Original Research Article

A Study on the Existence and Local Attractivity of Solutions to Fractional Order Nonlinear Random Integral Equations via Fixed Point Theory

Abstract

In this work, we examine whether a class of fractional order nonlinear random integral equations defined on the set of non-negative real numbers have a solution and whether such solutions are locally attractive. In addition to appropriate Caratheodory conditions on the random operators involved, the analysis is performed under nonlinear contraction conditions. Using a suitable fixed point theorem, we rigorously prove the existence of random a solution and investigate their local attractivity after introducing a set of structural assumptions on the nonlinear random operators. To demonstrate the application of the suggested framework and to validate the theoretical results, specific example is provided.

Keyword: Fixed point theorems, nonlinear integral equation, Random operator and equations, Initial value problems.
AMS (MOS) Subject Classification: 47H10, 45G10, 60H25, 34A12.

1 Introduction

Fractional calculus is a branch of mathematics that studies integrals and derivatives of arbitrary order as well as their applications. Fractional calculus is an extension of ordinary differentiation and integration to any order [8]. The concept of fractional calculus is arising in the mathematical modelling of system and process occurring in many engineering and scientific approaches such as physics, chemistry, aerodynamics, electrodynamics, economics, etc [9][12][11] [6]. Random parameters are coefficients or parameters that play a significant role in natural process. As a result, when we discuss any parameters or coefficients, the random analysis of random equation is obvious. The random equations have been researched in the literature for a long time by various mathematicians all over the world. Thus, studying a natural or physical phenomenon using random models or equations is an important branch of the analysis. Nonlinear random integral equations are more important in number of physical problems, many physical phenomena in life science, engineering and technology, biological problems, neutron transportation theory and specially to the application theoretical physics [5][13]. Integral equations are well known for their various helpful and essential application in describing a numerous event and problems of real world. The theory of integral equations is rapidly developing using the tools of functional analysis, topology and fixed-point theory. The study of nonlinear integral equations with unbounded intervals is relatively recent and has been expanded to include new aspect of attractivity and asymptotic attractivity of solution. There are two techniques to deals with these properties of solutions, namely, classical fixed-point theorems utilising hypotheses from analysis and topology and the fixed point theorems including the use of measure of non-compactness [2][4] [7]. The theory of fractional order integral equations has recently attracted a lot of attention and is an important part of nonlinear analysis. In this work, we study the existence and locally attractive solution of the fractional order nonlinear random integral equation.

2 Statement of the problem

Let α, $\beta \in (0,1)$ and $\mathscr{R}$ denote the real numbers where as $\mathscr{R}_+$ be the set of non-negative real numbers i.e. $\mathscr{R}_+ = [0,\infty) \subset \mathscr{R}$. Consider the fractional order nonlinear random integral equation (FONRIE)

$$u(t,\upsilon) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(\theta(s),\upsilon),\upsilon)\,ds + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1} g(s,u(\eta(s),\upsilon),\upsilon)\,ds \tag{2.1}$$

for all $t \in \mathscr{R}_+$, $\upsilon \in \Omega$. Where $f : \mathscr{R}_+ \times \mathscr{R} \times \Omega \to \mathscr{R}$; $g : \mathscr{R}_+ \times \mathscr{R} \times \Omega \to \mathscr{R}$; $\theta,\eta : \mathscr{R}_+ \to \mathscr{R}_+$ and by solution of the (2.1) we mean a function $u \in BM(\mathscr{R}_+,\mathscr{R})$ that satisfies (2.1) on $\mathscr{R}_+$. Where $BM(\mathscr{R}_+,\mathscr{R})$ is space of bounded measurable real valued functions defined on $\mathscr{R}_+$. define a supremum norm $\|.\|$ in $BM(\mathscr{R}_+,\mathscr{R})$ by $\|u\| = \sup_{t\in\mathscr{R}_+} \mid u(t) \mid$. Clearly $BM(\mathscr{R}_+,\mathscr{R})$ is a separable Banach algebra with this maximum norm. By $L^1(\mathscr{R}_+,\mathscr{R})$ norm $\|.\|$ is defined by $\|u\|_{L^1} = \int_0^t \mid u(t) \mid ds$.

3 Preliminaries

We begin by outlining the relevant notations, definitions and preliminary concepts needed for the main work.

Definition 3.1. *[4][7] Let $x : \Omega \to S$ be mapping with*

(i) $(\Omega, \mathscr{A}, \mu)$ be a complete probability measure space and $(\Omega, \mathscr{A})$ be a measurable space

(ii) $\mathscr{S}$ be a separable Banach space with σ-algebra β_s of all Borel subsets of $\mathscr{S}$

(iii) for any Borel subset B of S, $x^{-1}(B) = \{\upsilon \in \Omega : x(\upsilon) \in B\} \in \mathscr{A}$ then the mapping x is called random variable.

Definition 3.2. *[2][4] If $\upsilon \to g(\upsilon,s)$ is measurable for each $s \in S$ then the function $g : \Omega \times S \to S$ is known as random operator(RO) and this RO is denoted by $g(\upsilon)s = g(\upsilon,s)$*

Definition 3.3. *[3] A random operator $g : \Omega \times S \to S$ is said to be continuous random operator if $\lim_{n\to\infty} \|s_n - s\| = 0 \to \lim_{n\to\infty} \|g(\upsilon)s_n - g(\upsilon)s\| = 0$*

Definition 3.4. *[2] [4] Let $g : \Omega \times S \to S$ be random operator. A random variable $\rho : \Omega \to S$ is known as random fixed point of RO g, if $g(\upsilon)\rho(\upsilon) = \rho(\upsilon)$ for every $\upsilon \in \Omega$*

Definition 3.5. *[2] A RO $g : \Omega \times S \to S$ is known as totally bounded , if $g(\upsilon)(U)$ is totally bounded subset of S for any bounded subset of U of S and said to be compact if for every fixed $\upsilon \in \Omega$ the mapping $g(\Omega,.) : S \to S$ is compact; that is for every bounded subset U of S, $\overline{g(\upsilon,U)}$ is relatively compact subset of S equivalently $\overline{g(\upsilon,U)}$ is closure of $g(\upsilon,U)$ is compact.*

Definition 3.6. *[4] If the RO is continuous and totally bounded on S then the RO $g(\upsilon)$ is known as completely continuous RO on Banach space S*

Definition 3.7. *[10] A sequence of measurable functions $f_n(\upsilon)$ is said to be equicontinuous on S, if for every $\varepsilon > 0$ there exist $\delta > 0$ such that $\|f_n(\upsilon,t_1) - f_n(\upsilon,t_2)\| < \varepsilon$ whenever $| t_1 - t_2 |< \delta$ for all $t_1,t_2 \in S \quad n = 1,2,3,\ldots$*

Definition 3.8. *The Riemann-Liouville fractional integral of order $\zeta > 0$ of a continuous function $f \in L^1[0,T]$ is defined by*

$$I^\zeta f(t) = \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{(\zeta-1)} f(s)\,ds$$

Provided right-hand side is defined pointwise on $(0,\infty)$.

Theorem 3.9. *[10] (**Arzela-Ascoli theorem**) Any sequence g_n of functions in $C(J,\mathscr{R})$ that is uniformly bounded and equi-continuous has a convergent subsequence.*

Definition 3.10. *[1] Let $S = BC(\mathscr{R}_+,\mathscr{R})$ be the Banach space of bounded continuous functions with the supremum norm. Random solutions of random equations are said to be locally attractive on $\mathscr{R}_+$, if there exist a function $x_0 \in S$ and a constant $a > 0$ such that the closed ball $\bar{B}_a(x_0) = \{x \in S : \|x - x_0\|_\infty \le a\}$ satisfies the following property: for any two random solutions $x = x(t,\omega)$ and $y = y(t,\omega)$ of the random equation belonging to $\bar{B}_a(x_0) \cap \mathfrak{S}$, where $\mathfrak{S}$ is a non-empty subset of S, we have $\lim_{t\to\infty} | x(t,\upsilon) - y(t,\upsilon) |= 0$ for all $\upsilon \in \Omega$.*

Definition 3.11. *[4] Let S be a Banach space and let $\mathrm{g} : \Omega \times S \to S$ be a random operator.Suppose there exists a non-decreasing, continuous function $\varphi : \Omega \times \mathscr{R}_+ \to \mathscr{R}_+$ such that for every $\upsilon \in \Omega$,*

$$\|\mathrm{g}(\upsilon,s_1) - \mathrm{g}(\upsilon,s_2)\| \le \varphi_\upsilon(\|s_1 - s_2\|)$$

For all $s_1,s_2 \in S$, where $\varphi_\upsilon(r) = \varphi(\upsilon,r)$ and $\varphi(\upsilon,0) = 0$ then the random operator g is called a D-Lipschitzician random operator. Here are some particular situations $(1.)$ If $\varphi_\upsilon(r) = \alpha(\upsilon)r$, for $\alpha(\upsilon) > 0$, then the random operator $\mathrm{g}(\upsilon,.)$ is called Lipschitz continuous with Lipschitz constant $\alpha(\upsilon)$. $(2.)$ the random operator $\mathrm{g}(\upsilon,.)$ is a random contraction operator with a contraction constant $\alpha(\upsilon)$, if for each $\upsilon \in \Omega$, $\alpha(\upsilon) < 1$. $(3.)$ $\mathrm{g}(\upsilon)$ is referred to as a nonlinear $\mathscr{D}$-contraction random operator on S if $\varphi_\upsilon(r) = \varphi(\upsilon,r) < r$ for $r > 0$ and for every $\upsilon \in \Omega$.

Theorem 3.12. *[14] Let C be a closed, convex and bounded subset of a separable Banach space S and let $\mathscr{L} : \Omega \times S \to S$ and $\mathscr{M} : \Omega \times C \to S$ be two operators such that for $\upsilon \in \Omega$*

(a) $\mathscr{L}(\upsilon)$ is a nonlinear contraction

(b) $\mathscr{M}(\upsilon)$ is completely continuous and

(c) $\mathscr{L}(\upsilon)u + \mathscr{M}(\upsilon)w \in C$ for all $w \in C \implies u \in C$
Then the operator equation $\mathscr{L}(\upsilon)u + \mathscr{M}(\upsilon)u = u$ has a random solution.

Definition 3.13. *[15] The condition of $L(\upsilon)$-caratheodory is satisfied by a mapping $\beta : \mathscr{R}_+ \times \mathscr{R} \times \Omega \to \mathscr{R}$ if*

- $(t,\upsilon) \to \beta(t,u,\omega)$ *is measurable for each* $u \in \mathscr{R}$
- $u \to \beta(t,u,\upsilon)$ *is continuous for each* $t \in \mathscr{R}_+$, $\upsilon \in \Omega$
  Furthermore $L^1(\upsilon)$*-caratheodory if*
- *There exists measurable and bounded function* $q : \Omega \to L^1(\mathscr{R}_+)$ *such that* $\mid \beta(t,u,\upsilon) \mid \leq q(t,\upsilon)$ *a.e.,* $t \in \mathscr{R}_+$ *for all* $u \in \mathscr{R}$.

4 Main Work

We consider following hypothesis

(H_0) The function $\theta, \eta : \mathscr{R}_+ \to \mathscr{R}_+$ is continuous.

(H_1) The function $\upsilon \to f(t,u,\upsilon)$ is measurable for all $t \in \mathscr{R}_+$, $u \in \mathscr{R}$ and the function $t \to f(t,u,\upsilon)$ is Riemann integrable for each $u \in \mathscr{R}$, $\upsilon \in \Omega$.

(H_2) There exist a function $\mathscr{J} : \Omega \to L^1(\mathscr{R}_+,\mathscr{R})$ such that for each $\upsilon \in \Omega$,

$$\mid f(t,u(\theta(t),\upsilon),\upsilon) - f(t,w(\theta(t),\upsilon),\upsilon) \mid \leq \mathscr{J}(t,\upsilon) \mid u(\theta(t),\upsilon) - w(\theta(t),\upsilon) \mid \ a.e.$$

for $t \in \mathscr{R}_+$, $\upsilon \in \Omega$, $u, w \in \mathscr{R}$ with $K_2 = \sup\{\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathscr{J}(s,\upsilon)\,ds\} < 1\}$.

(H_3) There exist a function $\phi : \Omega \to L^1(\mathscr{R}_+)$ such that for each $\upsilon \in \Omega$, $\mid f(t,u,\upsilon) \mid \leq \phi(t,\upsilon)$; *a.e.*
for all $t \in \mathscr{R}_+$, $u \in \mathscr{R}$ and $\int_0^t (t-s)^{\alpha-1}\phi(s,\upsilon)\,ds = \phi_1(t,\upsilon)$, $\Phi = \sup\left\{\frac{1}{\Gamma(\alpha)}\phi_1(t,\upsilon)\right\}$
& $\lim_{t\to\infty} \phi_1(t,\upsilon) = 0$.

(H_4) The function $\upsilon \to g(t,u,\upsilon)$ is measurable for all $t \in \mathscr{R}_+,$, $u \in \mathscr{R}$.

(H_5) The function g is $L^1(\upsilon)$-caratheodory and there exist function $q_r : \Omega \to L^1(\mathscr{R}_+,\mathscr{R})\, a.e.$ such that for almost every $\upsilon \in \Omega$, $\mid g(t,u(\eta(t),\upsilon),\upsilon) \mid \leq q_r(t,\upsilon)$ for all $(t,u) \in \mathscr{R}_+ \times \mathscr{R}$.

Remark 4.1. *Note that the hypothesis H_4, H_5 hold then there exist continuous bounded function $\nu : \mathscr{R}_+ \times \Omega \to \mathscr{R}_+$ defined by $\nu(t,\upsilon) = \int_0^t (t-s)^{\alpha-1} q_r(s,\upsilon)\,ds$ with $K_1 = \sup\{\frac{1}{\Gamma(\zeta)}\nu(t,\upsilon) : t \in \mathscr{R}_+,\ \upsilon \in \Omega\}$ and vanish at $t = \infty$.*

Theorem 4.2. *Assuming that the hypothesis (H_0) to (H_5) are true, the fractional order nonlinear random integral equation (2.1) has a solution if there exist a real number $r > 0$ such that $\Phi + K_1 = r$. Moreover, random solution is locally attractive on $\mathscr{R}_+$.*

Proof. By solution of FONRIE (2.1) we mean a continuous function $u : \mathscr{R}_+ \times \Omega \to \mathscr{R}$ that satisfies FONRIE (2.1)on $\mathscr{R}_+$. Let $S = BM(\mathscr{R}_+,\mathscr{R})$ be a measurable Banach space and consider the closed ball $\overline{B_r(0)}$ in S centered at origin and of radius r, where $r = \Phi + K_1 > 0$. Let us define two operators $\mathscr{L} : \Omega \times S \to S$; $\mathscr{M} : \Omega \times \overline{B_r(0)} \to S$ by

$$\mathscr{L}(\upsilon)u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(\theta(s), \upsilon), \upsilon)\, ds \tag{4.1}$$

$$\mathscr{M}(\upsilon)u(t) = \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} g(s, u(\eta(s), \upsilon), \upsilon)\, ds \tag{4.2}$$

For all $t \in \mathscr{R}_+ \& \upsilon \in \Omega$
Then the FONRIE(2.1) is equivalent to the operator equation
$u(t,\upsilon) = \mathscr{L}(\upsilon)u(t) + \mathscr{M}(\upsilon)u(t)$; for all $t \in \mathscr{R}_+ \& \upsilon \in \Omega$ We shall show that the operators $\mathscr{L}(\upsilon)$, $\mathscr{M}(\upsilon)$ satisfies all the conditions of the theorem (3.12). This will be done in following steps.
Step-I: First, we show that $\mathscr{L}(\upsilon)$, $\mathscr{M}(\upsilon)$ are random operators on S, $\&\overline{B_r(0)}$ respectively. By hypothesis H_1 the function $\upsilon \to f(t, u(\theta(s), \upsilon), \upsilon)$ is measurable for all $t \in \mathscr{R}_+ \& u \in \mathscr{R}$. The product $(t-s)^{\alpha-1} f(s, u(\theta(s), \upsilon), \upsilon)$ of continuous and measurable functions is again measurable. We known that Riemann integral as a limit of a finite sum of measurable function is again measurable. Therefore, the function $\upsilon \to \frac{1}{\Gamma\alpha} \int_0^t (t-s)^{\alpha-1} f(s, u(\theta(s), \upsilon), \upsilon)\, ds$ is measurable. Then the function $\upsilon \to \mathscr{L}(\upsilon)u$ is measurable for all $u \in \mathscr{R}$. Hence $\mathscr{L}(\upsilon)$ is random operator on S. By hypothesis H_4, $\upsilon \to g(t, u, \upsilon)$ is measurable for all $t \in \mathscr{R}_+$, $u \in \mathscr{R}$. The product $(t-s)^{\zeta-1} g(s, u(\eta(s), \upsilon), \upsilon)$ of continuous and measurable functions is again measurable. We know that Riemann integral as a limit of a finite sum of measurable function is again measurable. Therefore, the function $\upsilon \to \frac{1}{\Gamma\zeta} \int_0^t (t-s)^{\zeta-1} g(s, u(\eta(s), \upsilon), \upsilon)\, ds$ is measurable. Then the function $\upsilon \to \mathscr{M}(\upsilon)$ is measurable for all $u \in \mathscr{R}$. Hence $\mathscr{M}(\upsilon)$ is random operator on $\overline{B_r(0)}$.
Step-II: Next, we show that $\mathscr{L}(\upsilon)$ is a nonlinear contraction.
Let $u, w \in S$

$$\begin{aligned}
\mid \mathscr{L}(\upsilon)u(t) - \mathscr{L}(\upsilon)w(t) \mid &= \Bigg| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(\theta(s), \upsilon), \upsilon)\, ds \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, w(\theta(s), \upsilon), \upsilon)\, ds \Bigg| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big| f(s, u(\theta(s), \upsilon), \upsilon) - f(s, w(\theta(s), \upsilon), \upsilon) \Big|\, ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathscr{J}(s, \upsilon) \mid u(\theta(s), \upsilon) - w(\theta(s), \upsilon) \mid\, ds
\end{aligned}$$

Taking the supremum over t in the above inequality, we obtain
$\therefore K_2 = \sup\{\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathscr{J}(s, \upsilon)\, ds\} < 1$

$$\|\mathscr{L}(\upsilon)u - \mathscr{L}(\upsilon)w\| \leq K_2 \|u(\upsilon) - w(\upsilon)\| \tag{4.3}$$

$\therefore$ $\mathscr{L}(\upsilon)$ is nonlinear contraction on S on to itself.
Step-III Now, show that $\mathscr{M}(\upsilon)$ is continuous on $\overline{B_r(0)}$. Let $\{u_n\}$ be convergent sequence of

point in $\overline{B_r(0)}$, converging to the point $u \in \overline{B_r(0)}$, then it is enough to prove that

$$\lim_{n\to\infty} \mathscr{M}(\upsilon)u_n(t) = \mathscr{M}(\upsilon)u(t), t \in \mathscr{R}_+$$

By Lebesgue dominated converging theorem we obtain

$$\begin{aligned}
\lim_{n\to\infty} \mathscr{M}(\upsilon)u_n(t) &= \lim_{n\to\infty} \left\{ \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} g(s, u_n(\eta(s),\upsilon),\upsilon)\, ds \right\} \\
&= \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} \lim_{n\to\infty} g(s, u_n(\eta(s),\upsilon),\upsilon)\, ds \\
&= \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} g(s, u(\eta(s),\upsilon),\upsilon)\, ds \\
&= \mathscr{M}(\upsilon)u(t) \qquad (4.4)
\end{aligned}$$

For every $t \in \mathscr{R}_+$ and $\upsilon \in \Omega$.
This shows that the sequence $\{\mathscr{M}(\upsilon)u_n\}$ converges to $\mathscr{M}(\upsilon)u$ on $\overline{B_r(0)}$.
Hence $\mathscr{M}$ is continuous on $\overline{B_r(0)}$.
Step-IV: To show that $\mathscr{M}(\upsilon)$ is compact operator on $\overline{B_r(0)}$.
It suffices to show that$\mathscr{M}(\upsilon)(\overline{B_r(0)})$ is uniformly bounded and equicontinuous set in S, for each $\upsilon \in \Omega$.
First, we show that $\mathscr{M}(\upsilon)(\overline{B_r(0)})$ is uniformly bounded for each $\upsilon \in \Omega$.
Let $u \in \overline{B_r(0)}$ be arbitrary thus by hypothesis H_5, g is $L^1(\upsilon)$-caratheodory

$$\begin{aligned}
| \mathscr{M}(\upsilon)u(t) | &= \left| \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} g(s, u(\eta(s),\upsilon),\upsilon)\, ds \right| \\
&\leq \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} \mid g(s, u(\eta(s),\upsilon),\upsilon) \mid\, ds \\
&\leq \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} q_r(s,\upsilon)\, ds \\
&\leq \frac{1}{\Gamma(\zeta)} \nu(t,\upsilon)
\end{aligned}$$

Taking supremum all over t

$$\|\mathscr{M}(\upsilon)u\| \leq K_1 \text{ for all} t \in \mathscr{R}_+ \qquad (4.6)$$

Hence $\mathscr{M}(\overline{B_r(0)})$ is uniformly bounded subset of S.
Step-V: Now we shall show that$\mathscr{M}(\overline{B_r(0)})$is equicontinuous set in S.

Let $t_1, t_2 \in \mathscr{R}_+$ with $t_1 > t_2$

$$\begin{aligned}
|\mathscr{M}(\upsilon)u(t_1) - \mathscr{M}(\upsilon)u(t_2)| &= \left| \frac{1}{\Gamma(\zeta)} \int_0^{t_1} (t_1 - s)^{\zeta-1} g(s, u(\eta(s), \upsilon), \upsilon)\, ds \right. \\
&\quad \left. - \frac{1}{\Gamma(\zeta)} \int_0^{t_2} (t_2 - s)^{\zeta-1} g(s, u(\eta(s), \upsilon), \upsilon)\, ds \right| \\
&\leq \left| \frac{1}{\Gamma(\zeta)} \int_0^{t_1} (t_1 - s)^{\zeta-1} q_r(s, \upsilon)\, ds \right. \\
&\quad \left. - \frac{1}{\Gamma(\zeta)} \int_0^{t_1} (t_1 - s)^{\zeta-1} q_r(s, \upsilon)\, ds \right| \\
&\leq \frac{1}{\Gamma(\zeta)} |\nu(t_1, \upsilon) - \nu(t_2, \upsilon)| \qquad (4.7)
\end{aligned}$$

Since $\nu(t, \upsilon) = \int_0^{t_1} (t_1 - s)^{\zeta-1} q_r(s, \upsilon)\, ds$ and ν is uniformly continuous function.
Hence, $|\mathscr{M}(\upsilon)u(t_1) - \mathscr{M}(\upsilon)u(t_2)| \to 0$ as $t_1 \to t_2$ for all $t_1, t_2 \in \mathscr{R}_+$ and $\upsilon \in \Omega$.
This shows that $\mathscr{M}(\overline{B_r(0)})$ is equicontinuous set in S.
Hence $\mathscr{M}(\overline{B_r(0)})$ is uniformly bounded and equicontinuous set in $S = BM(\mathscr{R}_+, \mathscr{R})$ and so $\mathscr{M}(\overline{B_r(0)})$ is relatively compact by the Arzela-Ascoli theorem. As consequence, $\mathscr{M}$ is continuous and compact operator on $\overline{B_r(0)}$.
$\therefore \mathscr{M}$ is completely continuous operator on $\overline{B_r(0)}$.
Step-VI: Next, we show that $\mathscr{L}(\upsilon)u + \mathscr{M}(\upsilon)w = u$ this implies $u \in \overline{B_r(0)}$ for all $w \in \overline{B_r(0)}$.
Let $u \in S$ be arbitrary, Then

$$\begin{aligned}
|\mathscr{L}(\upsilon)u(t) + \mathscr{M}(\upsilon)w(t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^{t} (t - s)^{\alpha-1} f(s, u(\theta(s), \upsilon), \upsilon)\, ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\zeta)} \int_0^{t} (t - s)^{\zeta-1} g(s, w(\eta(s), \upsilon), \upsilon)\, ds \right| \\
&\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{t} (t - s)^{\alpha-1} f(s, u(\theta(s), \upsilon), \upsilon)\, ds \right| \\
&\quad + \left| \frac{1}{\Gamma(\zeta)} \int_0^{t} (t - s)^{\zeta-1} g(s, w(\eta(s), \upsilon), \upsilon)\, ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t} (t - s)^{\alpha-1} |f(s, u(\theta(s), \upsilon), \upsilon)|\, ds \\
&\quad + \frac{1}{\Gamma(\zeta)} \int_0^{t} (t - s)^{\zeta-1} |g(s, w(\eta(s), \upsilon), \upsilon)|\, ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t} (t - s)^{\alpha-1} \phi(s, \upsilon)\, ds \\
&\quad + \frac{1}{\Gamma(\zeta)} \int_0^{t} (t - s)^{\zeta-1} q_r(s, \upsilon)\, ds \\
&\leq \frac{1}{\Gamma(\alpha)} \phi_1(t, \upsilon) + \frac{1}{\Gamma(\zeta)} \nu(t, \upsilon)
\end{aligned}$$

taking supremum over t

$$\|\mathscr{L}(\upsilon)u + \mathscr{M}(\upsilon)w\| = \Phi + K_1 = r; \text{ for all } w \in \overline{B_r(0)}$$

$\therefore \mathscr{L}(\upsilon)u + \mathscr{M}(\upsilon)w \in \overline{B_r(0)}$ implies $u \in \overline{B_r(0)}$ for all $w \in \overline{B_r(0)}$

hypothesis (c) of theorem (3.12) holds.
Hence by theorem (3.12) to the operator equation $u(t,\upsilon) = \mathscr{L}(\upsilon)u + \mathscr{M}(\upsilon)u$ conclude that FONRIE (2.1) has a random solution on $\mathscr{R}_+$.
Step-VII: Finally, we have to show that the locally attractivity of the solution for FONRIE (2.1) .
Let $u \,\&\, w$ be two solutions FONRIE (2.1), Then we have
$| u(t,\upsilon) - w(t,\upsilon) |$

$$
\begin{aligned}
&= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(\theta(s),\upsilon),\upsilon)\,ds + \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} g(s,u(\eta(s),\upsilon),\upsilon)\,ds \right.\\
&\quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,w(\theta(s),\upsilon),\upsilon)\,ds - \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} g(s,w(\eta(s),\upsilon),\upsilon)\,ds \right|\\
&\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(\theta(s),\upsilon),\upsilon)\,ds \right| + \left| \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} g(s,u(\eta(s),\upsilon),\upsilon)\,ds \right|\\
&\quad + \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,w(\theta(s),\upsilon),\upsilon)\,ds \right| + \left| \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} g(s,w(\eta(s),\upsilon),\upsilon)\,ds \right|\\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mid f(s,u(\theta(s),\upsilon),\upsilon) \mid ds + \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} \mid g(s,u(\eta(s),\upsilon),\upsilon) \mid ds\\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mid f(s,w(\theta(s),\upsilon),\upsilon) \mid ds + \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} \mid g(s,w(\eta(s),\upsilon),\upsilon) \mid ds\\
&\leq 2 \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(s,\upsilon)\,ds + \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} q_r(s,\upsilon)\,ds \right]\\
&\leq 2 \left[\frac{1}{\Gamma(\alpha)} \phi_1(t,\upsilon) + \frac{1}{\Gamma(\zeta)} \nu(t,\upsilon) \right]
\end{aligned}
$$

Taking limit in above inequality as $t \to \infty$ yields

$$
\begin{aligned}
\lim_{t\to\infty} \mid u(t,\upsilon) - w(t,\upsilon) \mid &\leq 2 \left[\frac{1}{\Gamma(\alpha)} \lim_{t\to\infty} \phi_1(t,\upsilon) + \frac{1}{\Gamma(\zeta)} \lim_{t\to\infty} \nu(t,\upsilon) \right]\\
&= 0
\end{aligned}
$$

$\because \lim_{t\to\infty} \phi_1(t,\upsilon) = 0 \,\&\, \lim_{t\to\infty} \nu(t,\upsilon) = 0$, for $\varepsilon > 0$ there are real number $T_1 > 0$ and $T_2 > 0$ such that $\phi_1(t,\upsilon) < \frac{\Gamma(\alpha)\varepsilon}{4}$ for all $t \geq T_1$ and $\nu(t,\upsilon) < \frac{\Gamma(\zeta)\varepsilon}{4}$ for all $t \geq T_2$. If we choose $T = max(T_1,T_2)$, Then from above inequality it follows that $| u(t,\upsilon) - w(t,\upsilon) | \leq \varepsilon$ for all $t \geq T$. Consequently, the FORIE (2.1) has random a solution and the solution is locally attractive on $\mathscr{R}$. □

Example 4.3. *Let us consider FONRIE*

$$
\begin{aligned}
u(t,\upsilon) = &\frac{1}{\Gamma(1/2)} \int_0^t (t-s)^{-1/2} \frac{\lambda}{1+\upsilon^2} (1+s)^{-p} tanh(u(s/2,\upsilon))\,ds\\
&+ \frac{1}{\Gamma(1/5)} \int_0^t (t-s)^{-4/5} \frac{\mu}{1+\upsilon^2} (1+s)^{-q} sin(u(1+s,\upsilon))\,ds \qquad (4.10)
\end{aligned}
$$

Where $p,q > 1$ and $\lambda,\mu > 0$

Step-by-step hypothesis verification
Here, $f(t,u,\upsilon) = \dfrac{\lambda}{1+\upsilon^2}(1+t)^{-p} tanh(u(s/2,\upsilon))$ and

$g(t,u,\upsilon) = \dfrac{\mu}{1+\upsilon^2}(1+t)^{-q} sin(u(1+s,\upsilon))$
H_1: For fixed t,u; $\upsilon \to f(t,u,\upsilon)$ is measurable and for fixed u,υ; $t \to f(t,u,\upsilon)$ is continuous on $\mathscr{R}_+$. Hence Riemann integrable.
H_2:

$$\begin{aligned}
\mid f(t,u,\upsilon) - f(t,w,\upsilon)\mid &= \left|\frac{\lambda}{1+\upsilon^2}(1+s)^{-p} tanhu - \frac{\lambda}{1+\upsilon^2}(1+s)^{-p} tanhw\right| \\
&= \frac{\lambda}{1+\upsilon^2}(1+s)^{-p} \mid tanhu - tanhw \mid \\
&= \frac{\lambda}{1+\upsilon^2}(1+s)^{-p} tanh^{'}(\gamma) \mid u-w \mid \qquad (\because\ \gamma \in (u,w)) \\
&\leq \frac{\lambda}{1+\upsilon^2}(1+s)^{-p} \mid u-w \mid
\end{aligned}$$

$\because\ \mathscr{J}(t,\upsilon) = \frac{\lambda}{1+\upsilon^2}(1+s)^{-p} \in L^{1(\mathscr{R}_+)}$

$$\begin{aligned}
\frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}\mathscr{J}(s,\upsilon)\,ds &= \frac{1}{\Gamma(1/2)}\int_0^t (t-s)^{-1/2}\frac{\lambda}{1+\upsilon^2}(1+s)^{-p}\,ds \\
&= \frac{1}{\Gamma(1/2)}\frac{\lambda}{1+\upsilon^2}\int_0^t (t-s)^{-1/2}(1+s)^{-p}\,ds \\
&= \frac{1}{\Gamma(1/2)}\frac{\lambda}{1+\upsilon^2}\left(\int_0^{t/2}(t-s)^{-1/2}(1+s)^{-p}\,ds\right. \\
&\quad \left. + \int_{t/2}^t (t-s)^{-1/2}(1+s)^{-p}\,ds\right)
\end{aligned}$$

Now, suppose $I_1 = \displaystyle\int_0^{t/2}(t-s)^{-1/2}(1+s)^{-p}\,ds$ and $I_2 = \displaystyle\int_{t/2}^t (t-s)^{-1/2}(1+s)^{-p}\,ds$

$$\begin{aligned}
I_1 &= \int_0^{t/2}(t-s)^{-1/2}(1+s)^{-p}\,ds \\
&\leq \int_0^{t/2}(t/2)^{-1/2}(1+s)^{-p}\,ds
\end{aligned}$$

$\because$ for $s \in [0,t/2]$, $t-s \geq t/2$ and $x \to x^{-1/2}$ is decreasing for $x > 0$, $\therefore (t/2)^{-1/2} \geq (t-s)^{-1/2}$

$$\begin{aligned}
I_1 &\leq \left(\frac{t}{2}\right)^{-1/2}\int_0^{t/2}(1+s)^{-p}\,ds \leq \left(\frac{t}{2}\right)^{-1/2}\int_0^{\infty}(1+s)^{-p}\,ds = \frac{\left(\frac{t}{2}\right)^{-1/2}}{p-1} \\
I_2 &= \int_{t/2}^t (t-s)^{-1/2}(1+s)^{-p}\,ds \leq \left(1+\frac{t}{2}\right)^{-p}\int_{t/2}^t (t-s)^{-1/2}\,ds \\
&\leq \left(1+\frac{t}{2}\right)^{-p}\int_0^{t/2}(x)^{-1/2}\,dx \leq \sqrt{2}\left(1+\frac{t}{2}\right)^{-p} t^{1/2}
\end{aligned}$$

$$\frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}\mathscr{J}(s,\upsilon)\,ds \le \frac{1}{\Gamma(1/2)}\frac{\lambda}{1+\upsilon^2}\left[\frac{\left(\frac{t}{2}\right)^{-1/2}}{p-1}+K\left(1+\frac{t}{2}\right)^{-p}t^{1/2}\right]$$

This is bounded for all $t\in\mathscr{R}_+$ it depends on λ.
$K_2=\sup\left\{\frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}\mathscr{J}(s,\upsilon)\,ds\right\}<1$, by choosing λ sufficiently small.
H_3: $\mid f(t,u,\upsilon)\mid=\left|\frac{\lambda}{1+\upsilon^2}(1+t)^{-p}tanhu\right|\le\frac{\lambda}{1+\upsilon^2}(1+t)^{-p}$
$\therefore\ \phi(t,\upsilon)=\frac{\lambda}{1+\upsilon^2}(1+t)^{-p}$ then

$$\phi_1(t,\upsilon)=\int_0^t (t-s)^{-1/2}\frac{\lambda}{1+\upsilon^2}(1+s)^{-p}\le\frac{\lambda}{1+\upsilon^2}\left[\frac{\left(\frac{t}{2}\right)^{-1/2}}{p-1}+K\left(1+\frac{t}{2}\right)^{-p}t^{1/2}\right]\to 0$$

For $t\to\infty$ for $p>1$ and $\Phi=\sup\left\{\frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}\phi_1(s,\upsilon)\,ds\right\}$ is finite and proportional to λ.
H_4: $g(t,u,\upsilon)$ is measurable.
H_5: $\mid g(t,u,\upsilon)\mid=\left|\frac{\mu}{1+\upsilon^2}(1+t)^{-q}sin(u)\right|\le\frac{\mu}{1+\upsilon^2}(1+t)^{-q}$ so $q_r(t,\upsilon)=\frac{\mu}{1+\upsilon^2}(1+t)^{-q}$

$$\nu(t,\upsilon)=\int_0^t (t-s)^{-4/5}\frac{\mu}{1+\upsilon^2}(1+s)^{-q}\,ds\le\frac{\mu}{1+\upsilon^2}\left[\frac{\left(\frac{t}{2}\right)^{-4/5}}{q-1}+J\left(1+\frac{t}{2}\right)^{-p}t^{1/5}\right]\to 0$$

as $t\to\infty$.
Hence $K_2=\sup\left\{\frac{1}{\Gamma\zeta}\nu(t,\upsilon)\right\}$ is finite and proportional to μ.

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