A Systematic Research on Various Types of Hausdorff Hypergraphs

Abstract

A hypergraph $H = (V, \mathcal{E})$ is said to be a *Hausdorff hypergraph* if for any two distinct vertices u, v of V there exist hyperedges e_1, $e_2 \in \mathcal{E}$ such that $u \in e_1$, $v \in e_2$ and $e_1 \cap e_2 = \emptyset$.

In this paper we have discussed hausdorff property of hypergraphs as well as minimal hausdorff hypergraph. Previous work on Hausdorff-type separation properties, driven by classical topology and graph theory, has included the study of specific types of hypergraphs to address vertex separability with disjoint hyperedges. Continuing from this stream of research, this work aims to conduct an in-depth study of Hausdorff hypergraphs, with special emphasis on minimal Hausdorff hypergraphs and their variants. We obtain results on minimal Hausdorff hypergraphs with respect to bounds of the number of hyperedges and study sufficient conditions for competition hypergraphs of digraphs and independent hypergraphs of graphs to be Hausdorff. Links with conformal hypergraphs, cyclomatic number, and acyclicity are considered in an attempt to cover various aspects of Hausdorff separation principles on hypergraphs, thereby illustrating the merits of new types of hypergraphs in terms of improved vertex distinguishability and their usefulness for modeling higher-order networks.

Keywords: Hausdorff hypergraph,Minimal Hausdorff,Conformal, Cyclotomic number, Competition hypergraph, Independent hypergraph.

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*Corresponding author

Email address: seenav@christcollegeijk.edu.in (Seena V*)

1. Introduction

Hypergraphs are generalization of graphs, hence many of the definitions of graphs carry verbatim to hypergraphs. The basic idea of the hypergraph concept is to consider such a generalization of a graph in which any subset of a given set may be an edge rather than two-element subsets [1]. A *hypergraph*[2] H is a pair $(V, \mathcal{E})$, where V is a set of elements called nodes or vertices, and $\mathcal{E}$ is a set of nonempty subsets of V called *hyperedges* or *edges.* Therefore, $\mathcal{E}$ is a subset of $P(X)\backslash\{\emptyset\}$, where $P(X)$ is the power set of X. In drawing hypergraphs, vertices are points in the plane, edges are closed curves separating a respective subset from the rest of vertices. The cardinality of the finite set V is denoted by $|V|$, is called the *order* of the hypergraph. The number of edges is usually denoted by m or $m(H)$. A hypergraph which contains no vertices and no edges is called an empty hypergraph. A *trivial hypergraph* is a hypergraph such that $V \neq \emptyset$ and $E = \emptyset$.

A *simple hypergraph*[3] is a hypergraph with the property if e_i, e_j are hyperedges of H with $e_i \subseteq e_j$, then $i = j$. In other words a hypergraph having no multiple edges is called *simple.* Hence simple hypergraphs do not have empty and multiple edges. Two vertices in a hypergraph are *adjacent*[1] if there is a hyperedge which contains both vertices. Two hyperedges in a hypergraph are *incident*[1] if their intersection is nonempty.

A *k-uniform hypergraph*[4] or a k-hypergraph is a hypergraph in which every edge consists of k vertices. So a 2-uniform hypergraph is a graph, a 3-uniform hypergraph is a collection of unordered triples, and so on. The *rank* [1] $r(H)$ of a hypergraph is the maximum cardinality of any of the edges in the hypergraph. The *co-rank* [1] $cr(H)$ of a hypergraph is the minimum cardinality of a hyperedge in the hypergraph. If $r(H) = cr(H) = k$, then H is k-uniform. The degree [5] $d_H(v)$ of a vertex v in a hypergraph H is the number of edges of H that containing the vertex v. H is k-regular if every vertex has degree k.

An edge of a hypergraph which contains no vertices is called an *empty edge*. The degree of an empty edge is trivially zero. A vertex of a hypergraph which is incident to no edges is called an *isolated vertex.*[1] The degree of an isolated vertex is trivially zero. A hyperedge e of H with $|e| = 1$ is called a *loop*; more specifically a hyperedge $e = \{v\}$ is a loop at the vertex v. A vertex of degree 1 is called a pendant vertex .

A simple hypergraph H with $|E_i| = 2$ for each $E_i \in \mathcal{E}$ is a simple graph.

Assumptions and scope of the paper. Throughout this paper, we will deal only with finite and simple hypergraphs, having no isolated vertices and no empty hyperedges. Unless otherwise said so, all digraphs considered here are finite and simple. To be clear, the term *Hausdorff hypergraph* is used in the context of vertex separability by disjoint hyperedges and bears no relation to the classical topological notion of Hausdorff spaces.

It is on the structural properties of Hausdorff hypergraphs, with a particular emphasis on the minimal ones, and conditions under which competition hypergraphs of digraphs and independent hypergraphs of graphs are Hausdorff.

The following results do not claim to fully characterize all Hausdorff hypergraphs, but rather aim at sufficient and, in some cases necessary, conditions for Hausdorff separation within specific classes of hypergraphs. It is assumed, to agree with the usual definition, that competition hypergraphs have hyperedges of cardinality at least two. Weighted hypergraphs, and infinite hypergraphs are beyond the scope of the present study and hence not considered here.

2. Hypergraphs

Definition 2.1. A Hausdorff hypergraph H is said to be a *minimal Hausdorff hypergraph* if the removal of any hyperedge from H results in a hypergraph that is no longer Hausdorff. Thus, every hyperedge of H is essential for preserving the Hausdorff separation property. .

Example 2.1. Consider the hypergraph $H = (V, E)$, where

$$V = \{v_{11}, v_{12}, v_{21}, v_{22}\}$$

and

$$E = \{e_1, e_2, f_1, f_2\},$$

with

$$e_1 = \{v_{11}, v_{12}\}, \quad e_2 = \{v_{21}, v_{22}\}, \quad f_1 = \{v_{11}, v_{21}\}, \quad f_2 = \{v_{12}, v_{22}\}.$$

For any two distinct vertices of H, there exist two disjoint hyperedges containing them separately, and hence H is a Hausdorff hypergraph. However, removal of any hyperedge from E destroys this property. Therefore, H is a minimal Hausdorff hypergraph.

Theorem 2.1. Let n be a composite positive integer. Then there exists a minimal Hausdorff hypergraph G on n vertices such that the number of hyperedges of G is less than or equal to

$$\min\{\, p + q : p, q \in \mathbb{N},\ pq = n \,\}.$$

Proof. Since n is composite, there exist integers $p, q \geq 2$ such that $pq = n$. Choose such a factorization for which $p + q$ is minimum.

Let

$$V(G) = \{v_{ij} : 1 \leq i \leq p,\ 1 \leq j \leq q\},$$

so that $|V(G)| = pq = n$.

Define the hyperedge set $E(G)$ as follows:

- For each $i = 1, 2, \ldots, p$, define

$$e_i = \{v_{i1}, v_{i2}, \ldots, v_{iq}\}.$$

- For each $j = 1, 2, \ldots, q$, define

$$f_j = \{v_{1j}, v_{2j}, \ldots, v_{pj}\}.$$

Thus,

$$E(G) = \{e_1, e_2, \ldots, e_p\} \cup \{f_1, f_2, \ldots, f_q\},$$

and hence $|E(G)| = p + q$.

Hausdorff property. Let $u = v_{i_1 j_1}$ and $v = v_{i_2 j_2}$ be any two distinct vertices of G.

- If $i_1 \neq i_2$, then

$$u \in e_{i_1}, \quad v \in e_{i_2}, \quad \text{and} \quad e_{i_1} \cap e_{i_2} = \varnothing.$$

- If $i_1 = i_2$ and $j_1 \neq j_2$, then

$$u \in f_{j_1}, \quad v \in f_{j_2}, \quad \text{and} \quad f_{j_1} \cap f_{j_2} = \varnothing.$$

Thus, for every pair of distinct vertices, there exist disjoint hyperedges containing them separately. Hence, G is a Hausdorff hypergraph.

Minimality. Suppose a hyperedge e_i is removed from G. Consider two distinct vertices v_{ij_1} and v_{ij_2} with $j_1 \neq j_2$. After removing e_i, the only hyperedges containing these vertices are f_{j_1} and f_{j_2}, which do not provide disjoint separation for these vertices. Hence, the Hausdorff property fails.

Similarly, removing any hyperedge f_j destroys the Hausdorff property. Therefore, the removal of any hyperedge from G results in a hypergraph that is no longer Hausdorff.

Thus, G is a minimal Hausdorff hypergraph with

$$|E(G)| = p + q \leq \min\{p + q : pq = n\}.$$

□

Remark 2.1. For every perfect square r^2 there exists a $r - uniform$ Hausdorff hypergraph.

Remark 2.2. For every perfect square r^2 there exists a $r-$ *partite* Hausdorff hypergraph.

3. Competition Hypergraphs

Definition 3.1. [6] If $D = (V, A)$ is a digraph, its *competition hypergraph* $CH(D)$ has vertex set V and $e \subseteq V$ is an edge of $CH(D)$ if and only if $|e| \geq 2$ and there is a vertex $v \in V$, such that $e = \{w \in V/(w, v) \in A\}$. In this case we say that $v \in V$ corresponds to $e \in \mathcal{E}(CH(D))$ and vice versa.

Theorem 3.1. A simple digraph D has a competition hypergraph only if $|V(D)| \geq 3$.

Proof. From the definition of competition hypergraphs e is an edge of $CH(D)$ if $|e| \geq 2$ and there is a vertex $v \in V$ such that $e = \{w \in V/(w, v) \in A)\}$ therefore, $|V(D)| \geq 3$. □

Theorem 3.2. If the competition hypergraph $CH(D)$ of a digraph D is Hausdorff, then

$$|V(D)| > 4.$$

Proof. Let $D = (V, A)$ be a digraph and let $CH(D)$ denote its competition hypergraph. Recall that a hyperedge of $CH(D)$ is of the form

$$e = \{w \in V : (w, x) \in A\},$$

for some $x \in V$, and satisfies $|e| \geq 2$. A hypergraph is Hausdorff if for any two distinct vertices $u, v \in V$ there exist disjoint hyperedges e_1 and e_2 such that $u \in e_1$, $v \in e_2$, and $e_1 \cap e_2 = \varnothing$.

Case 1: $|V(D)| \leq 3$. Since every hyperedge of $CH(D)$ has cardinality at least 2, it is impossible to find two disjoint hyperedges separating any pair of vertices when $|V(D)| \leq 3$. Hence, $CH(D)$ is not Hausdorff in this case.

Case 2: $|V(D)| = 4$. Let

$$V(D) = \{v_1, v_2, v_3, v_4\}.$$

Assume, for contradiction, that $CH(D)$ is Hausdorff. Consider the vertices v_1 and v_2. Then there must exist disjoint hyperedges e_1 and e_2 such that

$$v_1 \in e_1, \quad v_2 \in e_2, \quad e_1 \cap e_2 = \varnothing.$$

Since every hyperedge has size at least 2, the only possible disjoint choices are

$$\{v_1, v_3\} \text{ and } \{v_2, v_4\}, \quad \text{or} \quad \{v_1, v_4\} \text{ and } \{v_2, v_3\}.$$

Suppose $\{v_1, v_3\}$ is a hyperedge of $CH(D)$. Then there exists a vertex $x \in V(D)$ such that

$$(v_1, x), (v_3, x) \in A,$$

and hence $x \in \{v_2, v_4\}$. In either case, it is impossible for $\{v_2, v_4\}$ to arise as a hyperedge of $CH(D)$, yielding a contradiction. A similar contradiction arises in the case $\{v_1, v_4\}$ and $\{v_2, v_3\}$.

Thus, no pair of disjoint hyperedges can separate v_1 and v_2, contradicting the assumption that $CH(D)$ is Hausdorff. Therefore, $CH(D)$ is not Hausdorff when $|V(D)| = 4$.

Since $CH(D)$ fails to be Hausdorff for all cases $|V(D)| \leq 4$, we conclude that

$$|V(D)| > 4$$

is a necessary condition for the competition hypergraph of a digraph to be Hausdorff. □

In Figure 1, D is a digraph on 5 vertices and its competition hypergraph $CH(D)$ is Hausdorff .

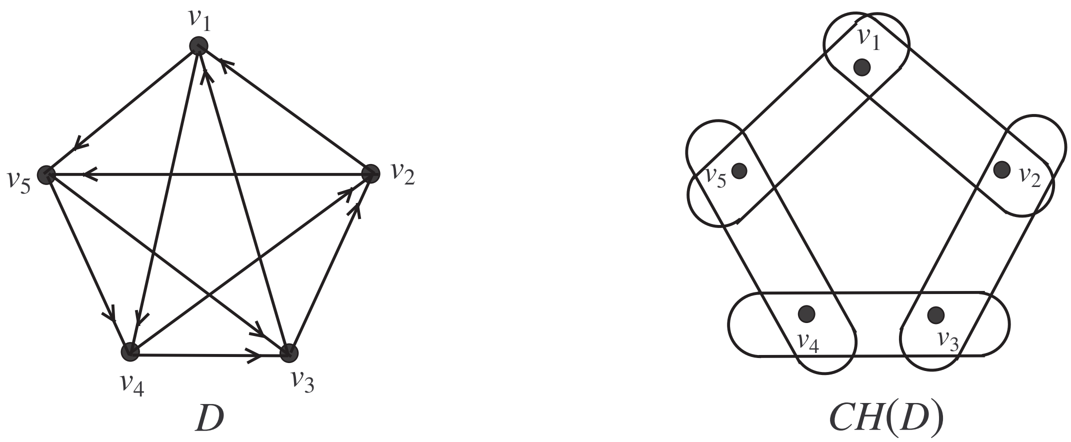

Figure 1: Digraph D and its competition hypergraph $CH(D)$.

Proposition 3.1. Let $D = (V, A)$ be a digraph. If given any two distinct vertices u and v of D, there exist two distinct vertices x and y such that $u \in P = \{w : (w, x) \in A\}$ and $v \in Q = \{z : (z, y) \in A\}$ and $P \cap Q = \emptyset$, then the competition hypergraph $CH(D)$ of D is Hausdorff .

Definition 3.2. A hypergraph is said to be *conformal*[7] if its hyperedges are exactly the cliques of its primal graph.

The hypergraph H in Example 3.1(Figure 2) is conformal but it is not Hausdorff .

Example 3.1.

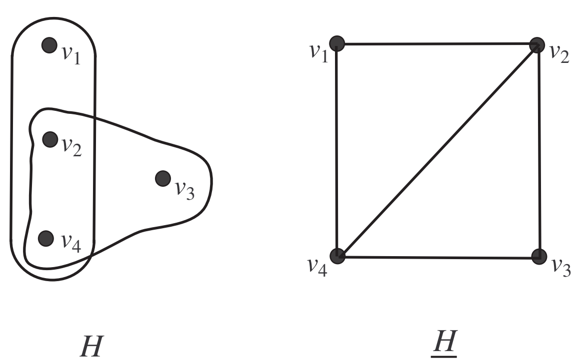

Figure 2: Hypergraph H and its primal graph $\underline{H}$

Example 3.2.

Consider the hypergraph $H = (V, \mathcal{E})$. Where $V = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{E} = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}, \{v_1, v_3\}, \{v_2, v_4\}\}$. Clearly $H = (V, \mathcal{E})$ is a Hausdorff hypergraph. Then $\underline{H} = (V, \mathcal{E})$, where $V = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{E} = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3, v_2v_4\}$ (see Figure 3). Note that $\underline{H}$ is itself a clique on four vertices and there exists no hyperedge containing all these four vertices. Therefore, H is not conformal.

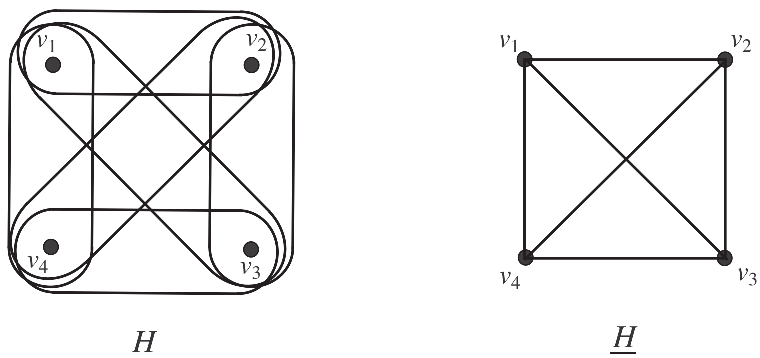

Figure 3: Hypergraph H and its primal graph $\underline{H}$.

Remark 3.1. Example 3.2 shows that a hypergraph H is Hausdorff need not imply it is conformal.

Definition 3.3. Let $H = (V, \mathcal{E})$ be a hypergraph with $\mathcal{E} = \{e_1, e_2, \dots, e_m\}$. The *cyclomatic number*[2] of H is the non-negative integer

$$\mu(H) = \sum_{i=1}^{m} |e_i| - |\bigcup_{i=1}^{m} e_i| - w_H$$

where w_H is the maximum weight of a forest $F \subset L(H)$.

Theorem 3.3. The line graph $L(H)$ of H_{mn} is a complete bipartite graph.

Theorem 3.4. The cyclomatic number of a minimal Hausdorff hypergraph on mn vertices is $mn - (m + n - 1)$

Proof. The minimal Hausdorff graph on mn vertices has m edges $e_1, e_2, \dots e_m$ and n edges $f_1, f_2, \dots f_n$. Then $L(H)$ is a complete bipartite graph on mn vertices, so $L(H) = K_{m,n}$. Weight of each edge is 1. Maximum weight of a forest in $L(H)$ is $(m+n-1.)$ Therefore, $\mu(H) = \sum_{i=0}^{m} |e_i| - |\bigcup_{i=1}^{m} e_i| - w(H)$ i.e $\mu(H) = 2mn - mn - (m + n - 1)$. This implies $\mu(H) = mn - (m + n - 1)$. □

There are many definitions for the *acyclicity of a hypergraph* [8]. The definition we use here is based on the Graham reduction [8], described below. Let $H = (V, \mathcal{E})$ be a given hypergraph. Graham's algorithm applies the following operations repeatedly to H until neither can be applied:

1. If a vertex $v \in V$has degree one, then delete v from the edge containing it.
2. If $e, f \in \mathcal{E}$ are distinct edges such that $e \subseteq f$, then delete e from $\mathcal{E}$.
3. If $e \in \mathcal{E}$ is empty, then delete e from $\mathcal{E}$.

The resulting hypergraph H is said to be Graham-reduced, and is called the *Graham reduction*[8] of H.

Definition 3.4. [8] A hypergraph is acyclic if its Graham reduction is empty. Otherwise it is called cyclic.

In a simple Hausdorff hypergraph without loops, degree of each vertex is ≥ 2, so we have:

Remark 3.2. All simple Hausdorff hypergraph without loops is cyclic.

Remark 3.3. If H is a hypergraph with all its edges are loops, then H is an acyclic hypergraph.

4. Independent Hypergraph

Now we define independent hypergraph as

Definition 4.1. Let $G = (V, E)$ be the given graph. An *independent hypergraph* $IH(G)$ is the hypergraph whose vertices are vertices of G and whose hyperedges are maximal independent subsets of G.

Example 4.1. Consider the graph $G = (V, E)$ where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_2v_4, v_3v_5\}$. The maximal independent subsets of G are $\big\{\{v_1, \}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_2, v_5\}, \{v_4, v_5\}\big\}$. The corresponding independent hypergraph is $IH(G) = (V, \mathcal{E})$ where $\mathcal{E} = \big\{\{v_1, \}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_2, v_5\}, \{v_4, v_5\}\big\}$. (See Figure 4)

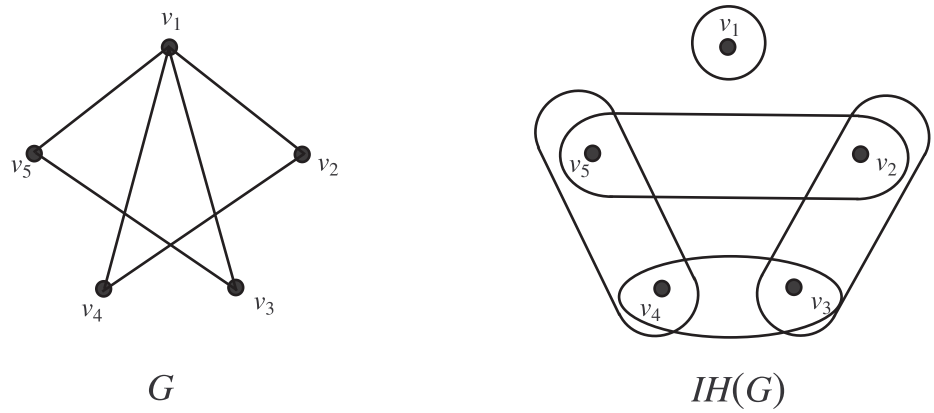

Figure 4: Graph G and its independent hypergraph $IH(G)$.

If $IH(G)$ is Hausdorff then we call $IH(G)$ as *Hausdorff independent hypergraph.*

Theorem 4.1. If G is a complete graph then $IH(G)$ is Hausdorff .

Proof. Since G is complete the only independent sets in G are singleton sets. Then the corresponding independent hypergraph $IH(G)$ is a hypergraph with only loops. Therefore, $IH(G)$ is Hausdorff . □

Theorem 4.2. If G is not complete then $IH(G)$ is Hausdorff only if $|V(G)| \geq 4$

Proof.

Case 1. $|V(G)| = 2$

In this case since G is not complete, it must be an empty graph. Let $G = (V, E)$ where $V = \{v_1, v_2\}$ and $E = \{\{v_1\}, \{v_2\}\}$. Then the maximal independent subset of G is $\{v_1, v_2\}$. The corresponding independent hypergraph $IH(G) = (V, \mathcal{E})$ where $\mathcal{E} = \{\{v_1, v_2\}\}$. Therefore, $IH(G)$ is not Hausdorff .

Case 2. $|V(G)| = 3$

Given that G is not complete. Then there arise three cases. G is an empty graph, or G contains one edge or G contain 2 edges

Subcase 1. G is an empty graph

Let $G = (V, E)$ where $V = \{v_1, v_2, v_3\}$ and $E = \emptyset$. Then G has only one maximal independent set, which is $\{v_1, v_2, v_3\}$. The corresponding independent hypergraph is $IH(G) = (V, \mathcal{E})$ where $V = \{v_1, v_2, v_3\}$ and $\mathcal{E} = \{\{v_1, v_2, v_3\}\}$ (see Figure 5). Note that $IH(G)$ is not Hausdorff .

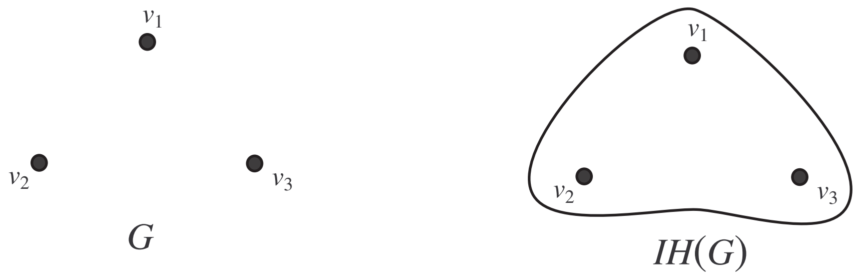

Figure 5: Graph G and its independent hypergraph $IH(G)$.

Subcase 2. G has only one edge

Let $G = (V, E)$ where $V = \{v_1, v_2, v_3\}$ and $E = \{v_1v_2\}$. Then the maximal independent sets in G are $\{v_1, v_3\}$ and $\{v_2, v_3\}$. The corresponding independent hypergraph is $IH(G) = (V, \mathcal{E})$ where $V = \{v_1, v_2, v_3\}$ and $\mathcal{E} = \{\{v_1, v_3\}, \{v_2, v_3\}\}$. (see Figure 6). Note that $IH(G)$ is not Hausdorff .

Figure 6: Graph G and its independent hypergraph $IH(G)$.

Subcase 3. G has two edges

Let $G = (V, E)$ where $V = \{v_1, v_2, v_3\}$ and $E = \{v_1v_2, v_1v_3\}$. Then the maximal independent sets in G are $\{v_1\}$ and $\{v_2, v_3\}$. The corresponding independent hypergraph is $IH(G) = (V, \mathcal{E})$ where $V = \{v_1, v_2, v_3\}$ and $\mathcal{E} = \{\{v_1\}, \{v_2, v_3\}\}$ (see Figure 7). Note that $IH(G)$ is not Hausdorff .

Figure 7: Graph G and its independent hypergraph $IH(G)$.

Case 3. $|V(G)| = 4$

Let $G = (V, E)$ where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_3, v_2v_4\}$. Then the maximal independent sets in G are $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_2, v_4\}$ and $\{v_3, v_4\}$. The corresponding independent hypergraph is $IH(G) = (V, \mathcal{E})$ where $V = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{E} = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\}$.(see Figure 8). Note that $IH(G)$ is Hausdorff .

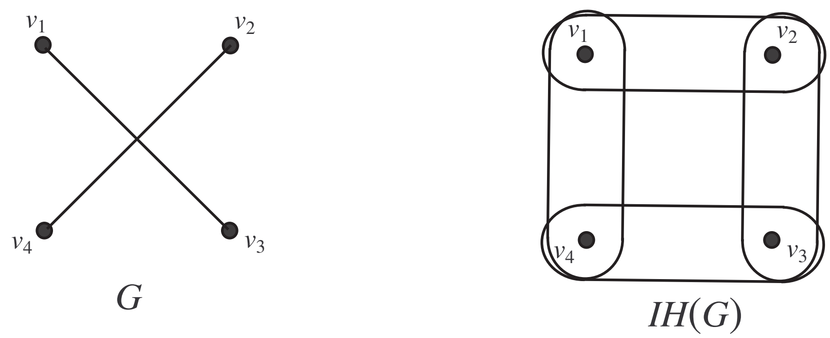

Figure 8: Graph G and its independent hypergraph $IH(G)$.

Thus we proved if G is not complete then $IH(G)$ is Hausdorff only if $|V(G)| \geq 4$ □

Theorem 4.3. Let $G = (V, E)$ be a bipartite graph with bipartion X and Y where $|X| = |Y| = n$ and degree of each vertex is $(n-1)$ and $x_iy_i \neq E$. Then the corresponding independent hypergraph is Hausdorff .

Proof. Given that $G = (V, E)$ is bipartite graph with bipartition X and Y where $|X| = |Y| = n$. Each vertex has degree $(n - 1)$ and $x_iy_i \neq E$ implies $\{x_i, y_i\}$ is a maximal independent set in G for every $i = 1, 2, 3 \ldots n$. Note that $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ are also maximal independent sets in G. The independent hypergraph $IH(G)$ corresponding to G is $(V, \mathcal{E})$, where $V = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ and $\mathcal{E} = \big\{\{x_1, x_2, \ldots, x_n\}, \{y_1, y_2, \ldots, y_n\}, \{x_1, y_1\}, \ldots, \{x_n, y_n\}\big\}$. Clearly $IH(G)$ is Hausdorff . Hence the theorem. □

Proposition 4.1. Given any two vertices x_1 and x_2 suppose there exists independent dominating sets A and B containing x_1 and x_2 respectively such that $V(A) \cap V(B) = \emptyset$ then the corresponding independent hypergraph is Hausdorff .

5. Application

In the field of medical research, it usually happens that different physicians give various combinations of drugs to patients with the same disease. To evaluate the potency and variety of such prescriptions, the situation can be modeled with hypergraph theory. Each patient is modeled as a vertex, and each combination of medicines prescribed is a hyperedge linking all patients using that combination. Two vertices are considered adjacent if the respective patients have at least one common medicine. Therefore, the hypergraph representation allows us to visualize and examine the crossover of medicines among patients. We say a medical sample is effective if, for any two patients, there exist disjoint hyperedges containing them separately. In this case, the hypergraph corresponding to the sample has the Hausdorff property, which implies that entirely separate sets of medicines can indicate any two patients. This modeling technique gives a clear mathematical framework to study variation in treatment, detect independent therapeutic paths, and study the separability of prescription strategies among patients coming from different regions or medical institutions.

Despite the numerous benefits of Hausdorff hypergraphs in handling the separability of treatment patterns, various methods currently existing in the area of medical diagnosis, using both graphs and hypergraphs, suffer from some drawbacks. Traditional graph models can only handle pairwise interactions and are unable to address the top-level interactions between patient sets and treatment sets. Though these models of hypergraphs can address the weakness of traditional models of a graph, they suffer from some drawbacks of having poor separation properties.

Although the Hausdorff property is useful in guaranteeing vertex septability with disjoint hyperedges, it also has some natural limitations. For medical data, it could be quite a challenge for the sets of prescribed drugs to be fully disjoint, giving rise to a case where the structure is not Hausdorff despite having some form of partial appartness. Additionally, the Hausdorff property is not concerned with the weight, varying dosages, or dynamic treatment patterns. Such natural limitations pose a challenge, prompting the need for either generalized Hausdorff properties or weighted hypergraph theories.

6. Conclusion

In this paper we have discussed hausdorff property of hypergraphs as well as minimal hausdorff hypergraph. We have also examined conditions under which competetion hypergraph and independence hypergraph become hausdorff. Also we have discussed some applications of Hausdorff hypergraph.

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