Original Research Article

# Visibility Polynomial of Lollipop, Bow, and Butterfly Graphs


## Abstract

The concept of mutual visibility in graphs serves as a useful framework for examining how information or influence propagates in networks subject to structural restrictions. Building on earlier work by Stefano and others, we study mutual-visibility sets in several fundamental graph classes and derive explicit formulae for their corresponding visibility polynomials. As part of this investigation, we employ tools such as shortest-separators and set-separators, which offer deeper insight into the relationship between graph topology and visibility constraints. Our results extend previous research on visibility polynomials, including those of graph joins and corona products, and provide new structural characterizations together with computational observations. This work contributes to the theoretical foundations of visibility-based invariants and enhances the understanding of visibility patterns in discrete network models.


*Keywords: mutual-visibility set; visibility polynomial*

2010 Mathematics Subject Classification: 05C30, 05C31, 05C76

## 1 Introduction

Let $G(V, E)$ be a simple graph and let $X \subseteq V$. Two vertices $u, v \in V$ are said to be $X$-visible Tian and Klavžar [2024] if there exists a shortest path $P$ from $u$ to $v$ such that the internal vertices of $P$ do not belong to $X$; that is, $V(P) \cap X \subseteq \{u, v\}$. A set $X$ is called a mutual-visibility set Di Stefano [2022] of $G$ if every pair of vertices in $X$ is $X$-visible. The maximum size of such a set in $G$ is referred to as the mutual-visibility number, denoted by $\mu(G)$.

The study of mutual visibility in graphs has attracted considerable interest due to its wide-ranging applications in both theory and practice. The earliest investigations can be traced back to Wu and Rosenfeld, who examined visibility-related problems in pebble graphs Rosenfeld and Wu [1994], Wu and Rosenfeld [1998]. A more formal graph-theoretic definition of mutual-visibility sets was later given by Stefano Di Stefano [2022]. Since then, the notion of mutual visibility has emerged as an effective framework for understanding how information, influence, or coordination propagate within networks subject to topological restrictions. This line of research has been explored extensively in the literature Korže and Vesel [2024, 2025], Tian and Klavžar [2024], Axenovich and Liu [2024], Cicerone et al. [2024], Brešar and Yero [2024], Cicerone et al. [2023a], Boruzanlı Ekinci and Bujtás [2024], Cicerone et al. [2023c], Kuziak and Rodríguez-Velázquez [2023], Bujtás et al. [2025a], and several extensions and variants of the concept have also been introduced Cicerone et al. [2023b].

Examining mutual-visibility sets of different orders provides valuable insights into how groups of agents can maintain simultaneous observation of one another—a feature that is essential for applications such as surveillance, target tracking, and distributed coordination. When visibility is restricted by obstacles, studying sets of various sizes becomes important for identifying how many agents can remain in line-of-sight and adapt their positions accordingly. A significant advancement in this area was made by Bujtás et al., who introduced the visibility polynomial in Bujtás et al. [2025b]. This polynomial invariant encodes the distribution of mutual-visibility sets of all orders in a graph. By enumerating these sets, one gains a richer understanding of the structural properties and visibility patterns inherent in the network.

In Tonny and Shikhi [2025a], the present authors studied the visibility polynomial of the join of two graphs. Also, an algorithm for computing the visibility polynomial of a graph has been identified, which has a time complexity of $O(n^3 2^n)$. This indicates that the problem is computationally intensive for larger graphs. In Tonny and Shikhi [2025b], the present authors investigate the visibility polynomial associated with the corona product of two graphs. As part of this investigation, the concept of set-separator with respect to two subsets of the vertex set of a graph is defined. Let $u, v \in V(G)$. A vertex $g \in V(G) \setminus \{u, v\}$ is said to be a shortest-separator with respect to $u$ and $v$ if every shortest $(u, v)$-path contains $g$. The collection of all shortest-separators is called the path-cut of $G$, denoted by $p_c(G)$. Let $A$ and $B$ be two disjoint subsets of $V(G)$. A vertex $g$ is said to be a set-separator with respect to $A$ and $B$ if $g$ is a shortest-separator for every $u \in A$ and $v \in B$.

In this paper, the mutual-visibility sets of some fundamental graph classes are characterized, and explicit formulae for the visibility polynomial associated with the graph classes are derived.

## 2 Notations and preliminaries

$G(V, E)$ represents an undirected simple graph with vertex set $V(G)$ and edge set $E(G)$. Unless otherwise stated, all graphs in this paper are assumed to be connected, so that there is at least one path between each pair of vertices. We follow the standard graph-theoretic definitions and notation as presented in Harary [1969].

A complete graph on $n$ vertices is a graph in which there is an edge between any pair of distinct vertices, and is denoted by $K_n$. A sequence of vertices $(u_0, u_1, u_2, \ldots, u_n)$ is referred to as a $(u_0, u_n)$-path in a graph $G$ if $u_i u_{i+1} \in E(G), \forall i \in \{0, 1, \ldots, (n-1)\}$. A cycle (or circuit) in a graph $G$ is a path $(u_0, u_1, u_2, \ldots, u_n)$ together with an edge $u_0 u_n$. If a graph $G$ on $n$ vertices itself is a path, it is denoted by $P_n$, and if the graph $G$ itself is a cycle, it is denoted by $C_n$. A lollipop graph $L_{m,n}$ is obtained by joining a complete graph $K_m$ to a path $P_n$ with a single edge. For integers $m, n \geq 0$ the bistar $R_{m,n}$ is obtained by taking two stars $K_{1,m}$ and $K_{1,n}$ with centres $c_1, c_2$ and leaf-sets $L_1$ (size $m$) and $L_2$ (size $n$), respectively, and joining $c_1$ and $c_2$ by a single edge. A shell graph $S_n$, where $n \geq 3$, is obtained from the cycle $C_n$ by adding $(n-3)$ chords incident with a common vertex, called the apex $c$. A bow graph $B_{m,n}$ is a double shell with the same apex in which each shell has any order. The butterfly graph $\mathcal{BF}_{m,n}$ is obtained from $B_{m,n}$ by adding two pendant vertices, each adjacent to the apex of $B_{m,n}$.

The distance $d_G(u, v)$ between two vertices $u$ and $v$ in $G$ is the length of the shortest $(u, v)$-path in $G$. The maximum distance between any pair of vertices of $G$ is called the diameter of $G$, denoted by $\operatorname{diam}(G)$. Let $G$ and $H$ be two graphs. Then the join, $G \vee H$ is the graph with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

## 3 Visibility polynomial of some graph classes

**Theorem 3.1** (Tonny and Shikhi [2025a]). *The visibility polynomial of the complete graph of order $n$ is given by $\mathcal{V}(K_n) = (1+x)^n$.*

**Lemma 3.2** (Tonny and Shikhi [2025b]). *Let $g$ be a set-separator with respect to the disjoint sets $A$ and $B$ of $V(G)$. If $S$ is a mutual-visibility set of $G$, containing $g$, then either $S \cap A = \emptyset$ or $S \cap B = \emptyset$. That is, either $S \subseteq \overline{A}$ or $S \subseteq \overline{B}$.*

**Theorem 3.3.** *The visibility polynomial of the lollipop graph $L_{m,n}$ is,*

$$\mathcal{V}(L_{m,n}) = (1+x)^{m-1}\big(1+(n+1)x\big) + \binom{n+1}{2}x^2$$

*Proof.* Let $L_{m,n}$ be the lollipop graph obtained by joining a complete graph $K_m$ to a path $P_n = (p_1, p_2 \dots, p_n)$ through a bridge. Let $c \in V(K_m)$ be a vertex incident with bridge, so that $cp_1 \in E(L_{m,n})$. Let $A = V(K_m) \setminus \{c\}$ and $B = V(P_n)$. Let $S$ be a mutual-visibility set of $L_{m,n}$ containing $c$. Since $c$ is a set-separator with respect to $A$ and $B$, by Lemma 3.2, $S$ cannot intersect both $A$ and $B$. Thus either $S \subseteq \{c\} \cup A$ or $S \subseteq \{c\} \cup B$.

In the first case, there is no restriction on $A$, so the contribution is $x(1+x)^{m-1}$. In the second case, $S$ consists of $c$ together with at most one path vertex, contributing $x + nx^2$. Accounting for $\{c\}$ only once, the total contribution to $\mathcal{V}(L_{m,n})$ from mutual-visibility sets containing $c$ is $x(1+x)^{m-1} + nx^2$.

Suppose $c \notin S$. If $S \cap A = \emptyset$, then $S \subseteq B$ and the admissible sets are exactly the mutual-visibility sets of $P_n$, contributing $\mathcal{V}(P_n) = 1 + nx + \binom{n}{2}x^2$ (see Bujtás et al. [2025b]).

If $S \cap A \neq \emptyset$, then necessarily $|S \cap B| \leq 1$. Indeed, if $S \cap B$ contained two distinct vertices $p_i, p_j \in P$ with $i < j$, consider the pair $(x, p_j)$ with $x \in S \cap A$. In $L_{m,n}$, the unique geodesic from $x$ to $p_j$ is $(x, c, p_1, p_2, \dots, p_j)$, whose internal vertices are $c, p_1, \dots, p_{j-1}$. Since $i < j$, the vertex $p_i$ lies among these internal vertices, and $p_i \in S$, contradicting the assumption that $S$ is a mutual-visibility set. Hence $|S \cap B| \leq 1$.

The choices for $S \cap A$ contribute $(1+x)^{m-1} - 1$ by Theorem 3.1, and the choices for $S \cap B$ contribute $1 + nx$. Thus this subcase contributes

$$\big((1+x)^{m-1} - 1\big)(1+nx).$$

Summing the two mutually-exclusive subcases, we obtain

$$\big((1+x)^{m-1} - 1\big)(1+nx) \; + \; \left(1 + nx + \binom{n}{2}x^2\right) = (1+x)^{m-1}(1+nx) + \binom{n}{2}x^2$$

Adding the contributions with $c \in S$ and with $c \notin S$, we obtain

$$\begin{aligned}\mathcal{V}(L_{m,n}) &= x(1+x)^{m-1} + nx^2 + (1+x)^{m-1}(1+nx) + \binom{n}{2}x^2 \\ &= (1+x)^{m-1}\big(1+(n+1)x\big) + \binom{n+1}{2}x^2\end{aligned}$$

□

**Theorem 3.4.** *The visibility polynomial of the bistar graph $R_{m,n}$ is,*

$$\mathcal{V}(R_{m,n}) = (1+x)^{m+n} + x\big((1+x)^m + (1+x)^n\big) + (m+n+1)x^2.$$

*Proof.* Let $R_{m,n}$ denote the bistar obtained from the stars $K_{1,m}$ and $K_{1,n}$ with centers $c_1$ and $c_2$, and leaf sets $L_1 = \{a_1, \ldots, a_m\}$ and $L_2 = \{b_1, \ldots, b_n\}$ of cardinalities $m$ and $n$, respectively, by adding the edge $c_1c_2$. Let $S \subseteq V(R_{m,n})$ be an arbitrary mutual-visibility set. We classify such sets according to whether they contain none, exactly one, or both of the centers $c_1$ and $c_2$.

**Case 1**: Assume that $c_1, c_2 \notin S$. Then every vertex of $S$ lies in $L_1 \cup L_2$. If $u, v \in L_1$ (or both in $L_2$), the unique shortest $(u, v)$-path is $(u, c_1, v)$ (resp. $(u, c_2, v)$), whose internal vertex is $c_1$ (resp. $c_2$). Since the corresponding centre does not belong to $S$, the internal vertex avoids $S$. If $u \in L_1$ and $v \in L_2$, then the shortest $(u, v)$-path is $(u, c_1, c_2, v)$, whose internal vertices are $c_1, c_2$, both outside $S$. Thus, every pair of vertices in any subset of $L_1 \cup L_2$ is $S$-visible, and consequently every subset of $L_1 \cup L_2$ is mutual-visible. The contribution of Case 1 to $\mathcal{V}(B_{m,n})$ is therefore $(1 + x)^{m+n}$.

**Case 2**: Assume that $c_1 \in S$ and $c_2 \notin S$. The vertex $c_1$ is a set-separator for $A = L_1$ and $B = \{c_2\} \cup L_2$. By Lemma 3.2, a mutual-visibility set containing $c_1$ cannot intersect both $L_1$ and $L_2$. Thus two possibilities occur:

- If $S$ contains $c_1$ and no vertex of $L_1$, then $S$ may contain any subset of $L_2$, giving $x(1 + x)^n$.
- If $S$ contains $c_1$ and no vertex of $L_2$, then $S$ contains exactly one vertex from $L_1$. Indeed, for distinct $a_i, a_j \in L_1$, the unique shortest $(a_i, a_j)$-path is $(a_i, c_1, a_j)$, whose internal vertex $c_1$ lies in $S$, contradicting mutual-visibility. Hence, there are $m$ such sets, contributing $mx^2$.

Therefore the contribution of Case 2 to $\mathcal{V}(R_{m,n})$ is $x(1 + x)^n + mx^2$.

**Case 3**: Assume that $c_2 \in S$ and $c_1 \notin S$. By symmetry with Case 2, the contribution of this case to $\mathcal{V}(R_{m,n})$ is

$$x(1 + x)^m + nx^2.$$

**Case 4**: Assume that $c_1, c_2 \in S$. In this situation, no leaf can belong to $S$. For instance, if $b \in L_2$, then the unique shortest $(c_1, b)$-path is $(c_1, c_2, b)$, whose internal vertex $c_2$ lies in $S$, contradicting mutual-visibility. Similarly, if $a \in L_1$, the unique shortest $(c_2, a)$-path has internal vertex $c_1 \in S$. Thus, the only mutual-visibility set in this case is $\{c_1, c_2\}$, contributing $x^2$.

Summing the contributions from the four cases gives

$$\mathcal{V}(R_{m,n}) = (1 + x)^{m+n} + \big(x(1 + x)^n + mx^2\big) + \big(x(1 + x)^m + nx^2\big) + x^2,$$

□

**Theorem 3.5** (Tonny and Shikhi [2025c]). *Let $B_{m,n}$ be the bow graph obtained as the union of two shell graphs $S_m$ and $S_n$ sharing the same apex $c$. Equivalently, $B_{m,n} = K_1 \vee \big(P_{m-1} \cup P_{n-1}\big)$ with apex $c$ and disjoint paths $P_{m-1} = (a_1, \ldots, a_{m-1})$ and $P_{n-1} = (b_1, \ldots, b_{n-1})$. Its visibility polynomial is*

$$\mathcal{V}(B_{m,n}) = (1 + x)^{m+n-2} + x + (m + n - 2)x^2 + (2m + 2n - 10)x^3$$

**Theorem 3.6.** *The visibility polynomial of the butterfly graph $\mathcal{BF}_{m,n}$ is,*

$$\mathcal{V}(\mathcal{BF}_{m,n}) = (1 + x)^{\,m+n} \; + \; x \; + \; (m + n)\, x^2 \; + \; (2m + 2n - 10)\, x^3.$$

*Proof.* Let $B_{m,n} = K_1 \vee \big(P_{m-1} \cup P_{n-1}\big)$ be the bow graph with apex $c$, and let $\mathcal{BF}_{m,n}$ be the graph obtained from $B_{m,n}$ by attaching two pendent edges at $c$, with new leaf vertices $p$ and $q$. Let $S$ be a mutual-visibility set of $\mathcal{BF}_{m,n}$ and define

$$U = S \cap \{p, q\}.$$

We decompose the visibility polynomial $\mathcal{V}(\mathcal{BF}_{m,n})$ into three disjoint contributions according to the cardinality of $U$, namely,

$$\mathcal{V}(\mathcal{BF}_{m,n}) = \mathcal{V}_0 + \mathcal{V}_1 + \mathcal{V}_2 \tag{3.1}$$

where $\mathcal{V}_i$ denotes the contribution of all mutual-visibility sets $S$ with $|U| = i$, for $i = 0, 1, 2$.

**Case 1**: Suppose $|U| = 0$, that is, $S$ contains neither $p$ nor $q$. Since attaching the leaves $p$ and $q$ at the apex $c$ does not alter any distances among the vertices of $B_{m,n}$, the collection of mutual-visibility sets in this case coincides with those of $B_{m,n}$. Therefore,

$$\mathcal{V}_0 = \mathcal{V}(B_{m,n}).$$

**Case 2**: Suppose $|U| = 1$, that is, $S$ contains exactly one of the leaves, say $p$. Accordingly, $S$ can be written as $S = \{p\} \cup X$ for some subset $X \subseteq V(B_{m,n})$.

If $c \in X$, then Lemma 3.2 applies with $g = c$, $A = \{p\}$ and $B = V(B_{m,n}) \setminus \{c\}$. Since $S$ contains both $p$ and $c$, the lemma forces $S \cap U = \emptyset$. Thus the only admissible set in this subcase is

$$S = \{p, c\},$$

which contributes $x^2$.

If $c \notin X$, then every vertex $u \in X$ is visible from $p$ via the path $(p, c, u)$, whose internal vertex $c$ lies outside $S$. Moreover, all pairs of vertices within $X$ remain mutually visible exactly as in $B_{m,n}$ with $c$ excluded. Therefore, $X$ is any subset of $V(B_{m,n}) \setminus \{c\}$ and hence the polynomial contribution of such choices of $X$ is $(1 + x)^{m+n-2}$. Therefore the contribution corresponding to sets of the form $S = \{p\} \cup X$ is

$$x(1 + x)^{m+n-2} + x^2.$$

By symmetry, the same contribution arises for sets of the form $S = \{q\} \cup X$. Hence the total contribution in this case is

$$\mathcal{V}_1 = 2\big[x(1 + x)^{m+n-2} + x^2\big].$$

**Case 3**: Suppose $|U| = 2$, that is, $S$ contains both leaves $p$ and $q$.

If $c \in S$, then the unique shortest $(p, q)$-path $(p, c, q)$ has internal vertex $c \in S$, contradicting mutual-visibility. Hence no mutual-visibility set can contain $p$, $q$ and $c$ simultaneously. Therefore $c \notin S$.

In this situation, every vertex $u \in V(B_{m,n}) \setminus \{c\}$ is visible from both $p$ and $q$ via the paths $(p, c, u)$ and $(q, c, u)$, whose internal vertex $c$ lies outside $S$. Moreover, all pairs of vertices within $V(B_{m,n}) \setminus \{c\}$ remain mutually visible exactly as in $B_{m,n}$ with $c$ excluded. Accordingly, $X$ may be chosen as an arbitrary subset of $V(B_{m,n}) \setminus \{c\}$. Consequently, the contribution in this case is

$$\mathcal{V}_2 = x^2(1 + x)^{m+n-2}.$$

Substituting $\mathcal{V}_0, \mathcal{V}_1$ and $\mathcal{V}_2$ in (3.1) , we obtain

$$\begin{aligned}
\mathcal{V}(\mathcal{BF}_{m,n}) &= \mathcal{V}(B_{m,n}) + 2\big[x(1 + x)^{m+n-2} + x^2\big] + x^2(1 + x)^{m+n-2} \\
&= (1 + x)^{m+n-2}(1 + 2x + x^2) + x + (m + n - 2)x^2 + (2m + 2n - 10)x^3 + 2x^2 \\
&= (1 + x)^{m+n-2}(1 + x)^2 + x + (m + n - 2)x^2 + (2m + 2n - 10)x^3 + 2x^2 \\
&= (1 + x)^{m+n} + x + (m + n)x^2 + (2m + 2n - 10)x^3,
\end{aligned}$$

which is the desired expression. □

# 4 Conclusion

In this paper, the visibility polynomial of certain graph classes is studied, namely lollipop graphs, bow graphs, and butterfly graphs, and obtained explicit expressions for each by analyzing their shortest-path visibility structures. This work adds to the understanding of visibility-based polynomial invariants and provides a foundation for further investigations into more general graph families, their structural properties, and possible connections between the visibility polynomial and other graph invariants.

## Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

## Competing Interests

Author has declared that no competing interests exist.

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