COEFFICIENT BOUNDS FOR A NEW SUBCLASS OF BI-UNIVALENT FUNCTIONS INVOLVING THE SĂLĂGEAN DEFERENTIAL OPERATOR

Abstract. The study of bi-univalent functions has attracted considerable attention in Geometric Function Theory due to its importance in coefficient problems and related applications. Motivated by recent developments involving differential operators, this paper introduces two new subclasses of bi-univalent functions belonging to the family Σ in the open unit disk by means of the Sălăgean differential operator. For these subclasses, sharp estimates for the coefficients $|b_2|$ and $|b_3|$ are obtained. The results presented in this work extend and generalize several earlier findings in the literature. Moreover, the proposed framework provides a useful basis for further investigations, including higher-order coefficient bounds, Fekete-Szegö inequalities, Hankel determinants, and other related problems associated with these subclasses.

Keywords: bi-univalent, Geometric function theory, Salagean differential operator, Coefficient Bounds

1. Introduction and Preliminaries

Let $\mathscr{A}$ represent the class of analytic functions expressed as

$$h(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n, \tag{1.1}$$

which are analytic within the open unit disk $\Delta = \{\zeta : |\zeta| < 1\}$. The subclass $\mathscr{S} \subset \mathscr{A}$ consists of those functions that are univalent in Δ.

Ding et al. [8] proposed the class $L_\lambda(\beta)$ of analytic functions, defined as

$$L_\lambda(\beta) = \left\{ h \in \mathscr{A} : \Re\left((1-\lambda)\frac{h(\zeta)}{\zeta} + \lambda h'(\zeta) \right) > \beta,\ \beta < 1,\ \lambda \geq 0 \right\}.$$

It follows directly that $L_{\lambda_1}(\beta) \subset L_{\lambda_2}(\beta)$, whenever $\lambda_1 > \lambda_2 \geq 0$. In particular, for $\lambda \geq 1$ and $0 \leq \beta < 1$,

$$L_\lambda(\beta) \subset L_1(\beta) = \{h \in \mathscr{A} : \Re h'(\zeta) > \beta\},$$

demonstrating that $L_\lambda(\beta)$ is a univalent subclass (see [6, 7, 12]).

For $h \in \mathscr{A}$, the Sălăgean differential operator $D^m : \mathscr{A} \to \mathscr{A}$ is defined recursively by

$$D^0 h(\zeta) = h(\zeta), \qquad D^1 h(\zeta) = \zeta h'(\zeta), \qquad D^2 h(\zeta) = \zeta (D^1 h(\zeta))',$$

and, more generally for $m \in \mathbb{N}$,

$$D^m h(\zeta) = \zeta + \sum_{n=2}^{\infty} n^m b_n \zeta^n, \quad \zeta \in \Delta \tag{1.2}$$

The operator D^m is known as Sălăgean differential operator (see[16]).

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If $h \in \mathscr{S}$, then h admits an inverse h^{-1} with the properties

$$h^{-1}(h(\zeta)) = \zeta, \quad \zeta \in \Delta,$$

and

$$h^{-1}(h(w)) = w, \quad |w| < r_0(h), \quad r_0(h) \geq \tfrac{1}{4}.$$

The inverse has the power series expansion

$$h^{-1}(w) = w - b_2 w^2 + (2b_2^2 - b_3)w^3 - (5b_2^3 - 5b_2 b_3 + b_4)w^4 + \cdots .$$

A function $h \in \mathscr{A}$ is called bi-univalent if both h and its inverse h^{-1} are univalent in the open unit disk Δ. The collection of all such functions is represented by Σ (see [19]).

Brannan and Taha [3] (see also [20]) introduced certain subclasses of Σ that are linked to the classical families of starlike and convex functions, namely $\mathscr{S}^*(\alpha)$ and $\mathscr{K}(\alpha)$ for $0 \leq \alpha < 1$. As an example, the class $\mathscr{S}^*_\Sigma[\alpha]$ representing strongly bi-starlike functions of order α is defined by functions that satisfy the following condition:

$$h \in \Sigma, \qquad \left| \arg\left(\frac{\zeta h'(\zeta)}{h(\zeta)}\right) \right| < \frac{\alpha\pi}{2}, \quad \zeta \in \Delta,$$

and

$$\left| \arg\left(\frac{w(g'(w))}{g(w)}\right) \right| < \frac{\alpha\pi}{2}, \quad w \in \Delta,$$

where g denotes the analytic continuation of h^{-1} in Δ. These classes, $\mathscr{S}^*_\Sigma(\alpha)$ and $\mathscr{K}_\Sigma(\alpha)$, motivated further studies on coefficient problems, in particular deriving bounds for $|b_2|$ and $|b_3|$. Later, Frasin and Aouf [9] refined such results using methods of Srivastava et al. [19].

The study of Salagean-type operators continues to attract attention in geometric function theory. Recent works include P. Sharma et.al [18], Sagsöz [17], Breaz and Cotîrla [4], Alharbi and Yamaguchi-Noshiro [1], A. Murugan et.al [13], H. Ö. Güney and S. Yalçin[10], Ibrahim [11], and Chang and Janteng [5]. These studies investigated coefficient estimates for generalized, q- and (p,q)-Salagean operators as well as mappings in leaf-like domains. A related extension by Naik and Sahoo [14] derived Fekete-Szegö inequalities for analytic functions in leaf-like domains.

Inspired by these earlier studies, we define two new subclasses of bi-univalent functions, namely $\mathscr{B}_\Sigma(m, \alpha, \lambda)$ and $\mathscr{B}_\Sigma(m, \beta, \lambda)$, associated with the family Σ through a weighted Sălăgean differential operator involving the parameters $(m, \alpha, \beta, \lambda)$. This generalized framework encompasses several existing subclasses as particular cases and offers greater flexibility in analytic characterization. Moreover, the coefficient bounds obtained for $|b_2|$ and $|b_3|$ extend and unify many previously known results in the literature.The study can find application in Refs. [21–24].

Definition 1.1. Let $h(\zeta)$ be given by (1.1). We say h belongs to $\mathscr{B}_\Sigma(m, \alpha, \lambda)$ if

$$h \in \Sigma \quad \text{and} \quad \left| \arg\left((1-\lambda)\frac{D^m h(\zeta)}{\zeta} + \lambda (D^m h(\zeta))'\right) \right| < \frac{\alpha\pi}{2}, \tag{1.3}$$

for $0 < \alpha \leq 1$, $\lambda \geq 1$, and $\zeta \in \Delta$, together with

$$\left| \arg\left((1-\lambda)\frac{D^m g(w)}{w} + \lambda (D^m g(w))'\right) \right| < \frac{\alpha\pi}{2}, \tag{1.4}$$

where g is the inverse of h, expanded as

$$g(w) = w - b_2 w^2 + (2b_2^2 - b_3)w^3 + \cdots . \tag{1.5}$$

Remark 1.2. *If we take $m = 0$, in definition (1.1) this reduces to the class $\mathscr{B}_\Sigma(\alpha, \lambda)$ studied in [9].*

Remark 1.3. *By taking $m = 0$ and $\lambda = 1$ in definition (1.1) it reduce to the class $\mathscr{H}_\Sigma^\alpha$ introduced in [19].*

Definition 1.4. A function $h(\zeta)$ given by (1.1) belongs to $\mathscr{B}_\Sigma(m, \beta, \lambda)$ if

$$h \in \Sigma \quad \text{and} \quad \Re\left((1-\lambda)\frac{D^m h(\zeta)}{\zeta} + \lambda(D^m h(\zeta))'\right) > \beta, \tag{1.6}$$

and

$$\Re\left((1-\lambda)\frac{D^m g(w)}{w} + \lambda(D^m g(w))'\right) > \beta, \tag{1.7}$$

where $0 \le \beta < 1$, $\lambda \ge 1$, and g is the inverse of h as in (1.5).

Remark 1.5. *For $m = 0$, in definition (1.4) this reduces to $\mathscr{B}_\Sigma(\beta, \lambda)$ studied in [9].*

Remark 1.6. *If we take $m = 0$, $\lambda = 1$,in definition (1.4) it becomes $\mathscr{H}_\Sigma(\beta)$, introduced in [19].*

We require the following lemma, which will be used throughout.

Lemma 1.7. *[15] If $f \in P$, then $|d_k| \le 2$ for all k, where P denotes the class of analytic functions in Δ having positive real part and given by the expansion*

$$f(\zeta) = 1 + d_1\zeta + d_2\zeta^2 + d_3\zeta^3 + \cdots, \quad \zeta \in \Delta.$$

2. Main Result

Theorem 2.1. *If $h(\zeta) \in \mathscr{B}_\Sigma(m, \alpha, \lambda)$ with $0 < \alpha \le 1$, $\lambda \ge 1$,then*

$$|b_2| \le \frac{2\alpha}{\sqrt{2^m(\lambda+1)^2 + \alpha\big(3^m(4\lambda+2) - 2^{2m}(\lambda+1)^2\big)}}, \tag{2.1}$$

and

$$|b_3| \le \frac{4\alpha^2}{2^{2m}(\lambda+1)^2} + \frac{2\alpha}{3^m(2\lambda+1)}. \tag{2.2}$$

Proof. From conditions (1.3) and (1.4), there exist analytic functions $p(\zeta)$ and $q(w)$ belonging to P such that

$$(1-\lambda)\frac{D^m h(\zeta)}{\zeta} + \lambda(D^m h(\zeta))' = [p(\zeta)]^\alpha, \tag{2.3}$$

and

$$(1-\lambda)\frac{D^m g(w)}{w} + \lambda(D^m g(w))' = [q(w)]^\alpha. \tag{2.4}$$

Here,

$$p(\zeta) = 1 + \sum_{n=1}^{\infty} p_n \zeta^n, \quad \zeta \in \Delta, \tag{2.5}$$

$$q(w) = 1 + \sum_{n=1}^{\infty} q_n w^n, \quad w \in \Delta. \tag{2.6}$$

Comparing coefficients between (2.3) and (2.4) yields

$$2^m(\lambda+1)b_2 = \alpha p_1, \tag{2.7}$$

$$3^m(2\lambda+1)b_3 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2, \tag{2.8}$$

$$-2^m(\lambda+1)b_2 = \alpha q_1, \tag{2.9}$$

$$3^m(2\lambda+1)(2b_2^2 - b_3) = \alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.10}$$

From (2.7) and (2.9), we deduce

$$p_1 = -q_1. \tag{2.11}$$

Hence,

$$2(2^{2m}(\lambda+1)^2)b_2^2 = \alpha^2(p_1^2 + q_1^2). \tag{2.12}$$

Combining (2.8), (2.10), and (2.12), one obtains

$$2(3^m(2\lambda+1))b_2^2 = \alpha(p_2+q_2) + \frac{\alpha(\alpha-1)}{2} \cdot \frac{2(2^{2m}(\lambda+1)^2)b_2^2}{\alpha^2}.$$

This simplifies to

$$b_2^2 = \frac{\alpha^2(p_2+q_2)}{2^m(\lambda+1)^2 + \alpha\big(3^m(4\lambda+2) - 2^{2m}(\lambda+1)^2\big)}.$$

Applying Lemma 1.7, which guarantees $|p_2|, |q_2| \leq 2$, establishes inequality (2.1).

To estimate $|b_3|$, subtracting (2.10) from (2.8) gives

$$2(3^m(2\lambda+1))(b_3 - b_2^2) = \alpha(p_2 - q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2 - q_1^2). \tag{2.13}$$

Using (2.11) and (2.12), this becomes

$$2(3^m(2\lambda+1))b_3 = \frac{\alpha^2(2\lambda+1)3^m(p_1^2+q_1^2)}{2^{2m}(\lambda+1)^2} + \alpha(p_2 - q_2).$$

Therefore,

$$b_3 = \frac{\alpha^2(p_1^2+q_1^2)}{2(2^{2m}(\lambda+1)^2)} + \frac{\alpha(p_2-q_2)}{2(3^m(2\lambda+1))}.$$

Applying Lemma 1.7 once more ($|p_1|, |q_1|, |p_2|, |q_2| \leq 2$) gives the estimate in (2.2). □

The following cases arise as direct consequences of Theorem 2.1.

Corollary 2.2. *([9]) For $m = 0$ and if $h(\zeta) \in \mathscr{B}_\Sigma(\alpha, \lambda)$, then*

$$|b_2| \leq \frac{2\alpha}{\sqrt{(\lambda+1)^2 + \alpha(1 + 2\lambda - \lambda^2)}}, \tag{2.14}$$

and

$$|b_3| \leq \frac{4\alpha^2}{(\lambda+1)^2} + \frac{2\alpha}{2\lambda+1}. \tag{2.15}$$

Corollary 2.3. *([19]) If $\lambda = 1$, $m = 0$, and $h(\zeta) \in \mathscr{H}_\Sigma^\alpha$, then*

$$|b_2| \leq \alpha\sqrt{\frac{2}{2+\alpha}}, \tag{2.16}$$

$$|b_3| \leq \frac{\alpha(3\alpha+2)}{3}. \tag{2.17}$$

Theorem 2.4. *If $h(\zeta) \in \mathscr{B}_\Sigma(m, \beta, \lambda)$ with $0 \leq \beta < 1$, $\lambda \geq 1$, then*

$$|b_2| \leq \sqrt{\frac{2(1-\beta)}{3^m(2\lambda+1)}}, \tag{2.18}$$

and

$$|b_3| \leq \frac{4(1-\beta)^2}{2^{2m}(\lambda+1)^2} + \frac{2(1-\beta)}{3^m(2\lambda+1)}. \tag{2.19}$$

Proof. From (1.6)(1.7), we can find analytic functions $p(\zeta), q(w) \in P$ such that

$$(1-\lambda)\frac{D^m h(\zeta)}{\zeta} + \lambda(D^m h(\zeta))' = \beta + (1-\beta)p(\zeta), \tag{2.20}$$

$$(1-\lambda)\frac{D^m g(w)}{w} + \lambda(D^m g(w))' = \beta + (1-\beta)q(w). \tag{2.21}$$

Using the expansions (2.5)(2.6) and comparing coefficients, we obtain

$$2^m(\lambda+1)b_2 = (1-\beta)p_1, \tag{2.22}$$

$$3^m(2\lambda+1)b_3 = (1-\beta)p_2, \tag{2.23}$$

$$-\,2^m(\lambda+1)b_2 = (1-\beta)q_1, \tag{2.24}$$

$$3^m(2\lambda+1)(2b_2^2 - b_3) = (1-\beta)q_2. \tag{2.25}$$

From (2.22) and (2.24), it follows that

$$p_1 = -q_1. \tag{2.26}$$

Consequently,

$$2(2^{2m}(\lambda+1)^2)b_2^2 = (1-\beta)^2(p_1^2 + q_1^2). \tag{2.27}$$

Moreover, combining (2.23) and (2.25), we find

$$2(3^m(2\lambda+1))b_2^2 = (1-\beta)(p_2 + q_2).$$

Using Lemma 1.7, which guarantees $|p_2|, |q_2| \leq 2$, leads to (2.18).

For $|b_3|$, subtract (2.25) from (2.23), giving

$$2(3^m(2\lambda+1))(b_3 - b_2^2) = (1-\beta)(p_2 - q_2),$$

which can be written as

$$b_3 = \frac{(1-\beta)^2(p_1^2+q_1^2)}{2(2^{2m}(\lambda+1)^2)} + \frac{(1-\beta)(p_2-q_2)}{2(3^m(2\lambda+1))}.$$

Applying Lemma 1.7, inequality (2.19) follows immediately. □

Corollary 2.5. *([9]) For $m = 0$ and if $h(\zeta) \in \mathscr{B}_\Sigma(\beta, \lambda)$, then*

$$|b_2| \leq \sqrt{\frac{2(1-\beta)}{2\lambda+1}}, \tag{2.28}$$

and

$$|b_3| \leq \frac{4(1-\beta)^2}{(\lambda+1)^2} + \frac{(1-\beta)}{2\lambda+1}. \tag{2.29}$$

Corollary 2.6. *([19]) For $\lambda = 1$, $m = 0$, and if $h(\zeta) \in \mathscr{H}_\Sigma(\beta)$, then*

$$|b_2| \leq \sqrt{\frac{2(1-\beta)}{3}}, \tag{2.30}$$

$$|b_3| \leq \frac{(1-\beta)(5-3\beta)}{3}. \tag{2.31}$$

3. Conclusion

In this paper, two new subclasses of bi-univalent functions belonging to the family Σ were introduced using the Sălăgean differential operator. For each subclass, sharp bounds for the initial coefficients $|b_2|$ and $|b_3|$ were obtained, extending and unifying several earlier results available in the literature. The derived estimates demonstrate the effectiveness of the proposed operator-based approach in addressing coefficient problems for bi-univalent functions. The results presented here provide a useful foundation for further research, including the study of higher-order coefficient bounds, Fekete–Szegö inequalities, Hankel determinants, and other related problems associated with these subclasses.

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