Original Research Article

Various Types of Hausdorff Hypergraphs

Abstract

A hypergraph $H = (V, \mathcal{E})$ is said to be a *Hausdorff hypergraph* if for any two distinct vertices u, v of V there exist hyperedges e_1, $e_2 \in \mathcal{E}$ such that $u \in e_1$, $v \in e_2$ and $e_1 \cap e_2 = \emptyset$.

In this paper we have discussed hausdorff property of hypergraphs as well as minimal hausdorff hypergraph. We have also examined conditions under which competetion hypergraph and independence hypergraph become hausdorff. Also we have discussed some applications of Hausdorff hypergraph.

Keywords: Hausdorff hypergraph,Minimal Hausdorff,Conformal, Cyclotomic number, Competition hypergraph, Independent hypergraph.

2010 MSC: 05C65

1. Introduction

Hypergraphs are generalization of graphs, hence many of the definitions of graphs carry verbatim to hypergraphs. The basic idea of the hypergraph concept is to consider such a generalization of a graph in which any subset of a given set may be an edge rather than two-element subsets [1]. A *hypergraph*[2] H is a pair $(V, \mathcal{E})$, where V is a set of elements called nodes or vertices, and $\mathcal{E}$ is a set of nonempty subsets of V called *hyperedges* or *edges.* Therefore, $\mathcal{E}$ is a subset of $P(X) \setminus \{\emptyset\}$, where $P(X)$ is the power set of X. In drawing hypergraphs, vertices are points in the plane, edges are closed curves separating a respective subset from the rest of vertices. The cardinality of the finite set V is denoted by $|V|$, is called the *order* of the hypergraph. The number of edges is usually denoted by m or $m(H)$. A hypergraph which contains no vertices and no edges is called an empty hypergraph. A *trivial hypergraph* is a hypergraph such that $V \neq \emptyset$ and $E = \emptyset$.

A *simple hypergraph*[3] is a hypergraph with the property if e_i, e_j are hyperedges of H with $e_i \subseteq e_j$, then $i = j$. In other words a hypergraph having no multiple edges is called *simple*. Hence simple hypergraphs do not have empty and multiple edges. Two vertices in a hypergraph are *adjacent*[1] if there is a hyperedge which contains both vertices. Two hyperedges in a hypergraph are *incident*[1] if their intersection is nonempty.

A *k-uniform hypergraph*[4] or a k-hypergraph is a hypergraph in which every edge consists of k vertices. So a 2-uniform hypergraph is a graph, a 3-uniform hypergraph is a collection of unordered triples, and so on. The *rank* [1] $r(H)$ of a hypergraph is the maximum cardinality of any of the edges in the hypergraph. The *co-rank* [1] $cr(H)$ of a hypergraph is the minimum cardinality of a hyperedge in the hypergraph. If $r(H) = cr(H) = k$, then H is k-uniform. The degree [5] $d_H(v)$ of a vertex v in a hypergraph H is the number of edges of H that containing the vertex v. H is k-regular if every vertex has degree k.

An edge of a hypergraph which contains no vertices is called an *empty edge*. The degree of an empty edge is trivially zero. A vertex of a hypergraph which is incident to no edges is called an *isolated vertex.*[1] The degree of an isolated vertex is trivially zero. A hyperedge e of H with $|e| = 1$ is called a *loop*; more specifically a hyperedge $e = \{v\}$ is a loop at the vertex v. A vertex of degree 1 is called a pendant vertex .

A simple hypergraph H with $|E_i| = 2$ for each $E_i \in \mathcal{E}$ is a simple graph.

Through out this paper we consider only simple hypergraph with no isolated vertices.

2. Hypergraphs

Definition 2.1. A Hausdorff hypergraph H is said to be a minimal Hausdorff hypergraph if we remove any hyperedge of H, then the resulting hypergraph is non-Hausdorff .

Theorem 2.1. Let H be a hypergraph with n vertices, where n is a composite number. Then there exists a minimal Huasdroff hypergraph G such that the number of edges of G is less than or equal to $\min\{p + q : pq = n\}$.

Proof. Let $\min\{p + q : pq = n\} = a + b$, where $b \leq a$. Note that $ab = n$. We label the n vertices of H as $v_{i1}, v_{i2}, \dots v_{ib}$, where $i = 1, 2, \dots a$. Construct a new hypergraph G by putting the vertices $v_{1j}, v_{2j}, \dots v_{aj}$ in a hyperedge say f_j. Then we get 'b' hyperedges $f_1, f_2, \dots f_b$. Similarly, place vertices $v_{i1}, v_{i2}, \ldots v_{ib}$ in a hyperedge say e_i. Then we get 'a' hyperedges $e_1, e_2, \ldots e_a$. Clearly $v_{ij} \in e_i \cap f_j$. Note that G is a hypergraph with vertex set $V = \{v_{i1}, v_{i2}, \ldots v_{ib}; i = 1, 2, \ldots, a\}$ and edge set $\{f_1, f_2, \ldots, f_b, e_1, e_2, \ldots, e_a\}$. For any two vertices v_{ir}, v_{is} there exist two hyperedges $f_r, f_s : f_r \cap f_s = \emptyset$. This is true for every $i = 1, 2, \ldots, a$. Similarly, for any two vertices v_{lj}, v_{mj} there exist two hyperedges $e_l, e_m : e_l \cap e_m = \emptyset$. This is true for every $j = 1, 2, \ldots, b$. Therefore, G is a Hausdorff hypergraph. If we remove any hyperedge of G, then G is not a Hausdorff hypergraph. Therefore, G is a minimal Huasdroff hypergraph. Clearly number of edges of G is $a + b$.

□

Remark 2.1. For every perfect square r^2 there exists a $r - uniform$ Hausdorff hypergraph.

Remark 2.2. For every perfect square r^2 there exists a $r - partite$ Hausdorff hypergraph.

3. Competition Hypergraphs

Definition 3.1. [6] If $D = (V, A)$ is a digraph, its *competition hypergraph* $CH(D)$ has vertex set V and $e \subseteq V$ is an edge of $CH(D)$ if and only if $|e| \geq 2$ and there is a vertex $v \in V$, such that $e = \{w \in V/(w, v) \in A\}$. In this case we say that $v \in V$ corresponds to $e \in \mathcal{E}(CH(D))$ and vice versa.

Theorem 3.1. A simple digraph D has a competition hypergraph only if $|V(D)| \geq 3$.

Proof. From the definition of competition hypergraphs e is an edge of $CH(D)$ if $|e| \geq 2$ and there is a vertex $v \in V$ such that $e = \{w \in V/(w, v) \in A)\}$ therefore, $|V(D)| \geq 3$. □

Theorem 3.2. The competition hypergraph of a digraph D is a Hausdorff hypergraph only if $|V(D)| > 4$.

Proof. A hypergraph $H = (V, \mathcal{E})$ is Hausdorff if for any two distinct vertices u and v of V there exist distinct hyperedges e_1 and e_2 of $\mathcal{E}$ such that $u \in e_1$ and $v \in e_2$ and $e_1 \cap e_2 = \emptyset$. Let $D = (V, A)$ be a digraph and $CH(D)$ be its competition hypergraph. From Theorem 3.1 it follows that $|V(D)| \geq 3$. Suppose that $|V(D)| = 4$. Let v_1, v_2, v_3, v_4 be the vertices of given digraph. Consider two distinct vertices of $CH(D)$, say v_1 and v_2. The possible disjoint hyperedges e_1 and e_2 containing v_1 and v_2 respectively are

1. $e_1 = \{v_1, v_3\}, e_2 = \{v_2, v_4\}$

2. $e_1 = \{v_1, v_4\}, e_2 = \{v_2, v_3\}$

The set $\{v_1, v_3\}$ is a hyperedge of $CH(D)$ if either $(v_1, v_2), (v_3, v_2) \in A$ or $(v_1, v_4), (v_3, v_4) \in A$. In both case $\{v_2, v_4\} \notin CH(D)$, which is a contradiction. A similar contradiction arises when $e_1 = \{v_1, v_4\}$ and $e_2 = \{v_2, v_3\}$. Therefore, if $|V(D)| = 4$, then the competition hypergraph $CH(D)$ of D is non Hausdorff . Hence the theorem. □

In Figure 1, D is a digraph on 5 vertices and its competition hypergraph $CH(D)$ is Hausdorff .

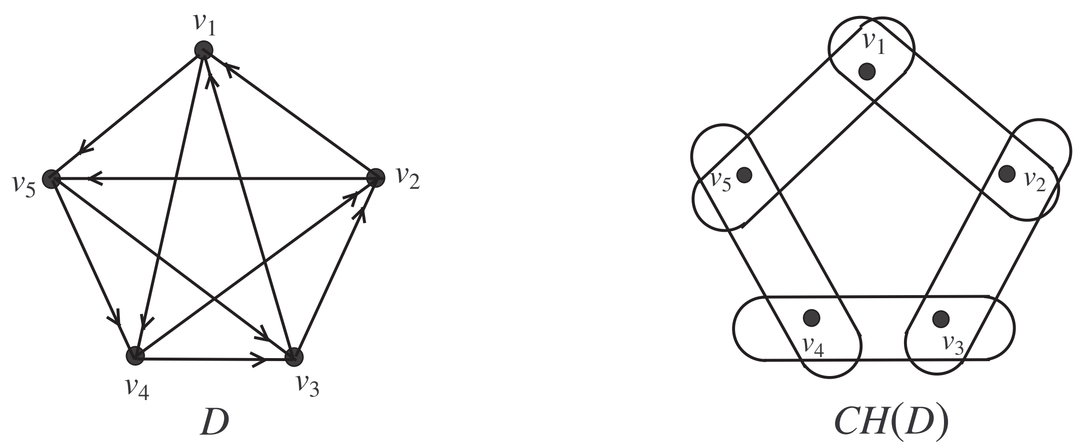

Figure 1: Digraph D and its competition hypergraph $CH(D)$.

Proposition 3.1. Let $D = (V, A)$ be a digraph. If given any two distinct vertices u and v of D, there exist two distinct vertices x and y such that $u \in P = \{w : (w, x) \in A\}$ and $v \in Q = \{z : (z, y) \in A\}$ and $P \cap Q = \emptyset$, then the competition hypergraph $CH(D)$ of D is Hausdorff .

Definition 3.2. A hypergraph is said to be *conformal*[7] if its hyperedges are exactly the cliques of its primal graph.

The hypergraph H in Example 3.1 is conformal but it is not Hausdorff .

Example 3.1.

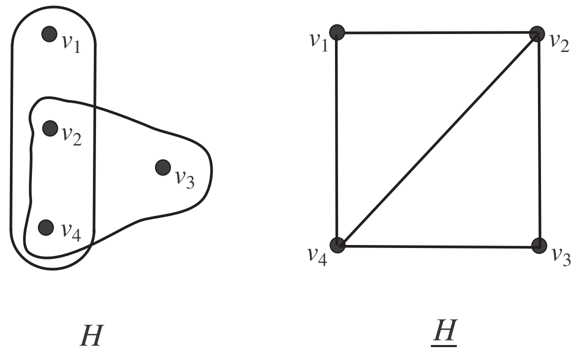

Figure 2: Hypergraph H and its primal graph $\underline{H}$

Example 3.2.

Consider the hypergraph $H = (V, \mathcal{E})$. Where $V = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{E} = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}, \{v_1, v_3\}, \{v_2, v_4\}\big\}$. Clearly $H = (V, \mathcal{E})$ is a Hausdorff hypergraph. Then $\underline{H} = (V, \mathcal{E})$, where $V = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{E} = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3, v_2v_4\}$ (see Figure 3). Note that $\underline{H}$ is itself a clique on four vertices and there exists no hyperedge containing all these four vertices. Therefore, H is not conformal.

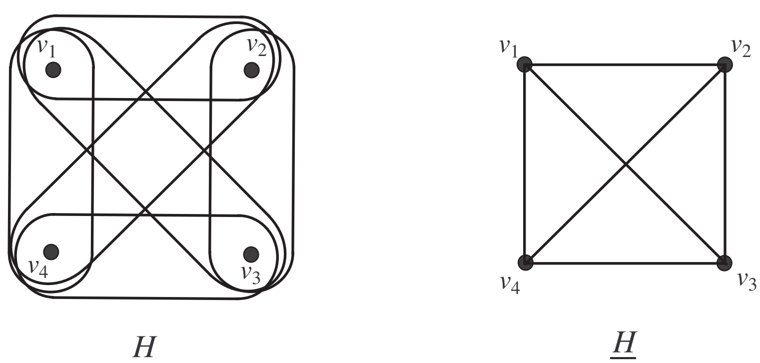

Figure 3: Hypergraph H and its primal graph $\underline{H}$.

Remark 3.1. Example 3.2 shows that a hypergraph H is Hausdorff need not imply it is conformal.

Definition 3.3. Let $H = (V, \mathcal{E})$ be a hypergraph with $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$. The *cyclomatic number*[2] of H is the non-negative integer

$$\mu(H) = \sum_{i=1}^{m} |e_i| - |\bigcup_{i=1}^{m} e_i| - w_H$$

where w_H is the maximum weight of a forest $F \subset L(H)$.

Theorem 3.3. The line graph $L(H)$ of H_{mn} is a complete bipartite graph.

Theorem 3.4. The cyclomatic number of a minimal Hausdorff hypergraph on mn vertices is $mn - (m + n - 1_)$

Proof. The minimal Hausdorff graph on mn vertices has m edges $e_1, e_2, \ldots e_m$ and n edges $f_1, f_2, \ldots f_n$. Then $L(H)$ is a complete bipartite graph on mn vertices, so $L(H) = K_{m,n}$. Weight of each edge is 1. Maximum weight of a forest in $L(H)$ is $(m+n-1.)$ Therefore, $\mu(H) = \sum_{i=0}^{m} |e_i| - |\bigcup_{i=1}^{m} e_i| - w(H)$ i.e $\mu(H) = 2mn - mn - (m + n - 1)$. This implies $\mu(H) = mn - (m + n - 1)$. □

There are many definitions for the *acyclicity of a hypergraph* [8]. The definition we use here is based on the Graham reduction [8], described below. Let $H = (V, \mathcal{E})$ be a given hypergraph. Graham's algorithm applies the following operations repeatedly to H until neither can be applied:

1. If a vertex $v \in V$has degree one, then delete v from the edge containing it.
2. If $e, f \in \mathcal{E}$ are distinct edges such that $e \subseteq f$, then delete e from $\mathcal{E}$.
3. If $e \in \mathcal{E}$ is empty, then delete e from $\mathcal{E}$.

The resulting hypergraph H is said to be Graham-reduced, and is called the *Graham reduction*[8] of H.

Definition 3.4. [8] A hypergraph is acyclic if its Graham reduction is empty. Otherwise it is called cyclic.

In a simple Hausdorff hypergraph without loops, degree of each vertex is ≥ 2, so we have:

Remark 3.2. All simple Hausdorff hypergraph without loops is cyclic.

Remark 3.3. If H is a hypergraph with all its edges are loops, then H is an acyclic hypergraph.

4. Independent Hypergraph

Now we define independent hypergraph as

Definition 4.1. Let $G = (V, E)$ be the given graph. An *independent hypergraph* $IH(G)$ is the hypergraph whose vertices are vertices of G and whose hyperedges are maximal independent subsets of G.

Example 4.1. Consider the graph $G = (V, E)$ where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_2v_4, v_3v_5\}$. The maximal independent subsets of G are $\{\{v_1,\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_2, v_5\}, \{v_4, v_5\}\}$. The corresponding independent hypergraph is $IH(G) = (V, \mathcal{E})$ where $\mathcal{E} = \{\{v_1,\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_2, v_5\}, \{v_4, v_5\}\}$.

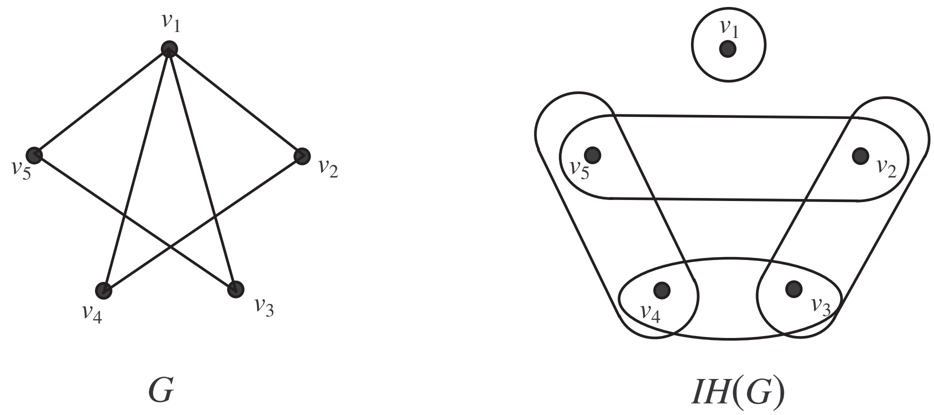

Figure 4: Graph G and its independent hypergraph $IH(G)$.

If $IH(G)$ is Hausdorff then we call $IH(G)$ as *Hausdorff independent hypergraph.*

Theorem 4.1. If G is a complete graph then $IH(G)$ is Hausdorff .

Proof. Since G is complete the only independent sets in G are singleton sets. Then the corresponding independent hypergraph $IH(G)$ is a hypergraph with only loops. Therefore, $IH(G)$ is Hausdorff . □

Theorem 4.2. If G is not complete then $IH(G)$ is Hausdorff only if $|V(G)| \geq 4$

Proof.

Case 1. $|V(G)| = 2$

In this case since G is not complete, it must be an empty graph. Let $G = (V, E)$ where $V = \{v_1, v_2\}$ and $E = \{\{v_1\}, \{v_2\}\}$. Then the maximal independent subset of G is $\{v_1, v_2\}$. The corresponding independent hypergraph $IH(G) = (V, \mathcal{E})$ where $\mathcal{E} = \{\{v_1, v_2\}\}$. Therefore, $IH(G)$ is not Hausdorff

.

Case 2. $|V(G)| = 3$

Given that G is not complete. Then there arise three cases. G is an empty graph, or G contains one edge or G contain 2 edges

Subcase 1. G is an empty graph

Let $G = (V, E)$ where $V = \{v_1, v_2, v_3\}$ and $E = \emptyset$. Then G has only one maximal independent set, which is $\{v_1, v_2, v_3\}$. The corresponding independent hypergraph is $IH(G) = (V, \mathcal{E})$ where $V = \{v_1, v_2, v_3\}$ and $\mathcal{E} = \{\{v_1, v_2, v_3\}\}$ (see Figure 5). Note that $IH(G)$ is not Hausdorff .

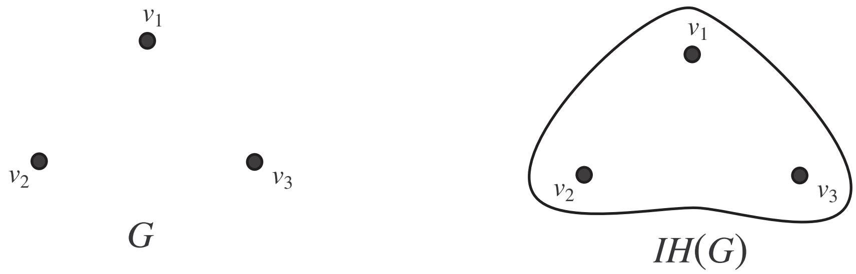

Figure 5: Graph G and its independent hypergraph $IH(G)$.

Subcase 2. G has only one edge

Let $G = (V, E)$ where $V = \{v_1, v_2, v_3\}$ and $E = \{v_1v_2\}$. Then the maximal independent sets in G are $\{v_1, v_3\}$ and $\{v_2, v_3\}$. The corresponding independent hypergraph is $IH(G) = (V, \mathcal{E})$ where $V = \{v_1, v_2, v_3\}$ and $\mathcal{E} = \{\{v_1, v_3\}, \{v_2, v_3\}\}$. (see Figure 6). Note that $IH(G)$ is not Hausdorff .

Figure 6: Graph G and its independent hypergraph $IH(G)$.

Subcase 3. G has two edges

Let $G = (V, E)$ where $V = \{v_1, v_2, v_3\}$ and $E = \{v_1v_2, v_1v_3\}$. Then the maximal independent sets in G are $\{v_1\}$ and $\{v_2, v_3\}$. The corresponding independent hypergraph is $IH(G) = (V, \mathcal{E})$ where $V = \{v_1, v_2, v_3\}$ and $\mathcal{E} = \{\{v_1\}, \{v_2, v_3\}\}$ (see Figure 7). Note that $IH(G)$ is not Hausdorff .

Figure 7: Graph G and its independent hypergraph $IH(G)$.

Case 3. $|V(G)| = 4$

Let $G = (V, E)$ where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_3, v_2v_4\}$. Then the maximal independent sets in G are $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_2, v_4\}$ and $\{v_3, v_4\}$. The corresponding independent hypergraph is $IH(G) = (V, \mathcal{E})$ where $V = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{E} = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\}$.(see Figure 8). Note that $IH(G)$ is Hausdorff .

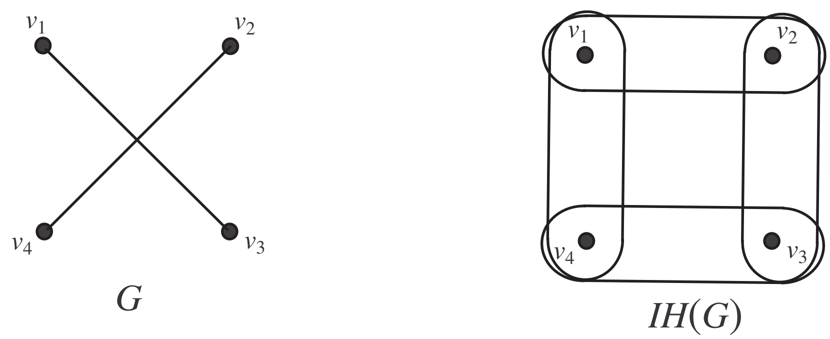

Figure 8: Graph G and its independent hypergraph $IH(G)$.

Thus we proved if G is not complete then $IH(G)$ is Hausdorff only if $|V(G)| \geq 4$ □

Theorem 4.3. Let $G = (V, E)$ be a bipartite graph with bipartion X and Y where $|X| = |Y| = n$ and degree of each vertex is $(n-1)$ and $x_iy_i \neq E$. Then the corresponding independent hypergraph is Hausdorff .

Proof. Given that $G = (V, E)$ is bipartite graph with bipartition X and Y where $|X| = |Y| = n$. Each vertex has degree $(n-1)$ and $x_iy_i \neq E$ implies $\{x_i, y_i\}$ is a maximal independent set in G for every $i = 1, 2, 3 \ldots n$. Note that $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ are also maximal independent sets in G. The independent hypergraph $IH(G)$ corresponding to G is $(V, \mathcal{E})$, where $V =$

$\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ and $\mathcal{E} = \big\{\{x_1, x_2, \ldots, x_n\}, \{y_1, y_2, \ldots, y_n\}, \{x_1, y_1\}, \ldots, \{x_n, y_n\}\big\}$. Clearly $IH(G)$ is Hausdorff . Hence the theorem. □

Proposition 4.1. Given any two vertices x_1 and x_2 suppose there exists independent dominating sets A and B containing x_1 and x_2 respectively such that $V(A) \cap V(B) = \emptyset$ then the corresponding independent hypergraph is Hausdorff .

5. Application

In the field of medical research, it usually happens that different physicians give various combinations of drugs to patients with the same disease. To evaluate the potency and variety of such prescriptions, the situation can be modeled with hypergraph theory. Each patient is modeled as a vertex, and each combination of medicines prescribed is a hyperedge linking all patients using that combination. Two vertices are considered adjacent if the respective patients have at least one common medicine. Therefore, the hypergraph representation allows us to visualize and examine the crossover of medicines among patients. We say a medical sample is effective if, for any two patients, there exist disjoint hyperedges containing them separately. In this case, the hypergraph corresponding to the sample has the Hausdorff property, which implies that entirely separate sets of medicines can indicate any two patients. This modeling technique gives a clear mathematical framework to study variation in treatment, detect independent therapeutic paths, and study the separability of prescription strategies among patients coming from different regions or medical institutions.

6. Conclusion

In this paper we have discussed hausdorff property of hypergraphs as well as minimal hausdorff hypergraph. We have also examined conditions under which competetion hypergraph and independence hypergraph become hausdorff. Also we have discussed some applications of Hausdorff hypergraph.

References

[1] M. Wahlström, Exact algorithms for finding minimum transversals in rank-3 hypergraphs, Journal of Algorithms 51 (2) (2004) 107–121.

[2] C. Berge, E. Minieka, Graphs and hypergraphs, Vol. 7, North-Holland publishing company Amsterdam, 1973.

[3] C. Berge, Hypergraphs: combinatorics of finite sets, Vol. 45, Elsevier, 1984.

[4] C. G. Heise, K. Panagiotou, O. Pikhurko, A. Taraz, Coloring d-embeddable k-uniform hypergraphs, Discrete & computational geometry 52 (4) (2014) 663–679.

[5] R. Tyshkevich, V. E. Zverovich, Line hypergraphs, Discrete Mathematics 161 (1) (1996) 265–283.

[6] M. Sonntag, H.-M. Teichert, Competition hypergraphs, Discrete applied mathematics 143 (1) (2004) 324–329.

[7] A. Bretto, Hypergraph theory, Springer, 2013.

[8] C. Berge, Hypergraphs: Combinatorics of Finite Sets, North-Holland, 1989.