2\sqrt{(1+3e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}})}},$$

$$\varsigma_{18} = \frac{e^{\frac{-\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}\beta^2\sqrt{e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}-1}}{b^4\alpha^3\sqrt{(1+3e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}})}}A_2,$$

Where

$$A_1 = \Big(h^{2\alpha}ab^2\beta - h^{2\alpha}b^3(\mu+\gamma-\theta) + 2h^\alpha ab\alpha\beta - 2h^\alpha b^2\alpha\gamma - 2a\alpha^2\beta(e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}-1) + 2b\alpha^2\gamma(e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}}-1)\Big),$$

$$A_2 = \Big(-h^{3\alpha}ab^3\beta + h^{3\alpha}b^4(\mu+\gamma-\theta) - 3h^{2\alpha}ab^2\alpha\beta + 3h^{2\alpha}b^3\alpha\gamma - 6h^{\alpha}ab\alpha^2\beta + 6h^{\alpha}b^2\alpha^2\gamma + 6a\alpha^3\beta(e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}} - 1) - 6b\alpha^3\gamma(e^{\frac{\gamma}{(\mu+\gamma-\theta)(\mu-\theta)^2}} - 1)\Big).$$

All ς_{ij} are same as $f(u_n, v_n)$ so we move for further calculations. Define κ as

$$\kappa = -Re\left[\frac{(1-2\lambda)\bar{\lambda}^2}{1-\lambda}\eta_{11}\eta_{20}\right] - \frac{1}{2}|\eta_{11}|^2 - |\eta_{02}|^2 + Re(\bar{\lambda}\eta_{21}), \tag{12}$$

Where

$$\eta_{20} = \frac{1}{8}[(F_{XX} - F_{YY} + 2G_{XY}) + i(G_{XX} - G_{YY} - 2F_{XY})],$$

$$\eta_{11} = \frac{1}{4}[(F_{XX} + F_{YY}) + i(G_{XX} + G_{YY}),$$

$$\eta_{02} = \frac{1}{8}[(F_{XX} - F_{YY} - 2G_{XY}) + i(G_{XX} - G_{YY} + 2F_{XY}),$$

$$\eta_{21} = \frac{1}{16}[(F_{XXX} + F_{XYY} + G_{XXY} + G_{YYY}) + i(G_{XXX} + G_{XYY} - F_{XXY} - F_{YYY}).$$

Here we state the following statement on the Neimark Sacker bifurcation as;

Theorem 4.1 *Suppose E_1 is the positive fixed point of the proposed system 9 then the model 9 undergoes the a Neimark Sacker bifurcation at the fixed point E_1. Moreover if $\kappa < 0$ ($\kappa > 0$) then the Neimark Sacker bifurcation of the proposed model at bifurcation parameter σ^* (defined as above) is supercritical (subcritical) and there exists a closed attracting (repelling)invariant curve bifurcates from fixed point E_1.*

5 Numerical Interpretation and Graphical Analysis:

In this section, we will interpret result numerically. For this let us fix some parameters such as $\alpha = 0.95$, $h = 0.15$, $\sigma = 0$, $\beta = 0.35$, $\gamma = 0.54$, $\theta = 0.56$ and $\mu = 0.51$. These values are taken for calculations based on previous studies.Initial quantities of healthy populations and infected one are taken in the ratio 7 : 3 that means $S_0 = 0.7$ and $I_0 = 0.3$. Three different orders are taken for graphical analysis such as $\alpha = 1$, $\alpha = 0.95$ and $\alpha = 0.9$, the discretization

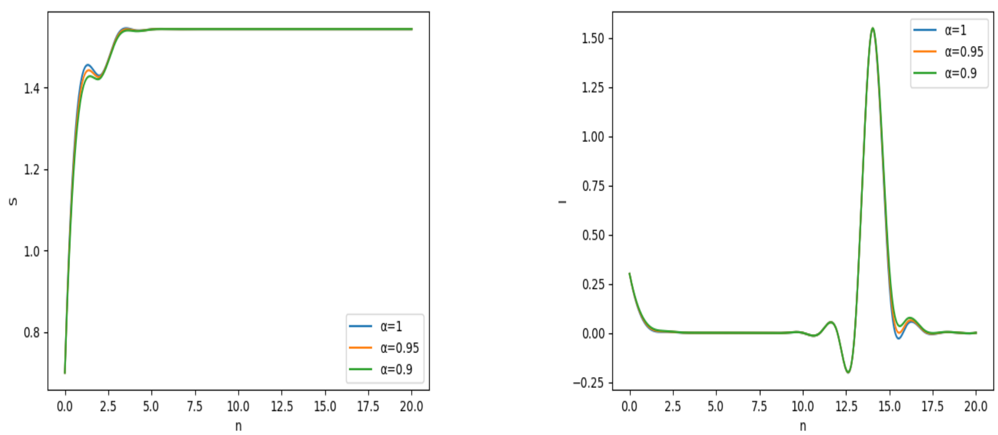

Figure 1: Graphical representation of discretized variable S and I for $\alpha = 1$, $\alpha = 0.95$ and $\alpha = 0.9$ for discretized parameter $h = 0.15$

parameter is also taken different in different graphs. The bifurcation parameter σ is calculated as -235.10985022. The eigen value $\lambda_{1,2} = 0.5 \pm i0.86602540$. The necessary conditions $\frac{d|\lambda_{1,2}(\sigma)|}{d\sigma}\Big|_{\sigma=\sigma^*} = 0.00425333093 \neq 0$ and is positive. The expression $F(X,Y)$ and $G(X,Y)$ are interpreted as follows;

$$F(X,Y) = -11225.882343X^2 - 28.0647058XY + 2157784157.872Y^2 - 682.10560980X^3 \\ - 1.7052640X^2Y - 1.226657957XY^2 - 594607717406.877Y^3 + 0(|X|^4 + |Y|^4).$$

$$G(X,Y) = 24.30474822X^2 + 0.060761870XY + 1.476801965X^3 \\ + 0.00369200491X^2Y + 0(|X|^4 + |Y|^4).$$

The coefficients

$$\eta_{11} = 539443232.9976543 + i6.0761870558,$$

$$\eta_{20} = -269724422.954222 + i10.05426999217,$$

$$\eta_{02} = -269724422.98460335 - i3.97808293658,$$

$$\eta_{21} = -42.631369862290 + i37160982338.128685,$$

which gives $\kappa = -4.36502287273e + 17$. Hence we can conclude that a stable limit cycle is observed with above defined parameters around the equilibrium point.

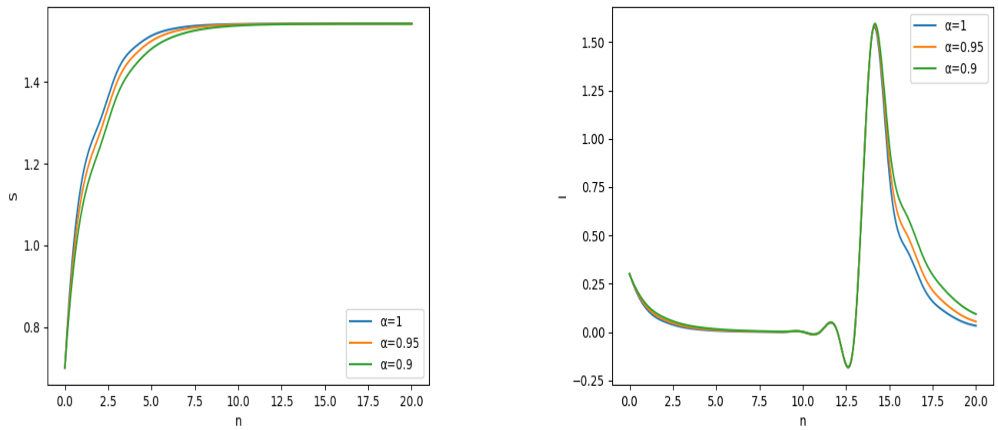

Figure 2: Graphical representation of discretized variable S and I for $\alpha = 1$, $\alpha = 0.95$ and $\alpha = 0.9$ for discretized parameter $h = 0.5$

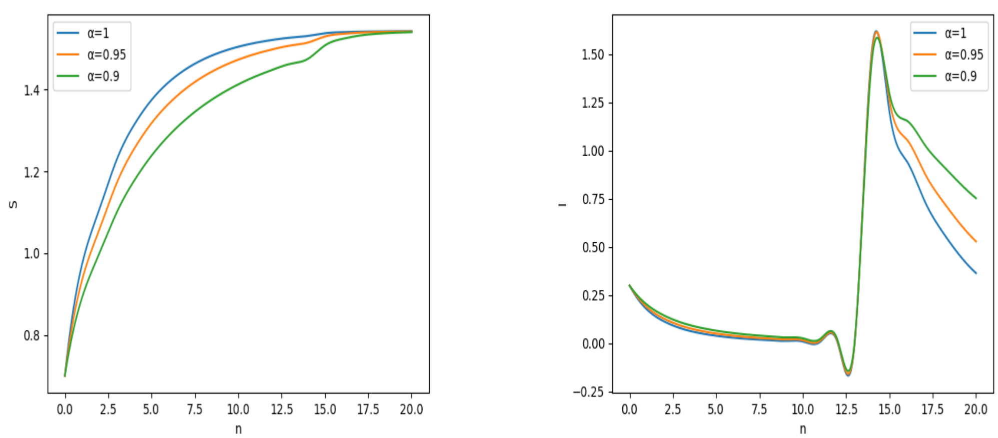

Figure 3: Graphical representation of discretized variable S and I for $\alpha = 1$, $\alpha = 0.95$ and $\alpha = 0.9$ for discretized parameter $h = 0.75$

6 Result and Discussion:

A COVID - 19 mathematical model discussing the healthy population class and infected population class has been put forward to talk about the discretization process and bifurcation analysis. Stability is examined for two distinct fixed points, one is trivial which is not stable, another is coexistence whose stability is communicated with the condition as $\sigma < \frac{\alpha}{(\mu+\gamma-\theta)(\mu-\theta)h^{\alpha}}$. Previously, numerical simulations and qualitative approach were discussed on the same model, here we came with the different approach such as discretization by using Conformable Fractional operator with bifurcation analysis namely, Neimark Sacker bifurcation. Fractional operator has allowed us to argue for different orders and discretization has provided us the result distinct result for different discretization parameters. Though the smaller discrete parameter $h = 0.15$ gives the most perfect result and as we go nearer to 1 it is showing variations in it. So for numeric calculations we kept the discrete parameter as $h = 0.15$ and the fractional order $\alpha = 0.95$.
For bifurcation analysis, the bifurcation parameter σ is designed for which we examined the Neimark Sacker bifurcation analysis. The result of bifurcation been proven positive for the defined parameters (as stated in earlier section). Discretization with fractional order allow us to trick with different order and discrete parameter. In graphical analysis, we observed different peak levels of infected populations for different discrete parameters.

7 Conclusion:

The study of COVID - 19 pandemic has been tried to revisit with the nonlinear system of equations with conformable order derivatives. The stability has been discussed around the bifurcation parameter σ. The COVID - 19 mathematical model is readdressed to study the Neimark-Sacker bifurcation analysis with the Conformable Derivatives. For the smaller discrete parameter h, the result has come out more accurate for fractional order $\alpha = 0.95$. Based on the calculations and defined parameters of the model, the graphical interpretation has been studied for the different fractional orders. Without using the numerical method, the discretization process can be carried out with the help of conformable derivatives for more accurate results of the mathematical models.

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