BOUNDS FOR THE k-FORCING PROPAGATION TIMES IN GRAPHS

Abstract. A subset Z_k of vertices in a graph is called a k-forcing set if its vertices are initially assigned a color while the remaining vertices start as uncolored. The graph undergoes a color propagation process based on the following rule: a colored vertex with at most k uncolored neighbors forces each of those neighbors to become colored. This process continues until all vertices in the graph are colored. The k-forcing number of a graph, denoted as $Z_k(G)$, represents the smallest possible size of a k-forcing set. The propagation time of a k-forcing set of graph G is the minimum number of steps that it takes to force all the vertices black, starting with the vertices in the k-forcing set and performing independent forces simultaneously. The minimum and maximum k-forcing propagation times of a graph are taken over all minimum k-forcing sets of the graph. We discuss the minimum k-forcing propagation time of graph G. Additionally, it delves into the precise determination of bounds for k-forcing propagation times. for certain well-known graphs. Also characterized regular graphs with k-forcing propagation time one.

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1. Introduction

Zero forcing is a graph coloring process that begins with vertices initially colored either white or black. The procedure repeatedly applies the following rule: if a black vertex has exactly one white (out-)neighbor, that neighbor is "forced" to change its color to black. A zero forcing set is an initial selection of black vertices capable of turning all vertices in the graph black through this process. The minimum cardinality of such a set is known as the zero forcing number of the graph G, denoted by $Z(G)$.

The propagation time of a zero forcing set refers to the number of steps required to completely color all the vertices of a graph black, starting from the initial zero forcing set. During this process, independent forces are applied simultaneously at each step, meaning that all vertices are capable of forcing a neighbor to change color do so in parallel until no further forcing is possible. The propagation time of a zero forcing set is discussed in [4]. The notion of propagation time, which measures how many rounds of simultaneous independent forces are required for a zero forcing set to color an entire graph, was implicit in [8] and later made explicit in [9]. Chilakamarri et al. [10] investigated this parameter (which they term the iteration index) for several graph families, including Cartesian products and a variety of grid graphs. Since controlling a whole network through sequential operations on a limited subset of particles is highly valuable [9], understanding the number of steps required to achieve full control—captured by the propagation time—is an essential aspect of the process. Similar kind of propagation is introduced in [11], [12], [13] and [14].

For a simple graph G and $k > 0$ a positive integer, the k-forcing number of G, denoted by $Z_k(G)$ is the minimum number of vertices that are needed to be initially colored black so that all vertices after a finite number of steps become colored black during the subsequent color changing rule.
Color changing rule : If a black colored vertex has at most k non-colored neighbors, then each of its non-colored neighbors becomes colored as black. When $k = 1$, this definition is same as that of the zero forcing number, denoted by $Z(G)$ see [5]. In [5] Amos et al. proposed the definition of k-forcing set. In [7] the authors presence a study on the k-forcing number in the context of splitting graphs. Yair Caro et al. in [6] proved a conjuncture post by Amos et al.in [5].

In this article, we extend this propagation time into k-forcing sets and introduce the minimum and the maximum k-forcing propagation time of a graph. Also found some bounds for k-forcing propagation time of a graph.

2. Motivation

The study of k-forcing propagation time in graphs is motivated by the deep connections to the spread of information, control processes, and monitoring in complex networks. The process of k-forcing extends the standard Zero Forcing rule by allowing a colored vertex to force up to k uncolored neighbors in one step, incorporating flexibility into the modeling of real diffusion processes. Indeed, many natural systems include agents that can influence several neighbors at once, due to the spread of influence in social networks, the propagation of signals in communicational systems, or activations in neural and biological networks. Consequently, the capability to determine how influence spreads within such a system may be a matter of both theoretical and practical concern.

From a graph-theoretic perspective, the k-forcing propagation time provides insight into the efficiency of dynamic coloring processes and the temporal behavior of forcing sets. It enables scholars to measure not only the minimum number of initial colored vertices required to eventually color the whole graph but also how fast this takes place. This time dimension enhances the classical study of forcing sets by allowing a deeper understanding of such structural properties as bottlenecks, centrality, and resilience of a network.

Furthermore, the study of k-forcing propagation time has algorithmic and computational implications. The determination of optimal forcing sets naturally induces challenging optimization problems: the search to predict their propagation behavior intersects with areas such as network design, resource placement, graph searching, and power domination. These results can guide strategy for efficient monitoring of power grids, minimizing control points in distributed systems, and improving protocols for fast and reliable information dissemination.

In general, there is a motivation to explore k-forcing propagation time because, by blending theoretical graph parameters with dynamic spread processes, it provides a versatile tool for analyzing and optimizing how influence moves through networks. Thus, this growing area combines both mathematical elegance and rich applications, constituting an important direction of research in modern graph theory.

3. k-forcing propagation time of graphs

In this section, we discuss the minimum and the maximum k-forcing propagation time for graph G. Also, find the exact value of minimum k-forcing propagation time for certain well-known graphs.

Definition 3.1. [4] *Let $G = (V, E)$ be a graph and Z a zero forcing set of G. Define $A^{(0)} = Z$, and for $t \geq 0$, $A^{(t+1)}$ is the set of vertices w for which there exists a vertex $b \in \bigcup_{s=0}^{t} A^{(s)}$ such that w is the only neighbor of b not in $\bigcup_{s=0}^{t} A^{(s)}$. The propagation time of Z in G, denoted $pt(G, Z)$, is the smallest integer t_0 such that $V = \bigcup_{s=0}^{t_0} A^{(s)}$.Two minimum zero forcing sets of the same graph may have different propagation times.The minimum propagation time of G is*

$$pt(G) = \min\{pt(G, Z)|Z \text{ is a minimum zero forcing set of } G\}.$$

and the maximum propagation time of G is

$$PT(G) = \max\{pt(G, Z)|Z \text{ is a minimum zero forcing set of } G\}.$$

We extend the concept of propagation time into k-forcing sets. We define the minimum and maximum propagation times for the k-forcing set of G as follows.

Definition 3.2. *Let $G = (V, E)$ be a graph and Z_k be a k-forcing set of G, where $k > 0$ is a fixed positive integer. Define $F^{(0)} = Z_k$, and for $p \geq 0$, $F^{(p+1)}$ is the set of vertices w such that there exists a vertex $b \in \bigcup_{s=0}^{p} F^{(s)}$ with at most k neighbors outside $\bigcup_{s=0}^{p} F^{(s)}$, and w is one of the neighbors of b not already in $\bigcup_{s=0}^{p} F^{(s)}$. The propagation time of Z_k in G, denoted $pt_k(G, Z_k)$, is the smallest integer p_0 such that $V = \bigcup_{s=0}^{p_0} F^{(s)}$.Two minimum k-forcing sets of the same graph may have different propagation times. The minimum k-forcing propagation time of G is defined as follows*

$$pt_k(G) = \min\{pt_k(G, Z_k)|Z_k \text{ is a minimum } k\text{-forcing set of } G\}.$$

Definition 3.3. *A subset Z_k of vertices of G is an efficient k-forcing set for G if Z_k is a minimum k-forcing set of G and $pt_k(G, Z_k) = pt_k(G)$.*

Definition 3.4. *The maximum k-forcing propagation time of G is*

$$PT_k(G) = \max\{pt_k(G, Z_k)|Z_k \text{ is a minimum } k\text{-forcing set of } G\}.$$

Consider the following example to show that two minimum k-forcing sets of the same graph may have different propagation times.

Example 3.5. *Let G be the graph in figure 1. Let $Z_2 = \{v_4\}$ and $Z'_2 = \{v_1\}$ are two minimum 2-forcing sets for G. Then $F^{(0)} = \{v_4\}$, $F^{(1)} = \{v_3, v_5\}$, $F^{(2)} = \{v_1, v_2, v_6, v_7\}$, so $pt_2(G, Z_2) = 2$. However $F'^{(0)} = \{v_1\}$, $F'^{(1)} = \{v_3\}$, $F'^{(2)} = \{v_2, v_4\}$, $F'^{(3)} = \{v_5\}$, $F'^{(4)} = \{v_6, v_7\}$, so $pt(G, Z'_2) = 4$. Hence $pt_2(G) = 2$, and $PT_2(G) = 4$.*

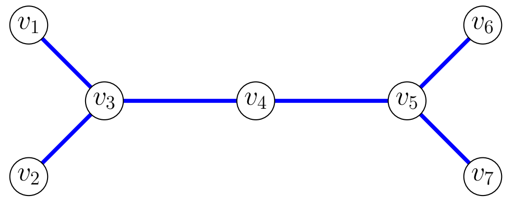

FIGURE 1. Graph for Example 3.5

From the definition, it is clear that $pt_k(G) \leq PT_k(G)$. When $k = 1$, this definition is same as that of the minimum and maximum propagation time of G, denoted by $pt(G)$ and $PT(G)$ respectively. The bounds for k-forcing propagation time are given below.

Proposition 3.6. *Let G be a graph with k-forcing number $Z_k(G)$, where $k > 1$. If n is the number of vertices in G, then*

$$\left\lceil \log_k \left(1 + \frac{(k-1)n}{Z_k(G)}\right) \right\rceil - 1 \leq pt_k(G) \leq PT_k(G) \leq n - Z_k(G).$$

Proof. Let G be a graph with k-forcing number $Z_k(G)$. If n is the number of vertices of G. Then total $n - Z_k(G)$ number of vertices being white in first stage. In each stage at-least one vertex colored black. Hence

$$PT_k(G) \leq n - Z_k(G).$$

In the first stage, at-most $kZ_k(G)$ vertices colored black, in the second stage at-most $k^2 Z_k(G)$ vertices colored black and so on. Hence, the lower bound for $pt_k(G)$ is the smallest positive integer p such that

$$Z_k(G)(1 + k + k^2 + \cdots + k^p) \geq n.$$

Solving this, we have

$$p = \left\lceil \log_k \left(1 + \frac{(k-1)n}{Z_k(G)}\right) \right\rceil - 1.$$

That is,

$$\left\lceil \log_k \left(1 + \frac{(k-1)n}{Z_k(G)}\right) \right\rceil - 1 \leq pt_k(G)$$

Therefore, we get

$$\left\lceil \log_k \left(1 + \frac{(k-1)n}{Z_k(G)}\right) \right\rceil - 1 \leq pt_k(G) \leq PT_k(G) \leq n - Z_k(G).$$

□

Remark: [4] When $k = 1$ we have the result

$$\left\lceil \frac{n - Z(G)}{Z(G)} \right\rceil \leq pt(G) \leq PT(G) \leq n - Z(G)$$

The following examples show that the bounds in the above proposition are sharp.

Example 3.7. *Let G be the graph in figure 2. Let $Z_3 = \{v_5\}$ be the minimum 3-forcing sets for G. Then $F^{(0)} = \{v_5\}$, $F^{(1)} = \{v_4, v_6, v_{10}\}$, $F^{(2)} = \{v_1, v_2, v_3, v_7, v_8, v_9, v_{11}, v_{12}, v_{13}\}$, so $pt_3(G) = pt_3(G, Z_3) = 2$. Also*

$$\left\lceil \log_3 \left(1 + \frac{(3-1)13}{Z_3(G)}\right) \right\rceil - 1 = 2.$$

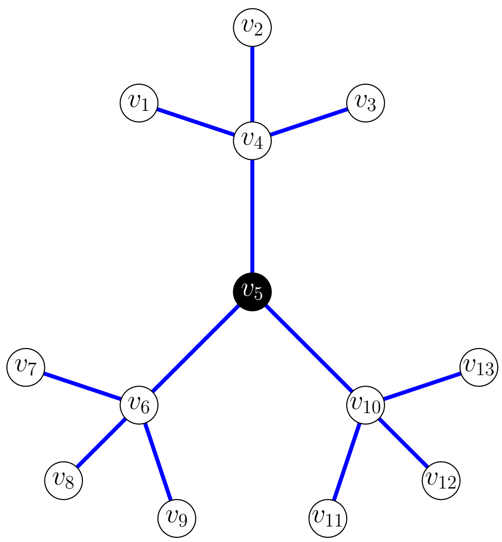

FIGURE 2. Graph for Example 3.7

Example 3.8. *Consider a path P_5 with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$, each v_i is adjacent with v_{i-1} and v_{i+1}, $1 < i < 5$. $Z_k = \{v_1\}$ is a minimum k-forcing set, where $k \geq 1$, and $PT_k(P_5) = 4 = 5 - Z_k(P_5)$.*

From proposition 3.6, we have the following result.

Proposition 3.9. *For a graph G, and k is an integer greater than one. If $pt_k(G) = 1$ then $|V(G)| \leq Z_k(G)(k+1)$. Here $Z_k(G)$ is the k-forcing number of G.*

Proof. Let G be a graph, and let $k > 1$ be an integer. Given that $pt_k(G) = 1$, then by proposition 3.6, we have

$$\left\lceil \log_k \left(1 + \frac{(k-1)n}{Z_k(G)}\right) \right\rceil - 1 \leq pt_k(G) = 1.$$

Solving this, we get $|V(G)| \leq Z_k(G)(k+1)$. □

The converse of this proposition is false. Let G be the graph obtained from K_4 by appending a leaf to one vertex. Then $Z_2(G) = 2$, $|V(G)| = 5 < Z_k(G)(k+1)$ and $pt_2(G) = 2$.

Let G be a k regular graph. Then every singleton set of vertices forms a k forcing set. Hence from proposition 3.6 we get,

$$\lceil \log_k (1 + (k-1)n) \rceil - 1 \leq pt_k(G). \tag{1}$$

Using this inequality, we have the following proposition.

Proposition 3.10. *For a k regular graph G, $pt_k(G) = 1$ iff $|G| = k + 1$.*

Proof. Let G be a k regular graph with $pt_k(G) = 1$. We know that the set $\{v\}$, where $v \in V(G)$, forms a k-forcing set for G. The vertex v is adjacent to k other vertices in G and $pt_k(G) = 1$. Hence $|G| = k + 1$.

Conversely, assume that $|G| = k + 1$, the singleton set $\{v\}$, where $v \in V(G)$, forms a k-forcing set for G. In the first stage, k vertices are colored black. Hence $pt_k(G) = 1$. □

For a k-regular graph G, the lower bound in inequality 1 is sharp only when $pt_k(G) = 1$. Hence, we can fine tune the lower bound by the following proposition.

Proposition 3.11. *Let G be a connected k-regular graph with $|V(G)| = n$. Then the lower bound for $pt_k(G)$ is,*

$$pt_k(G) \geq \begin{cases} \left\lceil \frac{n-1}{2} \right\rceil & \textit{for } k = 2 \\ \left\lceil \log_{k-1} \left(\frac{n(k-2)+2}{k} \right) \right\rceil & \textit{for } k > 2 \end{cases}$$

Proof. Let G be a k-regular graph. Then the set $\{v\}$, where $v \in V(G)$, forms a k-forcing set for G. The vertex v is adjacent to k other vertices in G, in the first step k vertices colored black, in the second step at-most $k(k-1)$ vertices colored black [since, in this step each black colored k vertices adjacent to k other vertices and out of which one already has the black vertex v], in the third step at-most $k(k-1)^2$ vertices colored black and so on. Hence, the lower bound for $pt_k(G)$ is the smallest positive integer p such that

$$1 + k + k(k-1) + k(k-1)^2 + \cdots + k(k-1)^{p-1} \geq n.$$

Solving this, we have

$$p = \begin{cases} \left\lceil \frac{n-1}{2} \right\rceil & \text{for } k = 2 \\ \left\lceil \log_{k-1} \left(\frac{n(k-2)+2}{k} \right) \right\rceil & \text{for } k > 2 \end{cases}$$

Therefore, we have

$$pt_k(G) \geq \begin{cases} \left\lceil \frac{n-1}{2} \right\rceil & \text{for } k = 2 \\ \left\lceil \log_{k-1} \left(\frac{n(k-2)+2}{k} \right) \right\rceil & \text{for } k > 2 \end{cases}$$

□

The following example shows that the lower bound in the above proposition is sharp.

Example 3.12. *Let G be the 3-regular graph in figure 3 with 22 vertices. Let $Z_3 = \{v_1\}$ be the minimum 3-forcing sets for G. Then $F^{(0)} = \{v_1\}$, $F^{(1)} = \{v_2, v_3, v_4\}$, $F^{(2)} = \{v_5, v_6, \ldots, v_{10}\}$, and $F^{(3)} = \{v_{11}, v_{12}, \ldots, v_{22}\}$ so $pt_3(G) = pt_3(G, Z_3) = 3$. Also*

$$\left\lceil \log_2 \left(\frac{22(3-2)+2}{3} \right) \right\rceil = 3.$$

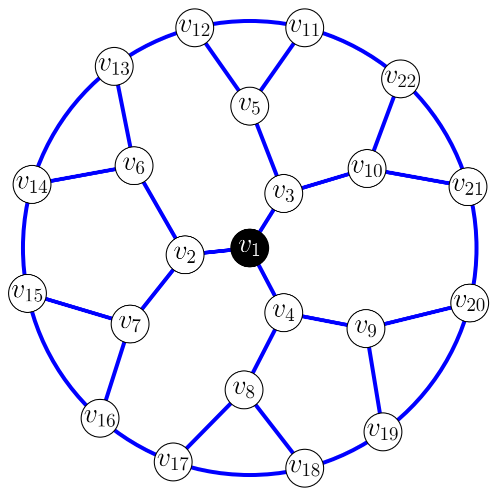

FIGURE 3. Graph for Example 3.12

We know that every k-forcing set is also a $k+1$ forcing set, But we cannot compare the k forcing propagation time and $k+1$ forcing propagation time of a graph. Consider the following proposition.

Proposition 3.13. *Let G be a graph, and let k, p be two positive integers with $k < p$. Then*

(i) *$pt_k(G) \leq pt_p(G)$ is not true, in general.*
(ii) *$pt_k(G) \geq pt_p(G)$ is not true, in general.*

Proof. (i) Consider a path P_5 with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$, each v_i is adjacent to v_{i-1} and v_{i+1}, $1 < i < 5$. We know that the set $A = \{v_1\}$ forms a zero forcing set and $pt(P_5) = 4$. But the set $B = \{v_3\}$ is a minimum k forcing set with $k \geq 2$. Hence $pt_k(P_5) = 2$. Therefore, we have $pt(P_5) > pt_2(P_5)$.

(ii) Consider the hyper cube graph Q_3 depicted in figure 4. We can see that the set $A = \{v_1, v_2, v_3, v_4\}$ is a minimum zero forcing set and $pt(Q_3) = 1$. It can be easily observed that any singleton set of vertices forms a 3-forcing set and is minimum. Hence $pt_3(Q_3) = 3$. Therefore $pt(Q_3) < pt_3(Q_3)$.

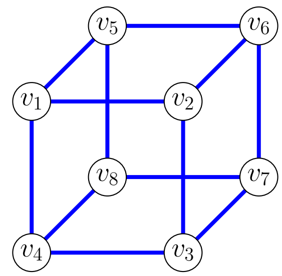

FIGURE 4. Hypercube graph Q_3

□

From the definition of k-forcing propagation time, we can easily observe that

Observation 3.14. *For any graph G, $pt_k(G) = 0$ if and only if G is a totally disconnected graph.*

Minimum k-forcing propagation time of a graph G and its sub graph H are not comparable, consider the following examples.

Let G be a graph. If H is a sub graph of G, then $pt_k(H) \leq pt_k(G)$ is not true, in general.

Consider the graph $G = K_4$ depicted in Figure 1.

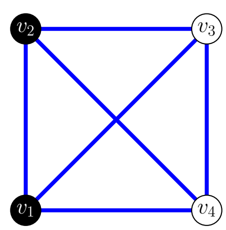

FIGURE 5. The Graph G.

Let $Z_2 = \{v_1, v_2\}$ be the minimum 2-forcing set for K_4 and $pt_2(G) = 1$. Now consider a sub graph $H = K_4 - e$ of G given below.

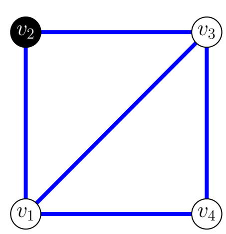

FIGURE 6. The Graph H.

Here $Z_2' = \{v_2\}$ be a minimum 2-forcing set for H and $pt_2(H) = 2$. Therefore, the assertion follows.

The converse of the above assertion is that it need not be true in general. That is, for a graph G and H is a sub graph of G, then $pt_k(H) \geq pt_k(G)$ is not true in general. Consider the following example.

Let G be the path graph P_5. The set $\{v_3\}$ is a minimum 2-forcing set and $pt_2(G) = 2$.

FIGURE 7. The Graph G

Consider the subgraph $H = P_3$ of G

FIGURE 8. The Graph H

Here $Z_2 = \{v_2\}$ be a minimum 2-forcing set for H and $pt_2(H) = 1$. Therefore, the assertion follows.

Next we investigate the k-forcing propagation time for path and cycle.

Theorem 3.15. *Let G be the path P_n, of order $n \geq 2$. Then $PT_k(P_n) = n-1$, $k \geq 1$.*

Proof. Let G be the path P_n with $V(G) = \{v_1, \ldots, v_n\}$. Then the set $\{v_1\}$ is a minimum k-forcing set, $k \geq 1$. In each time step only one forcing occurs. Hence $PT_k(P_n) = n-1$. □

Theorem 3.16. *Let G be the path P_n, of order $n \geq 2$. Then $pt_k(P_n) = \lceil \frac{n-1}{2} \rceil$, $k \geq 2$.*

Proof. Let G be the path P_n with $V(G) = \{v_1, \ldots, v_n\}$. Then the set $\{v_{\lceil \frac{n}{2} \rceil}$ is a minimum k-forcing set, $k \geq 2$. In each time step two forcing occurs simultaneously. Hence $pt_k(P_n) = \lceil \frac{n-1}{2} \rceil$. □

Theorem 3.17. *For the cycle C_n, of order $n \geq 3$, Then*

$$pt_k(C_n) = PT_k(C_n) = \begin{cases} \lceil \frac{n-2}{2} \rceil & \text{for } k = 1 \\ \lceil \frac{n-1}{2} \rceil & \text{otherwise} \end{cases}$$

Proof. Let C_n denote the cycle C_n with $V(C_n) = \{v_1, \ldots, v_n\}$. Then the set $\{v_1, v_2\}$ is a minimum zero forcing set. And the set $\{v_1\}$ is a minimum k-forcing set for $k > 1$. Both forcing sets are unique up to isomorphism and in each time step two forcing occurs simultaneously. Hence

$$pt_k(C_n) = PT_k(C_n) = \begin{cases} \lceil \frac{n-2}{2} \rceil & \text{for } k = 1 \\ \lceil \frac{n-1}{2} \rceil & \text{otherwise} \end{cases}$$

□

Theorem 3.18. *For any connected graph G, $PT_k(G) = |V(G)| - 1$ if and only if G is a path P_n.*

We can see that a graph G with $pt_k(G) = 0$ if and only if G is totally disconnect. And now we characterize graphs having minimum k-forcing propagation time $n - 1$.

Proof. Let G be a connected graph with $PT_k(G) = |V(G)| - 1$. then by proposition 3.6, we have $Z_k(G) = 1$. And only one forcing occurs in each time step. Hence G is a path P_n. Converse part is by proposition 3.15. □

4. Conclusion and Further Scopes

In this article, we found some bounds for the k-forcing propagation time of graphs. Also found the exact value of the k-forcing propagation time of graphs, such as paths, cycles. And we characterized regular graph with the k-forcing propagation time one.

Some questions remains open, such as the characterization of graphs where $pt_k(G) = 1$ and 2. Another open problem is to find the algorithm for finding maximum and minimum k-forcing propagation time for a connected graph G.

References

[1] AIM Special Work Group. Zero forcing sets and the minimum rank of graphs. Linear Algebra and its Applications, 428 (7): 1628 - 1648, 2008.

[2] Allan Frendrup, Michael A. Henning, Bert Randerath and Preben Dahl Vestergaard, *An upper bound on the kination number of a graph with minimum degree 2*, Science Direct, Discrete Mathematics 309 (2009), 639-646

[3] Hein van der Holst et al., *Zero Forcing Sets and the Minimum Rank of Graphs, Linear Algebra and its Applications*, 428 (2008),1628-1648.

[4] Leslie Hogben, My Huynh, Nicole Kingsley, Sarah Meyer, Shanise Walker, Michael Young, *Propagation time for zero forcing on a graph*,Discrete Applied Mathematics (2012)

[5] David Amos, Yair Caro, Randy Davila and Ryan Pepper, *Upper Bounds on the k-Forcing Number of a Graph*, Discrete Applied Mathematics, 181 (2015), 1 - 10.

[6] Yair Caro, Randy Davila, and Ryan Pepper *Extremal k-forcing sets in oriented graphs*, Discrete Applied Mathematics 262 (2019) 42–55.

[7] K. P. Premodkumar, Charles Dominic, Baby Chacko, *K- Forcing Number of Some Graphs and Their Splitting Graphs*, International Journal of Scientific Research in Mathematical and Statistical Sciences, 6(3), (2019),121 -127.

[8] Burgarth, Daniel, and Vittorio Giovannetti. *Full control by locally induced relaxation*, Physical review letters 99.10 (2007): 100501.

[9] Severini, Simone. *Nondiscriminatory propagation on trees*, Journal of Physics A: Mathematical and Theoretical 41.48 (2008): 482002.

[10] Chilakamarri, Kiran B., et al. *Iteration index of a zero forcing set in a graph*, arXiv preprint arXiv:1105.1492 (2011).

[11] Geneson, Jesse, and Leslie Hogben. *Expected propagation time for probabilistic zero forcing.* Australasian Journal of Combinatorics 83 (2022): 397.

[12] Crowd Math, P. A. *Propagation time for weighted zero forcing.* arXiv preprint arXiv:2005.07316 (2020).

[13] Carlson, Joshua, et al. *Throttling positive semidefinite zero forcing propagation time on graphs.* Discrete Applied Mathematics 254 (2019): 33-46.

[14] Rameh, Z., and E. Vatandoost. *Some Cayley graphs with propagation time 1.* Journal of the Iranian Mathematical Society 2.2 (2021): 111-122.