COEFFICIENT BOUNDS FOR A NEW SUBCLASS OF BI-UNIVALENT FUNCTIONS INVOLVING THE SĂLĂGEAN DEFERENTIAL OPERATOR

ABSTRACT. This paper introduced two distinct subclasses of the bi-univalent function family Σ in the open unit disk, formulated via Sălăgean differential operator. For these subclasses, precise estimates are derived for the coefficients $|b_2|$ and $|b_3|$.

1. INTRODUCTION AND MOTIVATION

Let $\mathscr{A}$ represent the class of analytic functions expressed as

$$h(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n, \tag{1.1}$$

which are analytic within the open unit disk $\Delta = \{\zeta : |\zeta| < 1\}$. The subclass $\mathscr{S} \subset \mathscr{A}$ consists of those functions that are univalent in Δ.

Ding et al. [8] proposed the class $L_\lambda(\beta)$ of analytic functions, defined as

$$L_\lambda(\beta) = \left\{ h \in \mathscr{A} : \Re\left((1-\lambda)\frac{h(\zeta)}{\zeta} + \lambda h'(\zeta) \right) > \beta,\ \beta < 1,\ \lambda \geq 0 \right\}.$$

It follows directly that $L_{\lambda_1}(\beta) \subset L_{\lambda_2}(\beta)$, whenever $\lambda_1 > \lambda_2 \geq 0$. In particular, for $\lambda \geq 1$ and $0 \leq \beta < 1$,

$$L_\lambda(\beta) \subset L_1(\beta) = \{h \in \mathscr{A} : \Re h'(\zeta) > \beta\},$$

demonstrating that $L_\lambda(\beta)$ is a univalent subclass (see [12, 6, 7]).

For $h \in \mathscr{A}$, the Sălăgean differential operator $D^m : \mathscr{A} \to \mathscr{A}$ is defined recursively by

$$D^0 h(\zeta) = h(\zeta), \qquad D^1 h(\zeta) = \zeta h'(\zeta), \qquad D^2 h(\zeta) = \zeta (D^1 h(\zeta))',$$

and, more generally for $m \in \mathbb{N}$,

$$D^m h(\zeta) = \zeta + \sum_{n=2}^{\infty} n^m b_n \zeta^n, \quad \zeta \in \Delta \tag{1.2}$$

The operator D^m is known as Sălăgean differential operator (see[16]).

If $h \in \mathscr{S}$, then h admits an inverse h^{-1} with the properties

$$h^{-1}(h(\zeta)) = \zeta, \quad \zeta \in \Delta,$$

and

$$h^{-1}(h(w)) = w, \quad |w| < r_0(h), \quad r_0(h) \geq \tfrac{1}{4}.$$

2010 Mathematics Subject Classification. 30C45, 30C50.

Key words and phrases. Analytic and univalent mappings, Bi-univalent functions, Starlike and convex subclasses, Coefficient inequalities, Sălăgean operator.

The inverse has the power series expansion

$$h^{-1}(w) = w - b_2 w^2 + (2b_2^2 - b_3)w^3 - (5b_2^3 - 5b_2 b_3 + b_4)w^4 + \cdots .$$

A function $h \in \mathscr{A}$ is called bi-univalent if both h and its inverse h^{-1} are univalent in the open unit disk Δ. The collection of all such functions is represented by Σ (see [19]).

Brannan and Taha [3] (see also [20]) introduced certain subclasses of Σ that are linked to the classical families of starlike and convex functions, namely $\mathscr{S}^*(\alpha)$ and $\mathscr{K}(\alpha)$ for $0 \leq \alpha < 1$. As an example, the class $\mathscr{S}^*_\Sigma[\alpha]$ representing strongly bi-starlike functions of order α is defined by functions that satisfy the following condition:

$$h \in \Sigma, \qquad \left| \arg\left(\frac{\zeta h'(\zeta)}{h(\zeta)} \right) \right| < \frac{\alpha\pi}{2}, \quad \zeta \in \Delta,$$

and

$$\left| \arg\left(\frac{w(g'(w))}{g(w)} \right) \right| < \frac{\alpha\pi}{2}, \quad w \in \Delta,$$

where g denotes the analytic continuation of h^{-1} in Δ. These classes, $\mathscr{S}^*_\Sigma(\alpha)$ and $\mathscr{K}_\Sigma(\alpha)$, motivated further studies on coefficient problems, in particular deriving bounds for $|b_2|$ and $|b_3|$. Later, Frasin and Aouf [9] refined such results using methods of Srivastava et al. [19].

The study of Salagean-type operators continues to attract attention in geometric function theory. Recent works include P. Sharma et.al [18], Sagsöz [17], Breaz and Cotîrla [4], Alharbi and Yamaguchi-Noshiro [1], A. Murugan et.al [13], H. Ö. Güney and S. Yalçin[10], Ibrahim [11], and Chang and Janteng [5]. These studies investigated coefficient estimates for generalized, q- and (p, q)-Salagean operators as well as mappings in leaf-like domains. A related extension by Naik and Sahoo [14] derived Fekete-Szegö inequalities for analytic functions in leaf-like domains.

Motivated by these investigations, we introduce two new subclasses of the bi-univalent function family Σ using a weighted Salagean operator characterized by the parameters $(m, \alpha, \beta, \lambda)$. This formulation unifies and extends earlier subclasses by allowing additional degrees of analytic freedom. The resulting coefficient estimates for $|b_2|$ and $|b_3|$ generalize known results as special cases.

We require the following lemma, which will be used throughout.

Lemma 1.1. [15] *If $f \in P$, then $|d_k| \leq 2$ for all k, where P denotes the class of analytic functions in Δ having positive real part and given by the expansion*

$$f(\zeta) = 1 + d_1\zeta + d_2\zeta^2 + d_3\zeta^3 + \cdots, \quad \zeta \in \Delta.$$

2. Coefficient Estimates for the class $\mathscr{B}_\Sigma(m, \alpha, \lambda)$

Definition 2.1. Let $h(\zeta)$ be given by (1.1). We say h belongs to $\mathscr{B}_\Sigma(m, \alpha, \lambda)$ if

$$h \in \Sigma \quad \text{and} \quad \left| \arg\left((1-\lambda)\frac{D^m h(\zeta)}{\zeta} + \lambda(D^m h(\zeta))' \right) \right| < \frac{\alpha\pi}{2}, \tag{2.1}$$

for $0 < \alpha \leq 1$, $\lambda \geq 1$, and $\zeta \in \Delta$, together with

$$\left| \arg\left((1-\lambda)\frac{D^m g(w)}{w} + \lambda(D^m g(w))' \right) \right| < \frac{\alpha\pi}{2}, \tag{2.2}$$

where g is the inverse of h, expanded as

$$g(w) = w - b_2 w^2 + (2b_2^2 - b_3)w^3 + \cdots . \tag{2.3}$$

Remark 2.2. *If we take $m = 0$, in definition (2.1) this reduces to the class $\mathscr{B}_\Sigma(\alpha, \lambda)$ studied in [9].*

Remark 2.3. *By taking $m = 0$ and $\lambda = 1$ in definition (2.1) it reduce to the class $\mathscr{H}_\Sigma^\alpha$ introduced in [19].*

Theorem 2.4. *If $h(\zeta) \in \mathscr{B}_\Sigma(m, \alpha, \lambda)$ with $0 < \alpha \leq 1$, $\lambda \geq 1$,then*

$$|b_2| \leq \frac{2\alpha}{\sqrt{2^m(\lambda+1)^2 + \alpha\big(3^m(4\lambda+2) - 2^{2m}(\lambda+1)^2\big)}}, \tag{2.4}$$

and

$$|b_3| \leq \frac{4\alpha^2}{2^{2m}(\lambda+1)^2} + \frac{2\alpha}{3^m(2\lambda+1)}. \tag{2.5}$$

Proof. From conditions (2.1) and (2.2), there exist analytic functions $p(\zeta)$ and $q(w)$ belonging to P such that

$$(1-\lambda)\frac{D^m h(\zeta)}{\zeta} + \lambda(D^m h(\zeta))' = [p(\zeta)]^\alpha, \tag{2.6}$$

and

$$(1-\lambda)\frac{D^m g(w)}{w} + \lambda(D^m g(w))' = [q(w)]^\alpha. \tag{2.7}$$

Here,

$$p(\zeta) = 1 + \sum_{n=1}^{\infty} p_n \zeta^n, \quad \zeta \in \Delta, \tag{2.8}$$

$$q(w) = 1 + \sum_{n=1}^{\infty} q_n w^n, \quad w \in \Delta. \tag{2.9}$$

Comparing coefficients between (2.6) and (2.7) yields

$$2^m(\lambda+1)b_2 = \alpha p_1, \tag{2.10}$$

$$3^m(2\lambda+1)b_3 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2, \tag{2.11}$$

$$-2^m(\lambda+1)b_2 = \alpha q_1, \tag{2.12}$$

$$3^m(2\lambda+1)(2b_2^2 - b_3) = \alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.13}$$

From (2.10) and (2.12), we deduce

$$p_1 = -q_1. \tag{2.14}$$

Hence,

$$2(2^{2m}(\lambda+1)^2)b_2^2 = \alpha^2(p_1^2 + q_1^2). \tag{2.15}$$

Combining (2.11), (2.13), and (2.15), one obtains

$$2(3^m(2\lambda+1))b_2^2 = \alpha(p_2+q_2) + \frac{\alpha(\alpha-1)}{2}\cdot\frac{2(2^{2m}(\lambda+1)^2)b_2^2}{\alpha^2}.$$

This simplifies to

$$b_2^2 = \frac{\alpha^2(p_2+q_2)}{2^m(\lambda+1)^2+\alpha\big(3^m(4\lambda+2)-2^{2m}(\lambda+1)^2\big)}.$$

Applying Lemma 1.1, which guarantees $|p_2|, |q_2| \leq 2$, establishes inequality (2.4).

To estimate $|b_3|$, subtracting (2.13) from (2.11) gives

$$2(3^m(2\lambda+1))(b_3-b_2^2) = \alpha(p_2-q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2). \tag{2.16}$$

Using (2.14) and (2.15), this becomes

$$2(3^m(2\lambda+1))b_3 = \frac{\alpha^2(2\lambda+1)3^m(p_1^2+q_1^2)}{2^{2m}(\lambda+1)^2} + \alpha(p_2-q_2).$$

Therefore,

$$b_3 = \frac{\alpha^2(p_1^2+q_1^2)}{2(2^{2m}(\lambda+1)^2)} + \frac{\alpha(p_2-q_2)}{2(3^m(2\lambda+1))}.$$

Applying Lemma 1.1 once more ($|p_1|, |q_1|, |p_2|, |q_2| \leq 2$) gives the estimate in (2.5). □

The following cases arise as direct consequences of Theorem 2.4.

Corollary 2.5. *([9]) For $m = 0$ and if $h(\zeta) \in \mathscr{B}_\Sigma(\alpha, \lambda)$, then*

$$|b_2| \leq \frac{2\alpha}{\sqrt{(\lambda+1)^2+\alpha(1+2\lambda-\lambda^2)}}, \tag{2.17}$$

and

$$|b_3| \leq \frac{4\alpha^2}{(\lambda+1)^2} + \frac{2\alpha}{2\lambda+1}. \tag{2.18}$$

Corollary 2.6. *([19]) If $\lambda = 1$, $m = 0$, and $h(\zeta) \in \mathscr{H}_\Sigma^\alpha$, then*

$$|b_2| \leq \alpha\sqrt{\frac{2}{2+\alpha}}, \tag{2.19}$$

$$|b_3| \leq \frac{\alpha(3\alpha+2)}{3}. \tag{2.20}$$

3. Coefficient Estimates for the class $\mathscr{B}_\Sigma(m, \beta, \lambda)$

Definition 3.1. A function $h(\zeta)$ given by (1.1) belongs to $\mathscr{B}_\Sigma(m, \beta, \lambda)$ if

$$h \in \Sigma \quad \text{and} \quad \Re\left((1-\lambda)\frac{D^m h(\zeta)}{\zeta} + \lambda(D^m h(\zeta))'\right) > \beta, \tag{3.1}$$

and

$$\Re\left((1-\lambda)\frac{D^m g(w)}{w} + \lambda(D^m g(w))'\right) > \beta, \tag{3.2}$$

where $0 \leq \beta < 1$, $\lambda \geq 1$, and g is the inverse of h as in (2.3).

Remark 3.2. *For $m = 0$, in definition (3.1) this reduces to $\mathscr{B}_\Sigma(\beta, \lambda)$ studied in* [9].

Remark 3.3. *If we take $m = 0$, $\lambda = 1$, in definition (3.1) it becomes $\mathscr{H}_\Sigma(\beta)$, introduced in* [19].

Theorem 3.4. *If $h(\zeta) \in \mathscr{B}_\Sigma(m, \beta, \lambda)$ with $0 \le \beta < 1$, $\lambda \ge 1$, then*

$$|b_2| \le \sqrt{\frac{2(1-\beta)}{3^m(2\lambda+1)}}, \tag{3.3}$$

and

$$|b_3| \le \frac{4(1-\beta)^2}{2^{2m}(\lambda+1)^2} + \frac{2(1-\beta)}{3^m(2\lambda+1)}. \tag{3.4}$$

Proof. From (3.1)(3.2), we can find analytic functions $p(\zeta), q(w) \in P$ such that

$$(1-\lambda)\frac{D^m h(\zeta)}{\zeta} + \lambda(D^m h(\zeta))' = \beta + (1-\beta)p(\zeta), \tag{3.5}$$

$$(1-\lambda)\frac{D^m g(w)}{w} + \lambda(D^m g(w))' = \beta + (1-\beta)q(w). \tag{3.6}$$

Using the expansions (2.8)(2.9) and comparing coefficients, we obtain

$$2^m(\lambda+1)b_2 = (1-\beta)p_1, \tag{3.7}$$

$$3^m(2\lambda+1)b_3 = (1-\beta)p_2, \tag{3.8}$$

$$-2^m(\lambda+1)b_2 = (1-\beta)q_1, \tag{3.9}$$

$$3^m(2\lambda+1)(2b_2^2 - b_3) = (1-\beta)q_2. \tag{3.10}$$

From (3.7) and (3.9), it follows that

$$p_1 = -q_1. \tag{3.11}$$

Consequently,

$$2(2^{2m}(\lambda+1)^2)b_2^2 = (1-\beta)^2(p_1^2 + q_1^2). \tag{3.12}$$

Moreover, combining (3.8) and (3.10), we find

$$2(3^m(2\lambda+1))b_2^2 = (1-\beta)(p_2+q_2).$$

Using Lemma 1.1, which guarantees $|p_2|, |q_2| \le 2$, leads to (3.3).

For $|b_3|$, subtract (3.10) from (3.8), giving

$$2(3^m(2\lambda+1))(b_3 - b_2^2) = (1-\beta)(p_2 - q_2),$$

which can be written as

$$b_3 = \frac{(1-\beta)^2(p_1^2+q_1^2)}{2(2^{2m}(\lambda+1)^2)} + \frac{(1-\beta)(p_2-q_2)}{2(3^m(2\lambda+1))}.$$

Applying Lemma 1.1, inequality (3.4) follows immediately. □

Corollary 3.5. *([9]) For $m = 0$ and if $h(\zeta) \in \mathscr{B}_\Sigma(\beta, \lambda)$, then*

$$|b_2| \leq \sqrt{\frac{2(1-\beta)}{2\lambda+1}}, \tag{3.13}$$

and

$$|b_3| \leq \frac{4(1-\beta)^2}{(\lambda+1)^2} + \frac{(1-\beta)}{2\lambda+1}. \tag{3.14}$$

Corollary 3.6. *([19]) For $\lambda = 1$, $m = 0$, and if $h(\zeta) \in \mathscr{H}_\Sigma(\beta)$, then*

$$|b_2| \leq \sqrt{\frac{2(1-\beta)}{3}}, \tag{3.15}$$

$$|b_3| \leq \frac{(1-\beta)(5-3\beta)}{3}. \tag{3.16}$$

4. Concluding Remarks and Applications

The coefficient bounds derived in this paper have potential applications in conformal mapping problems, distortion theory, and the study of analytic image domains in complex dynamical systems. They also provide useful estimates for analytic filters and potential functions in engineering mathematics where reversible analytic transformations are modeled by bi-univalent mappings. Future research may extend these results to q-calculus and fractional-order Salagean operators.

References

[1] A. Alharbi and M. Yamaguchi-Noshiro, *YamaguchiNoshiro type bi-univalent functions related with the SalageanErdlyKober operator*, Mathematics **10**(13) (2022), 2241.

[2] D.A. Brannan, J. Clunie, and W.E. Kirwan, *Coefficient estimates for a class of starlike functions*, Canad. J. Math. **22** (1970), 476–485.

[3] D.A. Brannan and T.S. Taha, *On some classes of bi-univalent functions*, in: S.M. Mazhar, A. Hamoui, N.S. Faour (Eds.), *Mathematical Analysis and its Applications*, KFAS Proceedings Series, Vol. 3, Pergamon Press, Oxford, 1988, pp. 53–60; see also *Studia Univ. Babes-Bolyai Math.* **31**(2) (1986), 70–77.

[4] D. Breaz and L.-I. Cotrla, *Coefficient estimates and FeketeSzeg inequalities for new m-fold symmetric bi-univalent functions defined using an operator*, J. Inequal. Appl. **2023**, Article 15.

[5] D. Chang and A. Janteng, *FeketeSzeg inequality for a subclass of bi-univalent functions by applying Salagean q-differential operator*, Malaysian J. Fundam. Appl. Sci. **19**(6) (2023), 857–864.

[6] M. Chen, *On the regular functions satisfying $\Re(f(z)/z) > \alpha$*, Bull. Inst. Math. Acad. Sinica **3** (1975), 65–70.

[7] P.N. Chichra, *New subclasses of the class of close-to-convex functions*, Proc. Amer. Math. Soc. **62** (1977), 37–43.

[8] S.S. Ding, Y. Ling, and G.J. Bao, *Some properties of a class of analytic functions*, J. Math. Anal. Appl. **195**(1) (1995), 71–81.

[9] B.A. Frasin and M.K. Aouf, *New subclasses of bi-univalent functions*, Appl. Math. Lett. **24** (2011), 1569–1573.

[10] H.Ö. Güney and S. Yalçin, *An application of Salagean operator concerning starlike and convex functions*, Axioms **11**(2) (2022), Article 50.

[11] M. Ibrahim, B. Khan, and A. Manickam, *A certain q-Salagean differential operator and its applications to subclasses of analytic and bi-univalent functions involving (p, q)-Chebyshev polynomials*, Commun. Math. (2024).

[12] T.H. MacGregor, *Functions whose derivative has a positive real part*, Trans. Amer. Math. Soc. **104** (1962), 532–537.

[13] A. Murugan, S.M. El-Deeb, M. Redn Almutiri, J.-S. Ro, and P. Sharma, *Certain new subclasses of bi-univalent functions associated with bounded boundary rotation involving Salagean operator*, *AIMS Math.* **9**(10) (2024), 27577–27592.

[14] A. Naik and S.C. Sahoo, *FeketeSzeg inequality estimate for analytic functions using Salagean-difference operator and leaf-like domain*, *Eur. J. Pure Appl. Math.* **18**(3) (2025), Article 634.

[15] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Gttingen, 1975.

[16] G.S. Salagean, *Subclasses of univalent functions*, *Complex AnalysisFifth Romanian-Finnish Seminar, Part I*, Lecture Notes in Math., Vol. 1013, Springer, Berlin (1983), 362–372.

[17] F. Sagsz and H. Orhan, *Coefficient inequalities for new subclasses of bi-univalent functions* f_δ, *Asian-European J. Math.* **15**(9) (2022), 2250160.

[18] P. Sharma, S. Sivasubramanian, A. Catas, and S.M. El-Deeb, *Initial coefficient bounds for bi-close-to-convex and bi-quasi-convex functions with bounded boundary rotation associated with q-Salagean operator*, *Mathematics* **13**(14) (2025), 2252.

[19] H.M. Srivastava, A.K. Mishra, and P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, *Appl. Math. Lett.* **23** (2010), 1188–1192.

[20] T.S. Taha, *Topics in Univalent Function Theory*, Ph.D. Thesis, University of London, 1981.