On Certain Relations of Powers
$(x^2 + y^2 + z^2 + d^2)^r = k(ax^2 + bx + c)^s(u^2 + v^2 + w^2)$

Abstract: Let $x, y, z, d, a, b, c, u, v, w$ and I be integers and suppose that k and r are non negative exponent. In this current study, we examine a diophantine equation relating a power relationship involving sum of four squares raised to power r and product of power of quadratic term and sums of three squares. In particular, this study, develops and introduces the diophantine equation $I = (x^2+y^2+z^2+d^2)^r = k(ax^2+bx+c)^s(u^2+v^2+w^2)$, particularly when $(r, s, k) = (3, 1, 1)$. This investigations involves determinations of the unknowns a, b, c, u, v and w for which the title equation has solution. The methodology involves transforming the given equation into under determine system of equations and solving via analytical method. Moreover, the study provides conjecture for the title equation.
Keywords: Diophantine Equations, Sum of Powers, polynomial equations, Title Equation

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1 Introduction

The contribution of Abel (1826) and Galois (1832) have shown that the general polynomial equations of degree higher than the fourth cannot be solved via radicals [1]. While Abel presented the proof of impossibility of solving these diophantine equations (Abel's impossibility theorem), Galois gave a more comprehensive proof using group theory. This does not mean that there is no algebraic solution to the general polynomial equations of degree five and above [2]. In fact, these equations are solved algebraically by employing symbolic coefficients: the general quintic is solved using the Bring radicals, while the general sextic can be solved in terms of Kampé de Fériet functions [3]. The study of polynomial identities seeking integer solutions, is a very significant area in the field of number theory. Historically, these equations have attracted the attention of number theorist due to their intrinsic challenge and significance in understanding the properties of integers. Despite the extensive studies of various Diophantine equations, including renowned challenges like Fermat's Last Theorem, Ramanujan Nagell equation, and Lebesgue

Nagell equation, as well as studies focusing on polynomials of degree less than 5, the specific examination of the diophantine equation $(x^2 + y^2 + z^2 + d^2)^r = k(ax^2 + bx + c)^s(u^2 + v^2 + w^2)$ remains largely unknown. Recent research has mainly focussed into the intricacies of diophantine equation with degrees less than 5, as referenced in [5,7,8,9,10]. For a comprehensive research on studies related to some of powers, readers are encouraged to explore[11,12,13,14,15,16]. Furthermore, the study by Bennett and Skinner [4] provides a rigorous analysis of Diophantine equations involving powers, offering insights into the distribution of solutions and the role of heights. Their methods, grounded in arithmetic geometry and modular forms, showcase the diversity of approaches to understanding such equations.
While the literature on Diophantine equations is extensive, the specific equation

$$(x^2 + y^2 + z^2 + d^2)^r = k(ax^2 + bx + c)^s(u^2 + v^2 + w^2),$$

has not been done.
In this paper, we provide a method to split a given sextic equation (sixth-degree polynomial equation) into a product of quadratic term and sums of three squares as factors. The quadratic term and the sums of three square polynomials are then expanded and equated to the given sextic equation to derive a system of non linear equations . The system is then solved via analytical method. In particular, this study will partially provide a solution to the title equation particularly when $(r, s, k) = (3, 1, 1)$.

2 Decomposition of Sextic Equation

Let the sextic equation whose solution is sought be:

$$x^6 + b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0 = 0, \tag{1}$$

where $b_0, b_1, b_2, b_3, b_4, b_5$ are real coefficients.
Consider another sextic equation of the form:

$$(x^2 + y^2 + z^2 + d^2)^3 = (ax^2 + bx + c)(u^2 + v^2 + w^2) = 0. \tag{2}$$

where u, v, w are unknown quadratic polynomials.
Notice that the sextic equation of equation (1) and (2) can be equated to derive a non linear systems of equations and is then solved analytically via method of inspection and mathematical intuitions. see theorem 3.1 and 3.2 below.
The study employs a methodology grounded in integer decomposition and factorization, taking a case-by-case approach alongside generalizations. Furthermore, the study provides conjecture for the title equation.

In the sequel we present the main results in the form of conjecture and theorems

3 Main Results

Conjecture 3.1. *For some integers $r \geq 2$ and $s \geq 1$, the diophantine equation*

$$(x^2 + y^2 + z^2 + d^2)^r = k(ax^2 + bx + c)^s(u^2 + v^2 + w^2)$$

has solution in integers.

Theorem 3.1. *Consider equation* (1) *satisfying the condition* $(r,s,k)=(3,1,1)$*. Then the diophantine equation*

$$(x^2+y^2+z^2+d^2)^3=(ax^2+bx+c)(u^2+v^2+w^2)$$

has solution in integers if $z-y=y-x=d=1$

Proof. Assume $z-y=y-x=1$ and consider the equation

$$(x^2+y^2+z^2+d^2)^3=(ax^2+bx+c)(u^2+v^2+w^2)\cdots(*).$$

The L.H.S of equation $(*)$,expressed as

$$(x^2+y^2+z^2+1)^3=(x^2+(x+1)^2+(x+2)^2+1)^3$$

simplifies as

$$27x^6+162x^5+486x^4+864x^3+972x^2+648x+216.$$

Without loss of generality, assume $u=\alpha_1x^2+\alpha_2x+\alpha_3, v=\alpha_4x^2+\alpha_5x+\alpha_6, w=\alpha_7x^2+\alpha_8x+\alpha_9$
Expanding the R.H.S of equation $(*)$ we' ve

$$(ax^2+bx+c)(u^2+v^2+w^2)=(ax^2+bx+c)((\alpha_1x^2+\alpha_2x+\alpha_3)^2+(\alpha_4x^2+\alpha_5x+\alpha_6)^2+(\alpha_7x^2+\alpha_8x+\alpha_9)^2)$$

Given the equation:

$$(ax^2+bx+c)\left[(\alpha_1x^2+\alpha_2x+\alpha_3)^2+(\alpha_4x^2+\alpha_4x+\alpha_6)^2+(\alpha_7x^2+\alpha_8x+\alpha_9)^2\right]$$

$$=27x^6+162x^5+486x^4+864x^3+972x^2+648x+216$$

Expanding the Squared Terms. Let

$$S(x)=(\alpha_1x^2+\alpha_2x+\alpha_3)^2+(\alpha_4x^2+\alpha_5x+\alpha_6)^2+(\alpha_7x^2+\alpha_8x+\alpha_9)^2$$

Expanding each square:

$$\begin{aligned}(\alpha_1x^2+\alpha_2x+\alpha_3)^2&=\alpha_1^2x^4+2\alpha_1\alpha_2x^3+(2\alpha_1\alpha_3+\alpha_2^2)x^2+2\alpha_2\alpha_3x+\alpha_3^2\\(\alpha_4x^2+\alpha_5x+\alpha_6)^2&=\alpha_4^2x^4+2\alpha_4\alpha_5x^3+(2\alpha_3\alpha_5+\alpha_4^2)x^2+2\alpha_4\alpha_5x+\alpha_5^2\\(\alpha_7x^2+\alpha_8x+\alpha_9)^2&=\alpha_7^2x^4+2\alpha_7\alpha_8x^3+(2\alpha_7\alpha_9+\alpha_8^2)x^2+2\alpha_8\alpha_9x+\alpha_9^2\end{aligned}$$

Summing them:

$$S(x)=(\alpha_1^2+\alpha_4^2+\alpha_7^2)x^4+2(\alpha_1\alpha_2+\alpha_4\alpha_5+\alpha_7\alpha_8)x^3+(\alpha_2^2+\alpha_5^2+\alpha_8^2+2\alpha_1\alpha_3+2\alpha_4\alpha_6+2\alpha_7\alpha_9)x^2$$
$$+2(\alpha_2\alpha_3+\alpha_5\alpha_6+\alpha_8\alpha_9)x+(\alpha_3^2+\alpha_6^2+\alpha_9^2)$$

Now multiplying $S(x)$ by ax^2+bx+c:
Thus, the expanded form of the right-hand side is:

$$a(\alpha_1^2+\alpha_4^2+\alpha_7^2)x^6+\left(2a(\alpha_1\alpha_2+\alpha_4\alpha_5+\alpha_7\alpha_8)+b(\alpha_1^2+\alpha_4^2+\alpha_7^2)\right)x^5$$
$$+\left(a(\alpha_2^2+\alpha_5^2+\alpha_8^2+2\alpha_1\alpha_3+2\alpha_4\alpha_6+2\alpha_7\alpha_9)+2b(\alpha_1\alpha_2+\alpha_4\alpha_5+\alpha_7\alpha_8)+c(\alpha_1^2+\alpha_4^2+\alpha_7^2)\right)x^4$$
$$+\left(2a(\alpha_2\alpha_3+\alpha_5\alpha_6+\alpha_8\alpha_9)+2b(\alpha_1\alpha_3+\alpha_4\alpha_6+\alpha_7\alpha_9)+b(\alpha_2^2+\alpha_5^2+\alpha_8^2)+2c(\alpha_1\alpha_2+\alpha_4\alpha_5+\alpha_7\alpha_8)\right)x^3$$
$$+\left(a(\alpha_3^2+\alpha_6^2+\alpha_9^2)+2b(\alpha_2\alpha_3+\alpha_5\alpha_6+\alpha_8\alpha_9)+2c(\alpha_1\alpha_4+\alpha_4\alpha_6+\alpha_7\alpha_9)+c(\alpha_2^2+\alpha_5^2+\alpha_8^2)\right)x^2$$
$$+\left(b(\alpha_3^2+\alpha_6^2+\alpha_9^2)+2c(\alpha_2\alpha_3+\alpha_5\alpha_6+\alpha_8\alpha_9)\right)x+c(\alpha_3^2+\alpha_6^2+\alpha_9^2)$$

The system of equations obtained by equating the coefficients of the left-hand side and right-hand side is:

$$\begin{cases} a(\alpha_1^2+\alpha_4^2+\alpha_7^2)=27, \\ 2a(\alpha_1\alpha_2+\alpha_4\alpha_5+\alpha_7\alpha_8)+b(\alpha_1^2+\alpha_4^2+\alpha_7^2)=162, \\ a(\alpha_2^2+\alpha_5^2+\alpha_8^2+2\alpha_1\alpha_3+2\alpha_4\alpha_6+2\alpha_7\alpha_9)+2b(\alpha_1\alpha_2+\alpha_4\alpha_5+\alpha_7\alpha_8)+c(\alpha_1^2+\alpha_4^2+\alpha_7^2)=486, \\ 2a(\alpha_2\alpha_3+\alpha_5\alpha_6+\alpha_8\alpha_9)+2b(\alpha_1\alpha_3+\alpha_4\alpha_6+\alpha_7\alpha_9)+b(\alpha_2^2+\alpha_5^2+\alpha_8^2)+2c(\alpha_1\alpha_2+\alpha_4\alpha_5+\alpha_7\alpha_8)=864, \\ a(\alpha_3^2+\alpha_6^2+\alpha_9^2)+2b(\alpha_2\alpha_3+\alpha_5\alpha_6+\alpha_8\alpha_9)+2c(\alpha_1\alpha_3+\alpha_4\alpha_6+\alpha_7\alpha_9)+c(\alpha_2^2+\alpha_5^2+\alpha_8^2)=972, \\ b(\alpha_3^2+\alpha_6^2+\alpha_9^2)+2c(\alpha_2\alpha_3+\alpha_5\alpha_6+\alpha_8\alpha_9)=648, \\ c(\alpha_3^2+\alpha_6^2+\alpha_9^2)=216. \end{cases}$$

Clearly, the system has 12 variables and 7 equation making it under determined system. Thus, we result to method of inspection and mathematical intuition to seek integer solution. We begin by equation (i), we determine a, α_1, α_4, and α_7 for which $a(\alpha_1^2+\alpha_4^2+\alpha_7^2)=27$. Assume $a=1, \Rightarrow (\alpha_1^2+\alpha_4^2+\alpha_7^2)=27$ and letting $\alpha_1=\alpha_4=\alpha_7=3$ we have $a(\alpha_1^2+\alpha_4^2+\alpha_7^2)=27$. Thus $(a,\alpha_1,\alpha_4,\alpha_7)=(1,3,3,3)$. Substituting for the values in equation (ii) reduces to $2(\alpha_2+\alpha_5+\alpha_8)+9b=54$. Need to determine the solution set $(\alpha_2,\alpha_5,\alpha_8)$ for which $2(\alpha_2+\alpha_5+\alpha_8)+9b=54$. Assume $\alpha_2=0, \alpha_5=6$ and $\alpha_8=12$ thus $9b=18, \Rightarrow b=2$. Now, substituting the solution set $(a,\alpha_1,\alpha_4,\alpha_7,\alpha_2,\alpha_5,\alpha_8)=(1,3,3,3,0,6,12)$. in equation (iii) we obtain $2(\alpha_3+\alpha_6+\alpha_9)+36b+9c=102$. Determining solution set $(\alpha_3,\alpha_6,\alpha_9,b,c)$ for which $2(\alpha_3+\alpha_6+\alpha_9)+36b+9c=102$. Let $\alpha_3=-6, \alpha_6=\alpha_9=6, \Rightarrow 9c=18, \Rightarrow c=2$. Since all the solution have been obtained the validity of the solution obtained can easily be verified in equation $(iv),(v),(vi)$ and (vii). Consequently, $ax^2+bx+c=x^2+2x+2$, $u=\alpha_1x^2+\alpha_2x+\alpha_3=3x^2+0x-6$, $v=\alpha_4x^2+\alpha_5x+\alpha_6=3x^2+6x+6$ and $w=\alpha_7x^2+\alpha_8x+\alpha_9=3x^2+12x+6$ and as such the result easily follows by proving the L.H.S is equal to the R.H.S. Consequently,

$$(x^2+y^2+z^2+d^2)^3=(x^2+2x+2)((3x^2+0x-6)^2+(3x^2+6x+6)^2+(3x^2+12x+6)^2)$$

concluding the proof. □

Theorem 3.2. *Consider equation* (1) *satisfying the condition* $(r,s,k)=(3,1,1)$. *Then the diophantine equation*

$$(x^2+y^2+z^2+d^2)^3=(ax^2+bx+c)(u^2+v^2+w^2)$$

has solution in integers if $z-y=y-x=d=2$

Proof. Assume $z-y=y-x=2$ and consider the equation

$$(x^2+y^2+z^2+d^2)^3=(ax^2+bx+c)(u^2+v^2+w^2)\cdots(*).$$

The L.H.S of equation $(*)$,expressed as

$$(x^2+y^2+z^2+4)^3=(x^2+(x+2)^2+(x+4)^2+4)^3$$

simplifies as

$$27x^6+324x^5+1944x^4+6912x^3+15552x^2+20736x+13824.$$

Without loss of generality, assume $u=\alpha_1x^2+\alpha_2x+\alpha_3, v=\alpha_4x^2+\alpha_5x+\alpha_6, w=\alpha_7x^2+\alpha_8x+\alpha_9$ Expanding the R.H.S of equation $(*)$ we' ve

$$(ax^2+bx+c)(u^2+v^2+w^2)=(ax^2+bx+c)((\alpha_1x^2+\alpha_2x+\alpha_3)^2+(\alpha_4x^2+\alpha_5x+\alpha_6)^2+(\alpha_7x^2+\alpha_8x+\alpha_9)^2)$$

Given the equation:

$$(ax^2+bx+c)\left[(\alpha_1x^2+\alpha_2x+\alpha_3)^2+(\alpha_4x^2+\alpha_4x+\alpha_6)^2+(\alpha_7x^2+\alpha_8x+\alpha_9)^2\right]$$

$$=27x^6+162x^5+486x^4+864x^3+972x^2+648x+216$$

Expanding the Squared Terms. Let

$$S(x)=(\alpha_1x^2+\alpha_2x+\alpha_3)^2+(\alpha_4x^2+\alpha_5x+\alpha_6)^2+(\alpha_7x^2+\alpha_8x+\alpha_9)^2$$

Expanding each square:

$$\begin{aligned}(\alpha_1x^2+\alpha_2x+\alpha_3)^2&=\alpha_1^2x^4+2\alpha_1\alpha_2x^3+(2\alpha_1\alpha_3+\alpha_2^2)x^2+2\alpha_2\alpha_3x+\alpha_3^2\\(\alpha_4x^2+\alpha_5x+\alpha_6)^2&=\alpha_4^2x^4+2\alpha_4\alpha_5x^3+(2\alpha_3\alpha_5+\alpha_4^2)x^2+2\alpha_4\alpha_5x+\alpha_5^2\\(\alpha_7x^2+\alpha_8x+\alpha_9)^2&=\alpha_7^2x^4+2\alpha_7\alpha_8x^3+(2\alpha_7\alpha_9+\alpha_8^2)x^2+2\alpha_8\alpha_9x+\alpha_9^2\end{aligned}$$

Summing them:

$$S(x)=(\alpha_1^2+\alpha_4^2+\alpha_7^2)x^4+2(\alpha_1\alpha_2+\alpha_4\alpha_5+\alpha_7\alpha_8)x^3+(\alpha_2^2+\alpha_5^2+\alpha_8^2+2\alpha_1\alpha_3+2\alpha_4\alpha_6+2\alpha_7\alpha_9)x^2$$
$$+2(\alpha_2\alpha_3+\alpha_5\alpha_6+\alpha_8\alpha_9)x+(\alpha_3^2+\alpha_6^2+\alpha_9^2)$$

Now multiplying $S(x)$ by ax^2+bx+c:
Thus, the expanded form of the right-hand side is:

$$\begin{aligned}&a(\alpha_1^2+\alpha_4^2+\alpha_7^2)x^6+\big(2a(\alpha_1\alpha_2+\alpha_4\alpha_5+\alpha_7\alpha_8)+b(\alpha_1^2+\alpha_4^2+\alpha_7^2)\big)x^5\\&+\big(a(\alpha_2^2+\alpha_5^2+\alpha_8^2+2\alpha_1\alpha_3+2\alpha_4\alpha_6+2\alpha_7\alpha_9)+2b(\alpha_1\alpha_2+\alpha_4\alpha_5+\alpha_7\alpha_8)+c(\alpha_1^2+\alpha_4^2+\alpha_7^2)\big)x^4\\&+\big(2a(\alpha_2\alpha_3+\alpha_5\alpha_6+\alpha_8\alpha_9)+2b(\alpha_1\alpha_3+\alpha_4\alpha_6+\alpha_7\alpha_9)+b(\alpha_2^2+\alpha_5^2+\alpha_8^2)+2c(\alpha_1\alpha_2+\alpha_4\alpha_5+\alpha_7\alpha_8)\big)x^3\\&+\big(a(\alpha_3^2+\alpha_6^2+\alpha_9^2)+2b(\alpha_2\alpha_3+\alpha_5\alpha_6+\alpha_8\alpha_9)+2c(\alpha_1\alpha_4+\alpha_4\alpha_6+\alpha_7\alpha_9)+c(\alpha_2^2+\alpha_5^2+\alpha_8^2)\big)x^2\\&+\big(b(\alpha_3^2+\alpha_6^2+\alpha_9^2)+2c(\alpha_2\alpha_3+\alpha_5\alpha_6+\alpha_8\alpha_9)\big)x+c(\alpha_3^2+\alpha_6^2+\alpha_9^2)\end{aligned}$$

The system of equations obtained by equating the coefficients of the left-hand side and right-hand side is:

$$\begin{cases}a(\alpha_1^2+\alpha_4^2+\alpha_7^2)=27,\\2a(\alpha_1\alpha_2+\alpha_4\alpha_5+\alpha_7\alpha_8)+b(\alpha_1^2+\alpha_4^2+\alpha_7^2)=324,\\a(\alpha_2^2+\alpha_5^2+\alpha_8^2+2\alpha_1\alpha_3+2\alpha_4\alpha_6+2\alpha_7\alpha_9)+2b(\alpha_1\alpha_2+\alpha_4\alpha_5+\alpha_7\alpha_8)+c(\alpha_1^2+\alpha_4^2+\alpha_7^2)=1944,\\2a(\alpha_2\alpha_3+\alpha_5\alpha_6+\alpha_8\alpha_9)+2b(\alpha_1\alpha_3+\alpha_4\alpha_6+\alpha_7\alpha_9)+b(\alpha_2^2+\alpha_5^2+\alpha_8^2)+2c(\alpha_1\alpha_2+\alpha_4\alpha_5+\alpha_7\alpha_8)=6912,\\a(\alpha_3^2+\alpha_6^2+\alpha_9^2)+2b(\alpha_2\alpha_3+\alpha_5\alpha_6+\alpha_8\alpha_9)+2c(\alpha_1\alpha_3+\alpha_4\alpha_6+\alpha_7\alpha_9)+c(\alpha_2^2+\alpha_5^2+\alpha_8^2)=15552,\\b(\alpha_3^2+\alpha_6^2+\alpha_9^2)+2c(\alpha_2\alpha_3+\alpha_5\alpha_6+\alpha_8\alpha_9)=20736,\\c(\alpha_3^2+\alpha_6^2+\alpha_9^2)=13824.\end{cases}$$

Clearly, the system has 12 variables and 7 equation making it under determined system. Thus we result to method of inspection and mathematical intuition to seek integer solution. We begin by equation (i), we determine a,α_1,α_4, and α_7 for which $a(\alpha_1^2+\alpha_4^2+\alpha_7^2)=27$. Assume $a=1,\Rightarrow(\alpha_1^2+\alpha_4^2+\alpha_7^2)=27$ and letting $\alpha_1=\alpha_4=\alpha_7=3$ we have $a(\alpha_1^2+\alpha_4^2+\alpha_7^2)=27$. Thus $(a,\alpha_1,\alpha_4,\alpha_7)=(1,3,3,3)$. Substituting for the values in equation (ii) reduces to $6(\alpha_2+\alpha_5+\alpha_8)+27b=324$. Need to determine the solution set

$(\alpha_2, \alpha_5, \alpha_8)$ for which $6(\alpha_2 + \alpha_5 + \alpha_8) + 27b = 324$. Assume $\alpha_2 = 0, \alpha_5 = 12$ and $\alpha_8 = 24$ thus $27b = 108, \Rightarrow b = 4$. Now, substituting the solution set $(a, b, \alpha_1, \alpha_4, \alpha_7, \alpha_2, \alpha_5, \alpha_8) = (1, 4, 3, 3, 3, 0, 12, 24)$. in equation (iii) we obtain $6(\alpha_3 + \alpha_6 + \alpha_9) + 27c = 360$. Determining solution set $(\alpha_3, \alpha_6, \alpha_9, b, c)$ for which $6(\alpha_3 + \alpha_6 + \alpha_9) + +27c = 360$. Let $\alpha_3 = -24, \alpha_6 = \alpha_9 = 24, \Rightarrow 27c = 216, \Rightarrow c = 8$. Since all the solution have been obtained the validity of the solution obtained can easily be verified in equation $(iv), (v), (vi)$ and (vii). Consequently, $ax^2 + bx + c = x^2 + 4x + 8$, $u = \alpha_1 x^2 + \alpha_2 x + \alpha_3 = 3x^2 + 0x - 24$, $v = \alpha_4 x^2 + \alpha_5 x + \alpha_6 = 3x^2 + 12x + 24$ and $w = \alpha_7 x^2 + \alpha_8 x + \alpha_9 = 3x^2 + 24x + 24$ and as such the result easily follows by proving the L.H.S is equal to the R.H.S. Consequently,

$$(x^2 + y^2 + z^2 + d^2)^3 = (x^2 + 4x + 8)((3x^2 + 0x - 24)^2 + (3x^2 + 12x + 24)^2 + (3x^2 + 24x + 24)^2)$$

concluding the proof. □

3.1 Examples

To argument the above theorem , this research will provide few examples to validate the results:

Example 1. *Case* (i) *when* $d = 1$
Let $I = (x^2 + y^2 + z^2 + d^2)^3 = (ax^2 + bx + c)(u^2 + v^2 + w^2) \cdots (*)$. *Assume,* $x = 1, y = 2, z = 3, d = 1, ax^2 + bx + b = x^2 + 2x + 2, u = (3x^2 + 0x - 6), v = (3x^2 + 6x + 6)$ *and* $(3x^2 + 12x + 6)$. *Replacing this values in equation* $(*)$ *we have the L.H.S as* $I = (1^2 + 2^2 + 3^2 + 1^2)^3 = 3375$. *Now, expanding R.H.S we have,* $I = (1^2 + 2*1 + 2)((3*1^2 + 0*1 - 6)^2 + (3*1^2 + 6*1 + 6)^2 + (3*1^2 + 12*1 + 6)^2) = 3375$. *Since, the L.H.S is equal to the R.H.S the result easily follows.*

Example 2. *Case* (i) *when* $d = 1$
Let $I = (x^2 + y^2 + z^2 + d^2)^3 = (ax^2 + bx + c)(u^2 + v^2 + w^2) \cdots (*)$. *Assume,* $x = 3, y = 4, z = 5, d = 1, ax^2 + bx + b = x^2 + 2x + 2, u = (3x^2 + 0x - 6), v = (3x^2 + 6x + 6)$ *and* $(3x^2 + 12x + 6)$. *Replacing this values in equation* $(*)$ *we have the L.H.S as* $I = (3^2 + 4^2 + 5^2 + 1^2)^3 = 132651$. *Now, expanding R.H.S we have,* $I = (3^2 + 2*3 + 2)((3*3^2 + 0*3 - 6)^2 + (3*3^2 + 6*3 + 6)^2 + (3*3^2 + 12*3 + 6)^2) = 132651$. *Since, the L.H.S is equal to the R.H.S the result easily follows.*

4 Conclusion

In conclusion the exploration of the Diophantine equation,$(x^2+y^2+z^2+d^2)^r = k(ax^2+bx+c)^s(u^2+v^2+w^2)$ with $(r, s, k) = (3, 1, 1)$, this study has made a major progress in unveiling integer solutions through the application of integer decomposition, factorization techniques, method of inspection and mathematical intuition. This research sheds light on the intricate relationships between power of sums of four squares , quadratic term and sums of three squares. While our findings are partial, they serve as a solid starting point for future research. To further advance our understanding, we recommend extending the analysis to different exponents, conducting in-depth case studies with specific variable constraints.

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