A Study on Dual Generalized Pierre Numbers

Abstract. In this study, we introduce a new family of number sequences, termed generalized dual Pierre numbers, defined over the bidimensional Clifford algebra of hyperbolic numbers. This algebraic framework enables the extension of classical integer sequences into a broader hypercomplex domain. As special cases, we examine the dual Pierre numbers and the dual Pierre Lucas numbers, thereby connecting our results with well-known recursive sequences.

For these sequences, we derive closed-form Binet-type formulas, construct their generating functions, and establish several summation identities. Furthermore, we develop matrix representations associated with the proposed sequences, which not only provide elegant proofs of recurrence relations but also suggest potential applications in linear algebra and operator theory.

Overall, this work enriches the theory of integer sequences by embedding them into the Clifford algebra framework, unveiling new structural connections, analytical techniques, and possible applications in both pure and applied mathematics.

2010 Mathematics Subject Classification. 11B37, 11B39, 11B83.

Keywords. Pierre numbers, Pierre-Lucas numbers, dual numbers, dual Pierre numbers.

1. Introduction

Dual numbers, first introduced by W.K. Clifford in 1873, represent a fascinating mathematical construct with a wide range of applications. They play a pivotal role in screw theory, the modeling of planar joints, and iterative techniques for displacement analysis in spatial mechanisms. Additionally, dual numbers are instrumental in the inertial force analysis of spatial systems and continue to find relevance in various branches of kinematics and robotics.Here are some general information about the applications of dual numbers.

- Engineering and Physics:

Used in electrical engineering and control systems.

Applied in wave analysis and signal processing.

Utilized in mechanical engineering for vibration analysis, among other applications.

- Mathematics and Geometry:

  Alongside complex numbers, dual numbers contribute to the extension of mathematical structures.

  Employed in geometry to represent various transformations.

- Computer Science:

  Found in graphics and image processing.

  Used in robotics and control systems for modeling and analysis.

- Finance and Economics:

  Applied in risk analysis and financial engineering.

  Utilized in option pricing and portfolio management.

- Optimization Problems:

  Used for finding solutions in optimization problems.

  Acts as a tool in linear programming and decision-making models.

- Quantum Mechanics:

  Employed in quantum computers and quantum mechanics for mathematical representation.

Next, we give some information raleted to hypercomplex number system and then we give some properities about dual number. As discussed in [10], the hypercomplex numbers systems are extensions of real numbers. Some examples of hypercomplex number systems ,which is commutative , are complex numbers, hyperbolic numbers and dual numbers.

- Complex numbers are formed by extending the real number system with the imaginary unit, denoted as "i", which satisfies the equation $i^2 = -1$. Complex numbers is defined as follows,

$$C = \{z = a + ib : a, b \in R, i^2 = -1\}.$$

- As discussed in [18], hyperbolic numbers extend the real number system with the hyperbolic unit j, where $j^2 = 1$. Hyperbolic numbers is defined as follows,

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\}.$$

- As discussed in[8], dual numbers extend the real number system by introducing a new element ε, where $\varepsilon^2 = 0$. Dual numbers is defined as follows,

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

Let $D = \{d = a + \varepsilon b : a, b \in R, \varepsilon^2 = 0, \varepsilon \neq 0\} \subseteq R \times R$ is a set called dual numbers and we define following process on D for every $d_1 = x + x^*\varepsilon$, $d_2 = y + y^*\varepsilon \in D$ as

$$\begin{aligned} + &: \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{D},\ d_1 + d_2 = (x + x^*\varepsilon) + (y + y^*\varepsilon) = (x + y) + (x^* + y^*)\varepsilon, \\ \cdot &: \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{D},\ d_1 \cdot d_2 = (x + x^*\varepsilon) \cdot (y + y^*\varepsilon) = xy + (xy^* + x^*y)\varepsilon, \\ d_1 &= (x + x^*\varepsilon) = (y + y^*\varepsilon) = d_2 \text{ if only if } x = x^*,\ y = y^*. \end{aligned}$$

Using above expressions we have following definations,

-
  - $(D, +)$ is an abelian grup,
  - $(D, +, \cdot)$ is commitative ring (where for every $d \in D$ we have $d \cdot 1 = d$ so that 1 is unit eleman on $\cdot$ process),
  - $(D, +, \cdot)$ is not field because for every $d \in D$ such that there is no element $d \cdot d' = d' \cdot d = 1$,
  - the D is a vector space on R,
  - $\widetilde{\mathbb{D}} = \{a + 0\varepsilon : a \in R\}$,which is subspace of D, is isomorph R,
  - $(1, \varepsilon)$ is basis of D,
  - for every $d = (x+x^*\varepsilon) \in D$ such that $\overline{d} = (x-x^*\varepsilon) \in D$, $\frac{1}{d} = (\frac{1}{x} + \frac{x^*}{x}\varepsilon) \in D$, $d \cdot \overline{d} = x^2, \overline{(\overline{d})} = d$
  - for every $d_1 = x + x^*\varepsilon$, $d_2 = y + y^*\varepsilon \in D, (y \neq 0)$, $\frac{d_1}{d_2} = (\frac{x}{y} + \frac{x^* - xy^*}{y^2}\varepsilon) \in D$, $\overline{(\frac{d_1}{d_2})} = (\frac{\overline{d_1}}{\overline{d_2}})$, $\overline{(d_1 + d_2)} = (\overline{d_1} + \overline{d_2})$ and $\overline{(d_1 \cdot d_2)} = (\overline{d_1} \cdot \overline{d_2})$. For more detail see [14]
- Dual hyperbolic number is a type of hypercomplex number, specifically a member of the hyperbolic number system. A dual hyperbolic number is defined by

  $$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon j a_3$$

  where $a_0, a_1, a_2, a_3 \in R$ are real numbers.

The set of all dual hyperbolic numbers are defined as

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon j a_3 : a_0, a_1, a_2, a_3 \in \mathbb{R},\ j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

where ε denotes the pure dual unit $(\varepsilon^2 = 0, \varepsilon \neq 0)$, j denotes the hyperbolic unit $(j^2 = 1)$, and εj denotes the dual hyperbolic unit $((j\varepsilon)^2 = 0)$.

The $\{1, j, \varepsilon, \varepsilon j\}$ is linear independent and $H_{\mathbb{D}} = sp\{1, j, \varepsilon, \varepsilon j\}$ so that $\{1, j, \varepsilon, \varepsilon j\}$ is a basis of $H_{\mathbb{D}}$.For more detail see [1]. The next properties are holds for the base elements $\{1, j, \varepsilon, \varepsilon j\}$ of dual hyperbolic numbers (commutative multiplications):$1.\varepsilon = \varepsilon, 1.j = j$, $\varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0$, $j^2 = j.j = 1, \varepsilon.j = j.\varepsilon$, $\varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0$, $j(\varepsilon j) = (\varepsilon j)j = \varepsilon$.

Let us now revisit the definition of generalized Pierre numbers.

A generalized Pierre sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$W_n = 2W_{n-1} - W_{n-4}, \tag{1.1}$$

with the initial values W_0, W_1, W_2, W_3 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-3)} - W_{-(n-4)},$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

The initial values of the generalized Pierre numbers for both positive and negative subscripts are presented in Table 1.

Table 1. A few generalized Pierre numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$2W_2 - W_3$
2	W_2	$2W_1 - W_2$
3	W_3	$2W_0 - W_1$
4	$2W_3 - W_0$	$4W_2 - W_0 - 2W_3$
5	$4W_3 - W_1 - 2W_0$	$4W_1 - 4W_2 + W_3$
6	$8W_3 - 2W_1 - W_2 - 4W_0$	$4W_0 - 4W_1 + W_2$
7	$15W_3 - 4W_1 - 2W_2 - 8W_0$	$W_1 - 4W_0 + 8W_2 - 4W_3$
8	$28W_3 - 8W_1 - 4W_2 - 15W_0$	$W_0 + 8W_1 - 12W_2 + 4W_3$
9	$52W_3 - 15W_1 - 8W_2 - 28W_0$	$8W_0 - 12W_1 + 6W_2 - W_3$
10	$96W_3 - 28W_1 - 15W_2 - 52W_0$	$6W_1 - 12W_0 + 15W_2 - 8W_3$
11	$177W_3 - 52W_1 - 28W_2 - 96W_0$	$6W_0 + 15W_1 - 32W_2 + 12W_3$
12	$326W_3 - 96W_1 - 52W_2 - 177W_0$	$15W_0 - 32W_1 + 24W_2 - 6W_3$
13	$600W_3 - 177W_1 - 96W_2 - 326W_0$	$24W_1 - 32W_0 + 24W_2 - 15W_3$

If we set $W_0 = 0, W_1 = 1, W_2 = 2, W_3 = 4$ then $\{W_n\}$ is the well-known Pierre sequence and if we set $W_0 = 4, W_1 = 2, W_2 = 4, W_3 = 8$ then $\{W_n\}$ is the well-known Pierre -Lucas sequence. In other words, Pierre sequence $\{P_n\}_{n\geq 0}$ and Pierre -Lucas sequence $\{C_n\}_{n\geq 0}$ are defined by the second-order recurrence relations

$$P_n = 2P_{n-1} - P_{n-4}, \qquad P_0 = 0, P_1 = 1, P_2 = 2,\ P_3 = 4, \quad n \geq 4, \tag{1.2}$$

and

$$C_n = 2C_{n-1} - C_{n-4}, \qquad C_0 = 4, C_1 = 2, C_2 = 4, C_3 = 8, \qquad n \geq 4. \tag{1.3}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = 2P_{-(n-3)} - P_{-(n-4)},$$

and

$$C_{-n} = 2C_{-(n-3)} - C_{-(n-4)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.2) and (1.3) hold for all integer n.

We can list some important properties of generalized Pierre numbers that are needed.

- Binet formula of generalized Pierre sequence can be calculated using its characteristic equation which is given as

$$z^4 - 2z^3 + 1 = (z^3 - z^2 - z - 1)(z - 1) = 0.$$

The roots of characteristic equation are

$$\begin{aligned}
\alpha &= \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3}, \\
\beta &= \frac{1 + \omega\sqrt[3]{19 + 3\sqrt{33}} + \omega^2\sqrt[3]{19 - 3\sqrt{33}}}{3}, \\
\gamma &= \frac{1 + \omega^2\sqrt[3]{19 + 3\sqrt{33}} + \omega\sqrt[3]{19 - 3\sqrt{33}}}{3}, \\
\delta &= 1,
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Using these roots and the recurrence relation, Binet formula can be given as

$$\begin{aligned}
W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \qquad (1.4) \\
&= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-1)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-1)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-1)} + \frac{p_4}{(1-\alpha)(1-\beta)(1-\gamma)} \\
&= A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4
\end{aligned}$$

where p_1, p_2, p_3 and p_4 are given below

$$\begin{aligned}
p_1 &= W_3 - (\beta+\gamma+\delta)W_2 + (\beta\gamma+\beta\delta+\gamma\delta)W_1 - \beta\gamma\delta W_0, \qquad (1.5) \\
p_2 &= W_3 - (\alpha+\gamma+\delta)W_2 + (\alpha\gamma+\alpha\delta+\gamma\delta)W_1 - \alpha\gamma\delta W_0, \\
p_3 &= W_3 - (\alpha+\beta+\delta)W_2 + (\alpha\beta+\alpha\delta+\beta\delta)W_1 - \alpha\beta\delta W_0, \\
p_4 &= W_3 - W_2 - W_1 - W_0
\end{aligned}$$

and

$$\begin{aligned}
A_1 &= \frac{W_3 - (\beta+\gamma+\delta)W_2 + (\beta\gamma+\beta\delta+\gamma\delta)W_1 - \beta\gamma\delta W_0}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}, \qquad (1.6) \\
A_2 &= \frac{W_3 - (\alpha+\gamma+\delta)W_2 + (\alpha\gamma+\alpha\delta+\gamma\delta)W_1 - \alpha\gamma\delta W_0}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\
A_3 &= \frac{W_3 - (\alpha+\beta+\delta)W_2 + (\alpha\beta+\alpha\delta+\beta\delta)W_1 - \alpha\beta\delta W_0}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\
A_4 &= \frac{W_3 - W_2 - W_1 - W_0}{-2}.
\end{aligned}$$

Binet formula of Pierre and Pierre Lucas sequences are

$$P_n = \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} - \frac{1}{2},$$

and

$$C_n = \alpha^n + \beta^n + \gamma^n + 1,$$

respectively.

The generating function for generalized Pierre numbers is

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 2W_0)x + (W_2 - 2W_1)x^2 + (W_3 - 2W_2)x^3}{1 - 2x + x^4}. \tag{1.7}$$

In the following section, we introduce the dual generalized Pierre numbers and investigate several of their fundamental properties.

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n.

Lemma 1. *[13, Lemma 1.4]. Suppose that $f_{W_n}(x) = \sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is the exponential generating function of the generalized Pierre sequence $\{W_n\}$. Then*

$$\begin{aligned}
\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \frac{(\alpha W_3 - \alpha(2-\alpha)W_2 + (-\alpha^2+\alpha+1)W_1 - W_0)}{2\alpha^2+2\alpha-2} e^{\alpha x} \\
&+ \frac{(\beta W_3 - \beta(2-\beta)W_2 + (-\beta^2+\beta+1)W_1 - W_0)}{2\beta^2+2\beta-2} e^{\beta x} \\
&+ \frac{(\gamma W_3 - \gamma(2-\gamma)W_2 + (-\gamma^2+\gamma+1)W_1 - W_0)}{2\gamma^2+2\gamma-2} e^{\gamma x} + (\frac{W_3 - W_2 + W_1 - W_0}{-2}) e^{x}.
\end{aligned}$$

The previous Lemma gives the following results as particular examples.

Corollary 2. *[13, Corollary 1.2].Exponential generating function of Pierre and Pierre-Lucas numbers are*

a): $\sum\limits_{n=0}^{\infty} P_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\frac{(\alpha^2+\alpha+1)\alpha^n}{2(\alpha^2+\alpha-1)} + \frac{(\beta^2+\beta+1)\beta^n}{2(\beta^2+\beta-1)} + \frac{(\gamma^2+\gamma+1)\gamma^n}{2(\gamma^2+\gamma-1)} - \frac{1}{2}) \frac{x^n}{n!}$

$= \frac{(\alpha^2+\alpha+1)}{2(\alpha^2+\alpha-1)} e^{\alpha x} + \frac{(\beta^2+\beta+1)}{2(\beta^2+\beta-1)} e^{\beta x} + \frac{(\gamma^2+\gamma+1)\gamma^n}{2(\gamma^2+\gamma-1)} e^{\gamma x} - \frac{1}{2} e^{x}.$

b): $\sum\limits_{n=0}^{\infty} C_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\alpha^n + \beta^n + \gamma^n + 1) \frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^{\gamma x} + e^{x}.$

Next, we provide an overview of selected publications in the literature that pertain to dual numbers.

- Göcen et al [12] presented the dual generalized Fibonacci matrices as

$$DWn = \begin{pmatrix} W_{n+1} + \epsilon W_{n+2} & W_n + \epsilon W_{n+1} \\ W_n + \epsilon W_{n+1} & W_{n-1} + \epsilon W_n \end{pmatrix} = \begin{pmatrix} W_{n+1} + \epsilon(Wn + 1 + Wn) & W_n + \epsilon W_{n+1} \\ W_n + \epsilon W_{n+1} & W_{n+1} - Wn + \epsilon W_n \end{pmatrix}$$

with initial conditions $DW_0 = \begin{pmatrix} W_1 + \epsilon(W_0 + W_1) & W_0 + \epsilon W_1 \\ W_0 + \epsilon W_1 & W_1 - W_0 + \epsilon W_0 \end{pmatrix}$,

$DW_1 = \begin{pmatrix} W_0 + W_1 + \epsilon(W_0 + 2W_1) & W_1 + \epsilon(W_0 + W_1) \\ W_1 + \epsilon(W_0 + W_1) & W_0 + \epsilon W_1 \end{pmatrix}$ as $\epsilon^2 = 0$

.

- Halici [16] studied Dual Fibonacci Octonions as

$$p=\sum_{s=0}^{7} F_{n+s}e_s$$

where Fibonacci given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.

- Aydın [9] studied Dual Jacobsthal Quaternions as

$$QJ_{k;n} = J_{k;n} + i_1 J_{k;n+1} + i_2 J_{k;n+2} + i_3 J_{k;n+3}$$

where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = 0$, $J_1 = 1$.

- Nurkan ,Guven,[7] studied Dual Fibonacci Quaternions as

$$\widetilde{Q}n = (F_n + F_{n+1}) + i(F_{n+1} + F_{n+2}) + j(F_{n+2} + F_{n+3}) + k(F_{n+3} + F_{n+4})$$

where Fibonacci given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.

- Gürses, Şentürk, Yüce[15] studied dual-generalized complex Fibonacci and Lucas numbers, respectively, as

$$\begin{aligned} \widetilde{\mathcal{F}}_n &= F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3}, \\ \widetilde{\mathcal{L}}_n &= L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3}, \end{aligned}$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$.

- Yılmaz and Soykan , [17] studied dual generalized Guglielmo numbers given by

$$\widetilde{W}_n = W_n + \varepsilon W_{n+1}$$

where generalized Guglielmo numbers are $W_n = 3W_{n-1} - 3W_{n-2} + W_{n-3}$ with the initial conditation $W_0, W_1, W_2 \quad (n \geq 2)$.

- Ayrılma and Soykan,[6] introduced On Dual Edouard Numbers are

$$DE_n = 7DE_{n-1} - 7DE_{n-2} + DE_{n-3}$$

where generalized Edouard numbers are $E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}$ with the initial conditation $E_0 = 0, E_1 = 1, E_2 = 7$

Following this, we provide details on dual hyperbolic sequences as they are presented in literature.

- Demirci and Soykan,[2] studied hyperbolic generalized Adrien numbers given by

$$HA_n = 3HA_{n-1} - HA_{n-2} - HA_{n-4}$$

where generalized Adrien numbers are $A_n = 3A_{n-1} - A_{n-2} + A_{n-4}$ with the initial conditation $A_0 = 0, A_1 = 1, A_2 = 3, A_3 = 8, \ \ n \geq 4$.

- Kalca and Soykan,[11] studied dual hyperbolic generalized Pandita numbers given by

$$\hat{P}_n = 2\hat{P}_{n-1} - \hat{P}_{n-2} + \hat{P}_{n-3} - \hat{P}_{n-4}$$

where generalized Pandita numbers are $P_n = 2P_{n-1} - P_{n-2} + P_{n-3} - P_{n-4}$ with the initial conditation $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 3, \ \ n \geq 4$.

In this paper, we define the dual generalized Pierre numbers in the next section and give some properties of them.

- Soykan et al [4] presented dual hyperbolic generalized Pell numbers given by

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}$$

where generalized Pell numbers, with the initial values V_0, V_1 not all being zero, are given by $V_n = 2V_{n-1} + V_{n-2}$, $V_0 = a$, $V_1 = b$ $(n \geq 2)$.

- Dikmen and Altınsoy,[3] studied On Third Order Hyperbolic Jacobsthal Numbers given by

$$\begin{aligned} \widehat{J}_n^{(3)} &= J_n^{(3)} + hJ_{n+1}^{(3)}, \\ \widehat{j}_n^{(3)} &= j_n^{(3)} + hj_{n+1}^{(3)} \end{aligned}$$

where Jacobsthal numbers, respectively, given by $J_n^{(3)} = J_{n-1}^{(3)} + J_{n-2}^{(3)} + 2J_{n-3}^{(3)}$, $J_0^{(3)} = 0$, $J_1^{(3)} = 1$, $J_2^{(3)} = 1$, $j_n^{(3)} = j_{n-1}^{(3)} + j_{n-2}^{(3)} + 2j_{n-3}^{(3)}$, $j_0^{(3)} = 2$, $j_1^{(3)} = 1$, $j_2^{(3)} = 5$.

- Doğan and Soykan,[5] studied hyperbolic generalized Pierre numbers given by

$$HP_n = 2HP_{n-1} - HP_{n-4}$$

where generalized Pierre numbers are $P_n = 2P_{n-1} - P_{n-4}$ with the initial conditation $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 4, \ \ n \geq 4$.

2. Dual Generalized Pierre Numbers and their Generating Functions and Binet's Formulas

In this section, we introduce the dual generalized Pierre numbers and derive their corresponding generating functions and Binet formulas. We now define the dual generalized Pierre numbers over the algebra $\mathbb{H}_{\mathbb{D}}$of dual dual numbers.The n th dual generalized Pierre number is

$$DW_n = W_n + \epsilon W_{n+1}. \tag{2.1}$$

The sequence $\{DW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$DW_{-n} = W_{-n} + \epsilon W_{-n+1},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.2) holds for all integer n.

Note that

$$\begin{aligned} DW_0 &= W_0 + \epsilon W_1, \\ DW_1 &= W_1 + \epsilon W_2, \\ DW_2 &= W_2 + \epsilon W_3, \\ DW_3 &= W_3 + \epsilon W_4 = W_3 + \epsilon(2W_3 - W_0). \end{aligned}$$

It can be easily shown that

$$DW_n = 2DW_{n-1} - DW_{n-4} \tag{2.2}$$

and

$$DW_{-n} = 2DW_{-(n-3)} - DW_{-(n-4)}.$$

The initial values of the dual generalized Pierre numbers for both positive and negative subscripts are listed in Table 2.

A few dual generalized Pierre numbers

n	DW_n	DW_{-n}
0	DW_0	DW_0
1	DW_1	$2DW_2 - DW_3$
2	DW_2	$2DW_1 - DW_3$
3	DW_3	$2DW_0 - DW_1$
4	$2DW_3 - DW_0$	$4DW_2 - DW_0 - 2DW_3$
5	$4DW_3 - DW_1 - 2DW_0$	$4DW_1 - 4DW_2 + DW_3$
6	$8DW_3 - DW_2 - 2DW_1 - 4DW_0$	$4DW_0 - 4DW_1 + DW_2$
7	$15DW_3 - 2DW_2 - 4DW_1 - 8DW_0$	$DW_1 - 4DW_0 + 8DW_2 - 4DW_3$
8	$28DW_3 - 4DW_2 - 8DW_1 - 15DW_0$	$DW_0 + 8DW_1 - 12DW_2 + 4DW_3$
9	$52DW_3 - 8DW_2 - 15DW_1 - 28DW_0$	$8DW_0 - 12DW_1 + 6DW_2 - DW_3$
10	$96DW_3 - 15DW_2 - 28DW_1 - 52DW_0$	$6DW_1 - 12DW_0 + 15DW_2 - 8DW_3$
11	$177DW_3 - 15DW_2 - 28DW_1 - 96DW_0$	$6DW_0 + 15DW_1 - 32DW_2 + 12DW_3$
12	$326DW_3 - 52DW_2 - 96DW_1 - 177DW_0$	$15DW_0 - 32DW_1 + 24DW_2 - 6DW_3$
13	$600DW_3 - 96DW_2 - 177DW_1 - 326DW_0$	$24DW_1 - 32DW_0 + 24DW_2 - 15DW_3$

As special cases, the nth dual dual Pierre numbers and the nth dual dual Pierre Lucas numbers are given as

$$DP_n = P_n + \epsilon P_{n+1} \tag{2.3}$$

and

$$DC_n = C_n + \epsilon C_{n+1} \tag{2.4}$$

respectively. The sequences $\{DP_n\}_{n\geq 0}$ and $\{DC_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$DP_{-n} = P_{-n} + \epsilon P_{-n+1},$$

and

$$DC_{-n} = C_{-n} + \epsilon C_{-n+1},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.3) and (2.4) holds for all integer n.

For dual Pierre numbers (taking $W_n = P_n,\ P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 4,$) we get

$$\begin{aligned} DP_0 &= \epsilon, \\ DP_1 &= 2\epsilon + 1, \\ DP_2 &= 4\epsilon + 2, \end{aligned}$$

and for dual Pierre Lucas numbers (taking $W_n = C_n$, $C_0 = 4, C_1 = 2, C_2 = 4, C_3 = 8,$) we get

$$
\begin{aligned}
DC_0 &= 2\epsilon + 4, \\
DC_1 &= 4\epsilon 2, \\
DC_2 &= 8\epsilon + 4.
\end{aligned}
$$

Selected values of the dual Pierre numbers and dual Pierre Lucas numbers for both positive and negative subscripts are presented in Table 3 and Table 4, respectively.

Table 3.dual Pierre numbers

n	DP_n	DP_{-n}
0	ϵ	ϵ
1	$2\epsilon + 1$	0
2	$4\epsilon + 2$	0
3	$8\epsilon + 4$	-1
4	$15\epsilon + 8$	$-\epsilon$
5	$28\epsilon + 15$	0

Table 4. dual Pierre- Lucas numbers

n	DC_n	DC_{-n}
0	$2\epsilon + 4$	$2\epsilon + 4$
1	$4\epsilon + 2$	4ϵ
2	$8\epsilon + 4$	0
3	$12\epsilon + 8$	6
4	$22\epsilon + 12$	$-4 + 6\epsilon$
5	$40\epsilon + 22$	-4ϵ

We now present the Binet formula for the dual generalized Pierre numbers, and for the remainder of the paper, we adopt the following notational conventions.

$$\widehat{\alpha} = 1 + \epsilon\alpha, \tag{2.5}$$

$$\widehat{\beta} = 1 + \epsilon\beta, \tag{2.6}$$

$$\widehat{\gamma} = 1 + \epsilon\gamma, \tag{2.7}$$

$$\widehat{\delta} = \widehat{1} = 1 + \epsilon. \tag{2.8}$$

Note that we have the following identities:

$$\begin{aligned}
\widehat{\alpha}^2 &= 1+2\alpha\epsilon,\\
\widehat{\beta}^2 &= 1+2\epsilon\beta,\\
\widehat{\alpha}\widehat{\beta} &= 1+(\alpha+\beta)\epsilon,\\
\widehat{\gamma}^2 &= 1+\gamma^2+2\epsilon\gamma,\\
\widehat{\delta}^2 &= \widehat{1}^2 = 2+2\epsilon,\\
\widehat{\gamma}\widehat{\delta} &= 1+\epsilon(1+\gamma).
\end{aligned}$$

THEOREM 3. *(Binet's Formula) For any integer n, the n th dual generalized Pierre number is*

$$DW_n = \widehat{\alpha}A_1\alpha^n + \widehat{\beta}A_2\beta^n + \widehat{\gamma}A_3\gamma^n + \widehat{\delta}A_4 \tag{2.9}$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}, \widehat{\delta}$ are given as (2.5)-(2.8)

Proof. Using Binet's formula of the generalized Pierre numbers given below

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4$$

where A_1, A_2, A_3, A_4 are given (1.6),we get

$$\begin{aligned}
DW_n &= W_n + \epsilon W_{n+1},\\
&= A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4 + (A_1\alpha^{n+1} + A_2\beta^{n+1} + A_3\gamma^{n+1} + A_4)\epsilon\\
&= \alpha A_1\alpha^n + \widehat{\beta}A_2\beta^n + \widehat{\gamma}A_3\gamma^n + \widehat{\delta}A_4.
\end{aligned}$$

This proves (2.9).□

As special cases, for any integer n, the Binet's Formula of nth dual Pierre number is

$$DP_n = \frac{(\alpha^2+\alpha+1)\alpha^n\widehat{\alpha}}{2(\alpha^2+\alpha-1)} + \frac{(\beta^2+\beta+1)\beta^n\widehat{\beta}}{2(\beta^2+\beta-1)} + \frac{(\gamma^2+\gamma+1)\gamma^n\widehat{\gamma}}{2(\gamma^2+\gamma-1)} - \frac{\widehat{1}}{2} \tag{2.10}$$

and the Binet's Formula of nth dual Pierre Lucas number is

$$DC_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n + \widehat{\gamma}\gamma^n + \widehat{1}. \tag{2.11}$$

Next, we present generating function.

THEOREM 4. *The generating function for the dual generalized Pierre numbers is*

$$f_{DW_n}(x) = \sum_{n=0}^{\infty} DW_n x^n = \frac{DW_0 + (DW_1 - 2DW_0)x + (DW_2 - 2DW_1)x^2 + (DW_3 - 2DW_2)x^3}{1-2x+x^4}.$$

Proof. We assume that $f_{DW_n}(x)$ is the generating function of the dual generalized Pierre numbers and then we can write

$$f_{DW_n}(x) = \sum_{n=0}^{\infty} DW_n x^n.$$

Then, using the definition of the dual generalized Pierre numbers, and substracting $xf(x)$ and $x^2f(x)$ from $f(x)$, we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
(1-2x+x^4)f_{DW_n}(x) &= \sum_{n=0}^{\infty} DW_n x^n - 2x\sum_{n=0}^{\infty} DW_n x^n + x^4\sum_{n=0}^{\infty} DW_n x^n \\
&= \sum_{n=0}^{\infty} DW_n x^n - 2\sum_{n=0}^{\infty} DW_n x^{n+1} + \sum_{n=0}^{\infty} DW_n x^{n+4} \\
&= \sum_{n=0}^{\infty} DW_n x^n - 2\sum_{n=1}^{\infty} DW_{(n-1)} x^n + \sum_{n=4}^{\infty} DW_{(n-4)} x^n \\
&= (DW_0 + DW_1x + DW_2x^2 + DW_3x^3) - 2(DW_0x + DW_1x^2 + DW_2x^3) \\
&\quad + \sum_{n=4}^{\infty}(DW_n - 2DW_{n-1} + DW_{n-4})x^n \\
&= DW_0 + (DW_1 - 2DW_0)x + (DW_2 - 2DW_1)x^2 + (DW_3 - 2DW_2)x^3.
\end{aligned}$$

As special cases, the generating functions for the dual Pierre and dual Pierre Lucas numbers are

$$\sum_{n=0}^{\infty} DP_n x^n = \frac{\epsilon + x}{1-2x+x^4}$$

and

$$\sum_{n=0}^{\infty} DC_n x^n = \frac{2\epsilon + 4 - 6x - 4\epsilon x^3}{1-2x+x^4}$$

respectively.

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty} DW_n \frac{x^n}{n!}$ of the sequence DW_n.

Lemma 5. *Suppose that $f_{DW_n}(x) = \sum\limits_{n=0}^{\infty} DW_n \frac{x^n}{n!}$ is the exponential dual generating function of the generalized Pierre sequence $\{DW_n\}$.*

Then $\sum\limits_{n=0}^{\infty} DW_n \frac{x^n}{n!}$ is given by

$$\sum_{n=0}^{\infty} DW_n \frac{x^n}{n!} == A_1 e^{\alpha x}\widehat{\alpha} + A_2 e^{\beta x}\widehat{\beta} + A_3 e^{\gamma x}\widehat{\gamma} + A_4 e^{x}\widehat{1}.$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}, \widehat{\delta}$ are given as (2.5)-(2.8)

Proof. Using Binet's formula

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4.$$

where A_1, A_2, A_3, A_4 are given as in (1.6) we get

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{W}_n\frac{x^n}{n!} &= \sum_{n=0}^{\infty}W_n\frac{x^n}{n!}+\epsilon\sum_{n=0}^{\infty}W_{n+1}\frac{x^n}{n!}\\ &= \sum_{n=0}^{\infty}(A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4)\frac{x^n}{n!}+\epsilon\sum_{n=0}^{\infty}(A_1\alpha^{n+1}+A_2\beta^{n+1}+A_3\gamma^{n+1}+A_4)\frac{x^n}{n!}\\ &= (A_1e^{\alpha x}+A_2e^{\beta x}+A_3e^{\gamma x}+A_4e^{x})+\epsilon(A_1\alpha e^{\alpha x}+A_2\beta e^{\beta x}+A_3\gamma e^{\gamma x}+A_4e^{x})\\ &= A_1e^{\alpha x}(1+\epsilon\alpha)+A_2e^{\beta x}(1+\epsilon\beta)+A_3e^{\gamma x}(1+\epsilon\gamma)+A_4e^{x}(1+\epsilon)\\ &= A_1e^{\alpha x}\widehat{\alpha}+A_2e^{\beta x}\widehat{\beta}+A_3e^{\gamma x}\widehat{\gamma}+A_4e^{x}\widehat{1}.\end{aligned}$$

This proves (5). □

The previous Lemma gives the following results as particular examples.

COROLLARY 6. *Exponential generating function of hiperbolic Pierre and hiperbolic Pierre-Lucas numbers are*

a):

$$\begin{aligned}\sum_{n=0}^{\infty}DP_n\frac{x^n}{n!} &= (\frac{(\alpha^2+\alpha+1)}{2(\alpha^2+\alpha-1)}e^{\alpha x}+\frac{(\beta^2+\beta+1)}{2(\beta^2+\beta-1)}e^{\beta x}+\frac{(\gamma^2+\gamma+1)}{2(\gamma^2+\gamma-1)}e^{\gamma x}-\frac{1}{2}e^{x})\\ &\quad +\epsilon(\frac{\alpha(\alpha^2+\alpha+1)}{2(\alpha^2+\alpha-1)}e^{\alpha x}+\frac{\beta(\beta^2+\beta+1)}{2(\beta^2+\beta-1)}e^{\beta x}+\frac{\gamma(\gamma^2+\gamma+1)}{2(\gamma^2+\gamma-1)}e^{\gamma x}-\frac{1}{2}e^{x}).\end{aligned}$$

b):

$$\sum_{n=0}^{\infty}DC_n\frac{x^n}{n!}=e^{\alpha x}+e^{\beta x}+e^{\gamma x}+e^{x}+\epsilon(\alpha e^{\alpha x}+\beta e^{\beta x}+\gamma e^{\gamma x}+e^{x}).$$

3. Obtaining Binet Formula From Generating Function

Next, we derive the Binet formula for the generalized dual Pierre numbers $\{DW_n\}$ by utilizing their corresponding generating function.

THEOREM 7. *Binet's formula of generalized dual Pierre numbers:*

$$DW_n=\frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}, \tag{3.1}$$

where

$$\begin{aligned} q_1 &= DW_0\alpha^3+(DW_1-2DW_0)\,\alpha^2+(DW_2-2DW_1)\,\alpha+DW_3-2DW_2,\\ q_2 &= DW_0\beta^3+(DW_1-2DW_0)\,\beta^2+(DW_2-2DW_1)\,\beta+DW_3-2DW_2,\\ q_3 &= DW_0\gamma^3+(DW_1-2DW_0)\,\gamma^2+(DW_2-2DW_1)\,\gamma+DW_3-2DW_2,\\ q_4 &= DW_0\delta^3+(DW_1-2DW_0)\,\delta^2+(DW_2-2DW_1)\,\delta+DW_3-2DW_2.\end{aligned}$$

Proof. Let

$$h(x) = x^4 - 2x + 1.$$

Then for some α, β, γ and δ we write

$$h(x) = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x).$$

i.e.,

$$x^4 - 2x + 1 = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x). \tag{3.2}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$ are the roots of $h(x)$. This gives α, β, γ and δ as the roots of

$$h(\frac{1}{x}) = 1 - \frac{2}{x} + \frac{1}{x^4} = 0.$$

This implies $x^4 - 2x + 1 = 0$. Now, by it follows that

$$\sum_{n=0}^{\infty} DW_n x^n = \frac{(DW_3 - 2DW_2)\, x^3 + (DW_2 - 2DW_1)\, x^2 + (DW_1 - 2DW_0)\, x + DW_0}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)}.$$

Then we write

$$\frac{(DW_3 - 2DW_2)\, x^3 + (DW_2 - 2DW_1)\, x^2 + (DW_1 - 2DW_0)\, x + DW_0}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)} = \frac{B_1}{(1 - \alpha x)} + \frac{B_2}{(1 - \beta x)} + \frac{B_3}{(1 - \gamma x)} + \frac{B_4}{(1 - \delta x)}. \tag{3.3}$$

So

$$\begin{aligned} & (DW_3 - 2DW_2)\, x^3 + (DW_2 - 2DW_1)\, x^2 + (DW_1 - 2DW_0)\, x + DW_0 \\ = \; & B_1(1 - \beta x)(1 - \gamma x)(1 - \delta x) + B_2(1 - \alpha x)(1 - \gamma x)(1 - \delta x) \\ & + B_3(1 - \alpha x)(1 - \beta x)(1 - \delta x) + B_3(1 - \alpha x)(1 - \beta x)(1 - \gamma x). \end{aligned}$$

If we consider $x = \frac{1}{\alpha}$,we get $DW_0 + \frac{1}{\alpha}(DW_1 - 2DW_0) + \frac{1}{\alpha^2}(DW_2 - 2DW_1) + \frac{1}{\alpha^3}(DW_3 - 2DW_2)$ $= -B_1\left(\frac{1}{\alpha}\beta - 1\right)\left(\frac{1}{\alpha}\gamma - 1\right)\left(\frac{1}{\alpha}\delta - 1\right).$

This gives

$$\begin{aligned} B_1 &= \alpha^3 (DW_0 + \frac{1}{\alpha^2}(DW_2 - 2DW_1) + \frac{1}{\alpha^3}(DW_3 - 2DW_2) + \frac{1}{\alpha}(DW_1 - 2DW_0) \\ &= \frac{DW_0\alpha^3 + (DW_1 - 2DW_0)\, \alpha^2 + (DW_2 - 2DW_1)\, \alpha + DW_3 - 2DW_2}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} B_2 &= \frac{DW_0\beta^3 + (DW_1 - 2DW_0)\, \beta^2 + (DW_2 - 2DW_1)\, \beta + DW_3 - 2DW_2}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)}, \\ B_3 &= \frac{DW_0\gamma^3 + (DW_1 - 2DW_0)\, \gamma^2 + (DW_2 - 2DW_1)\, \gamma + DW_3 - 2DW_2}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)}, \\ B_4 &= \frac{DW_0\delta^3 + (DW_1 - 2DW_0)\, \delta^2 + (DW_2 - 2DW_1)\, \delta + DW_3 - 2DW_2}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}. \end{aligned}$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty} DW_n x^n = B_1(1-\alpha x)^{-1} + B_2(1-\beta x)^{-1} + B_3(1-\gamma x)^{-1} + B_4(1-\delta x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} DW_n x^n = B_1 \sum_{n=0}^{\infty} \alpha^n x^n + B_2 \sum_{n=0}^{\infty} \beta^n x^n + B_3 \sum_{n=0}^{\infty} \gamma^n x^n + B_4 \sum_{n=0}^{\infty} \delta^n x^n = \sum_{n=0}^{\infty} (B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n) x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$DW_n = B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n.$$

Theorem 8. *For all integers m, n the following identity holds:*

$$DW_{m+n} = P_{m-2}DW_{n+3} - P_{m-5}DW_{n+2} - P_{m-4}DW_{n+1} - P_{m-3}DW_n. \tag{3.4}$$

Proof. First we assume that $m, n \geq 0$ Using Theorem (3.4) ,it can be proved by using induction on m. If $m = 0$ we get

$$DW_n = P_{-2}DW_{n+3} - P_{-5}DW_{n+2} - P_{-4}DW_{n+1} - P_{-3}DW_n.$$

which is true since $P_{-2} = 0, P_{-3} = -1, P_{-4} = 0, P_{-5} = 0$. Suppose that the equality holds for $m \leq k$. For $m = k+1$, we obtain

$$\begin{aligned} DW_{k+1+n} &= 2DW_{n+k} + -DW_{n+k-3}, \\ &= 2\left(P_{k-2}DW_{n+3} - P_{k-5}DW_{n+2} - P_{k-4}DW_{n+1} - P_{k-3}DW_n\right) \\ &\quad - \left(2P_{k-5}DW_{n+3} - P_{k-8}DW_{n+2} - P_{k-6}DW_{n+1} - P_{k-6}DW_n\right) \end{aligned}$$

by mathematical induction on m, this proves,Theorem 8.

The other cases of m, n can be proved smilarly for all integers m, n. □

Taking $DW_n = DP_n$ or $DW_n = DC_n$ in above Theorem, respectively, we obtain:

Corollary 9.

$$\begin{aligned} DP_{m+n} &= P_{m-2}DP_{n+3} - P_{m-5}DP_{n+2} - P_{m-4}DP_{n+1} - P_{m-3}DP_n, \\ DC_{m+n} &= P_{m-2}DC_{n+3} - P_{m-5}DC_{n+2} - P_{m-4}DC_{n+1} - P_{m-3}DC_n. \end{aligned}$$

4. SIMSON'S FORMULA

In this section, we present Simpson's formula for the dual generalized Pierre numbers, which constitutes a special case of [21,Theorem 4.1].

Theorem 10. *(Simpson's formula for dual generalized Pierre numbers) For all integers n we have,*

$$\begin{vmatrix} DW_{n+3} & DW_{n+2} & DW_{n+1} & DW_n \\ DW_{n+2} & DW_{n+1} & DW_n & DW_{n-1} \\ DW_{n+1} & DW_n & DW_{n-1} & DW_{n-2} \\ DW_n & DW_{n-1} & DW_{n-2} & DW_{n-3} \end{vmatrix} = \begin{vmatrix} DW_3 & DW_2 & DW_1 & DW_0 \\ DW_2 & DW_1 & DW_0 & DW_{-1} \\ DW_1 & DW_0 & DW_{-1} & DW_{-2} \\ DW_0 & DW_{-1} & DW_{-2} & DW_{-3} \end{vmatrix}$$

$= (DW_3 - DW_2 - DW_1 - DW_0)(DW_3^3 - DW_2^3 - DW_1^3 - DW_0^3 + (-5DW_2 + DW_1 + DW_0)DW_3^2$
$+(7DW_3 - 3DW_0 - DW_1)DW_2^2 + (3DW_3 + DW_2 - DW_0)DW_1^2 + (DW_3 + DW_2 + DW_1)DW_0^2 + 4(-DW_2DW_3 - DW_0DW_3 + DW_0DW_2)DW_1)$.

Proof. Using Theorem 10 ,it can be proved by using induction or use [21,Theorem 4.1]

From the Theorem 10 ,we get the following Corollary.

Corollary 11. *For all integers n, the Simson's formulas of dual dual Pierre numbers and dual dual Pierre Lucas numbers are given as,*

a): $\begin{vmatrix} DP_{n+3} & DP_{n+2} & DP_{n+1} & DP_n \\ DP_{n+2} & DP_{n+1} & DP_n & DP_{n-1} \\ DP_{n+1} & DP_n & DP_{n-1} & DP_{n-2} \\ DP_n & DP_{n-1} & DP_{n-2} & DP_{n-3} \end{vmatrix} = 1 + 2\epsilon,$

b): $\begin{vmatrix} DC_{n+3} & DC_{n+2} & DC_{n+1} & DC_n \\ DC_{n+2} & DC_{n+1} & DC_n & DC_{n-1} \\ DC_{n+1} & DC_n & DC_{n-1} & DC_{n-2} \\ DC_n & DC_{n-1} & DC_{n-2} & DC_{n-3} \end{vmatrix} = -176 - 352\epsilon,$

respectively.

5. Linear Sums

In this section, we present the summation formulas for the dual generalized Pierre numbers corresponding to both positive and negative subscripts.

We now present the summation formulas for the generalized Pierre numbers.

Theorem 12. *For the dual dual Pierre numbers, we have the following formulas:*

(a): $\sum_{k=0}^{n} W_k = \frac{1}{2}(-(n+3))W_{n+3} + (n+4)W_{n+2} + (n+3)W_{n+1} + (n+4)W_n + 3W_3 - 4W_2 - 3W_1 - 2W_0).$

(b): $\sum_{k=0}^{n} W_{2k} = \frac{1}{2}(-(n+2)W_{2n+2} + (n+3)W_{2n+1} + (n+3)W_{2n} + (n+2)W_{2n-1} + 2W_3 - 2W_2 - 3W_1 - W_0).$

(c): $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{2}(-(n+1)W_{2n+2} + (n+3)W_{2n+1} + (n+2)W_{2n} + (n+2)W_{2n-1} + 2W_3 - 3W_2 - W_1 - 2W_0).$

Proof. For the proof, see Soykan [19, Theorem 3.10]. □

Theorem 13. *For the dual Pierre numbers, we have the following formulas:*

(a): $\sum\limits_{k=0}^{n} DW_k = \frac{1}{2}(-(n+3))DW_{n+3} + (n+4)DW_{n+2} + (n+3)DW_{n+1} + (n+4)DW_n + 3DW_3 - 4DW_2 - 3DW_1 - 2DW_0)$.

(b): $\sum\limits_{k=0}^{n} DW_{2k} = \frac{1}{2}(-(n+2)DW_{2n+2} + (n+3)DW_{2n+1} + (n+3)DW_{2n} + (n+2)DW_{2n-1} + 2DW_3 - 2DW_2 - 3DW_1 - DW_0)$.

(c): $\sum\limits_{k=0}^{n} DW_{2k+1} = \frac{1}{2}(-(n+1)DW_{2n+2} + (n+3)DW_{2n+1} + (n+2)DW_{2n} + (n+2)DW_{2n-1} + 2DW_3 - 3DW_2 - DW_1 - 2DW_0)$.

Proof. Use Theorem 12 and the definition of DW_n. $\square$

As a special case of Theorem 13, we state the following Corollary.

COROLLARY 14. *For $n \geq 0$, dual dual Pierre numbers have the following properties:*

(a): $\sum\limits_{k=0}^{n} DP_k = \frac{1}{2}(-(n+3))DP_{n+3} + (n+4)DP_{n+2} + (n+3)DP_{n+1} + (n+4)DP_n + 1)$.

(b): $\sum\limits_{k=0}^{n} DP_{2k} = \frac{1}{2}(-(n+2))DP_{2n+2} + (n+3)DP_{2n+1} + (n+3)DP_{2n} + (n+2)DP_{2n-1} + \epsilon + 1)$.

(c): $\sum\limits_{k=0}^{n} DP_{2k+1} = \frac{1}{2}(-(n+1)DP_{2n+2} + (n+3)DP_{2n+1} + (n+2)DP_{2n} + (n+2)DP_{2n-1} + 1)$.

As a second special case of the above theorem, we obtain the following summation formulas for the dual Pierre Lucas numbers:

COROLLARY 15. *For $n \geq 0$, the dual Pierre Lucas numbers satisfy the following properties.*

(a): $\sum\limits_{k=0}^{n} DC_k = \frac{1}{2}(-(n+3)DC_{n+3} + (n+4)DC_{n+2} + (n+3)DC_{n+1} + (n+4)DC_n - 12\epsilon - 6)$.

(b): $\sum\limits_{k=0}^{n} DC_{2k} = \frac{1}{2}(-(n+2)DC_{2n+2} + (n+3)DC_{2n+1} + (n+3)DC_{2n} + (n+2)DC_{2n-1} - 6\epsilon - 2)$.

(c): $\sum\limits_{k=0}^{n} DC_{2k+1} = \frac{1}{2}(-(n+1)DC_{2n+2} + (n+3)DC_{2n+1} + (n+2)DC_{2n} + (n+2)DC_{2n-1} - 8\epsilon - 6)$.

Next, we present the ordinary generating functions corresponding to selected special cases of the dual generalized Pierre numbers.

THEOREM 16. *The ordinary generating functions of the sequences DW_{2n}, DW_{2n+1} are given as follows:*

(a): $\sum_{n=0}^{\infty} DW_{2n}x^n = \frac{DW_3(2x^2)+DW_2(x^3-4x^2+x)-DW_1(2x^3)+DW_0(x^2-4x+1)}{x^4+2x^2-4x+1}$.

(b): $\sum_{n=0}^{\infty} DW_{2n+1}x^n = \frac{DW_3(x^3+x)-DW_2(2x^3)-DW_1(x^2-4x+1)-DW_0(2x^2)}{x^4+2x^2-4x+1}$.

Proof. Similarly, the proof can be constructed as in[4]

From the preceding theorem, we derive the following Corollary, which provides a summation formula for the dual Pierre numbers. (Take $DW_n = DP_n$ with $DP_0 = \epsilon, DP_1 = 2\epsilon + 1, DP_2 = 4\epsilon + 2, DP_3 = 8\epsilon + 4$)

COROLLARY 17. *$n \geq 0$ the dual Pierre numbers exhibit the following properties.*

(a): $\sum_{n=0}^{\infty} DP_{2n}x^n = \frac{\epsilon+2x+\epsilon x^2}{x^4+2x^2-4x+1}$.

(b): $\sum_{n=0}^{\infty} DP_{2n+1}x^n = \frac{(-2\epsilon-1)+(8+16\epsilon)x+(-4\epsilon-1)x^2}{x^4+2x^2-4x+1}$.

6. Matrices related with dual Generalized Pierre Numbers

We define the square matrix A of order 4 as

$$A = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $detA = 1$. Note that

$$A^n = \begin{pmatrix} P_{n+1} & -P_{n-2} & -P_{n-1} & -P_n \\ P_n & -P_{n-3} & -P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-4} & -P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-5} & -P_{n-4} & -P_{n-3} \end{pmatrix}$$

for the proof see [20].

Then we give the following lemma.

Lemma 18. *For $n \geq 0$ the following identitiy is true:*

$$\begin{pmatrix} DW_{n+3} \\ DW_{n+2} \\ DW_{n+1} \\ DW_n \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}.$$

Proof. The identitiy(18) can be proved by mathematical induction on n. If $n = 0$ we obtain

$$\begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}$$

which is true. Assuming that the given identity holds for $n = k$, the following identity is consequently valid.

$$\begin{pmatrix} DW_{k+3} \\ DW_{k+2} \\ DW_{k+1} \\ DW_k \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}.$$

For $n = k+1$, we get

$$\begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} DW_{k+3} \\ DW_{k+2} \\ DW_{k+1} \\ DW_k \end{pmatrix}$$
$$= \begin{pmatrix} DW_{k+4} \\ DW_{k+3} \\ DW_{k+2} \\ DW_{k+1} \end{pmatrix}.$$

Thus, the proof completed .□

We define

$$N_{DW} = \begin{pmatrix} DW_3 & DW_2 & DW_1 & DW_0 \\ DW_2 & DW_1 & DW_0 & DW_{-1} \\ DW_1 & DW_0 & DW_{-1} & DW_{-2} \\ DW_0 & DW_{-1} & DW_{-2} & DW_{-3} \end{pmatrix}, \tag{6.1}$$

$$E_{DW} = \begin{pmatrix} DW_{n+3} & DW_{n+2} & DW_{n+1} & DW_n \\ DW_{n+2} & DW_{n+1} & DW_n & DW_{n-1} \\ DW_{n+1} & DW_n & DW_{n-1} & DW_{n-2} \\ DW_n & DW_{n-1-} & DW_{n-2} & DW_{n-3} \end{pmatrix}. \tag{6.2}$$

Now, we have the following theorem with N_{DW} and E_{DW},

THEOREM 19. *Using N_{DW} and E_{DW} ,we get*

$$A^n N_{DW} = E_{DW}.$$

Proof. Note that we get

$$
\begin{aligned}
A^n N_{DW} &= \begin{pmatrix} P_{n+1} & -P_{n-2} & -P_{n-1} & -P_n \\ P_n & -P_{n-3} & -P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-4} & -P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-5} & -P_{n-4} & -P_{n-3} \end{pmatrix} \begin{pmatrix} DW_3 & DW_2 & DW_1 & DW_0 \\ DW_2 & DW_1 & DW_0 & DW_{-1} \\ DW_1 & DW_0 & DW_{-1} & DW_{-2} \\ DW_0 & DW_{-1} & DW_{-2} & DW_{-3} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
a_{11} &= P_{n+1}DW_3 - P_{n-2}DW_2 - P_{n-1}DW_1 - P_n DW_0 = DW_{n+3}, \\
a_{12} &= P_{n+1}DW_2 - P_{n-2}DW_1 - P_{n-1}DW_0 - P_n DW_{-1} = DW_{n+2}, \\
a_{13} &= P_{n+1}DW_1 - P_{n-2}DW_0 - P_{n-1}DW_{-1} - P_n DW_{-2} = DW_{n+1}, \\
a_{14} &= P_{n+1}DW_0 - P_{n-2}DW_{-1} - P_{n-1}DW_{-2} - P_n DW_{-3} = DW_n, \\
a_{21} &= P_n DW_3 - P_{n-3}DW_2 - P_{n-2}DW_1 - P_{n-1}DW_0 = DW_{n+2}, \\
a_{22} &= P_n DW_2 - P_{n-3}DW_1 - P_{n-2}DW_0 - P_{n-1}DW_{-1} = DW_{n+1}, \\
a_{23} &= P_n DW_1 - P_{n-3}DW_0 - P_{n-2}DW_{-1} - P_{n-1}DW_{-2} = DW_n, \\
a_{24} &= P_n DW_0 - P_{n-3}DW_{-1} - P_{n-2}DW_{-2} - P_{n-1}DW_{-3} = DW_{n-1}, \\
a_{31} &= P_{n-1}DW_3 - P_{n-4}DW_2 - P_{n-3}DW_1 - P_{n-2}DW_0 = DW_{n+1}, \\
a_{32} &= P_{n-1}DW_2 - P_{n-4}DW_1 - P_{n-3}DW_0 - P_{n-2}DW_{-1} = DW_n, \\
a_{33} &= P_{n-1}DW_1 - P_{n-4}DW_0 - P_{n-3}DW_{-1} - P_{n-2}DW_{-2} = DW_{n-1}, \\
a_{34} &= P_{n-1}DW_0 - P_{n-4}DW_{-1} - P_{n-3}DW_{-2} - P_{n-2}DW_{-3} = DW_{n-2}, \\
a_{41} &= P_{n-2}DW_3 - P_{n-5}DW_2 - P_{n-4}DW_1 - P_{n-3}DW_0 = DW_n, \\
a_{42} &= P_{n-2}DW_2 - P_{n-5}DW_1 - P_{n-4}DW_0 - P_{n-3}DW_{-1} = DW_{n-1}, \\
a_{43} &= P_{n-2}DW_1 - P_{n-5}DW_0 - P_{n-4}DW_{-1} - P_{n-3}DW_{-2} = DW_{n-2}, \\
a_{44} &= P_{n-2}DW_0 - P_{n-5}DW_{-1} - P_{n-4}DW_{-2} - P_{n-3}DW_{-3} = DW_{n-3}.
\end{aligned}
$$

Using the theorem (8) the proof is done. □

By taking $DW_n = DP_n$ with DP_0, DP_1, DP_2, DP_3 in (6.1) and (6.2)

$DW_n = C_n$ with DC_0, DC_1, DC_2, DC_3 in (6.1) and (6.2) respectively, we get:

$$N_{DP} = \begin{pmatrix} 8\epsilon+4 & 4\epsilon+2 & 2\epsilon+1 & \epsilon \\ 4\epsilon+2 & 2\epsilon+1 & \epsilon & 0 \\ 2\epsilon+1 & \epsilon & 0 & 0 \\ \epsilon & 0 & 0 & -1 \end{pmatrix},$$

$$E_{DP} = \begin{pmatrix} DP_{n+3} & DP_{n+2} & DP_{n+1} & DP_{n} \\ DP_{n+2} & DP_{n+1} & DP_{n} & DP_{n-1} \\ DP_{n+1} & DP_{n} & DP_{n-1} & DP_{n-2} \\ DP_{n} & DP_{n-1} & DP_{n-2} & DP_{n-3} \end{pmatrix},$$

$$N_{DC} = \begin{pmatrix} 12\epsilon+8 & 8\epsilon+4 & 4\epsilon+2 & 2\epsilon+4 \\ 8\epsilon+4 & 4\epsilon+2 & 2\epsilon+4 & 4\epsilon \\ 4\epsilon+2 & 2\epsilon+4 & 4\epsilon & 0 \\ 2\epsilon+4 & 4\epsilon & 0 & 6 \end{pmatrix},$$

$$E_{DC} = \begin{pmatrix} DC_{n+3} & DC_{n+2} & DC_{n+1} & DC_{n} \\ DC_{n+2} & DC_{n+1} & DC_{n} & DC_{n-1} \\ DC_{n+1} & C_{n} & DC_{n-1} & DC_{n-2} \\ DC_{n} & DC_{n-1} & DC_{n-2} & DC_{n-3} \end{pmatrix}.$$

From Theorem [19], we can write the following corollary.

Corollary 20. *The following identities are hold:*

- **a):** $A^n N_{DP} = E_{DP}$.
- **b):** $A^n N_{DC} = E_{DC}$.

7. Conclusion

Sequences defined by recurrence relations have long served as a cornerstone of mathematical inquiry, prized for their intrinsic structure and broad applicability across diverse fields including physics, engineering, architecture, biology, and the arts. Classical second-order integer sequences such as the Fibonacci, Lucas, Pell, and Jacobsthal numbers—illustrate this versatility. The Fibonacci sequence, introduced by Leonardo of Pisa in his 1202 treatise Liber Abaci through a rabbit population model, has since become a fundamental tool for exploring recursive behavior and uncovering mathematical identities.

In this study, we advance the classical framework by extending it to fourth-order recurrence systems, introducing the dual Pierre numbers together with two distinguished subclasses. For these newly defined sequences, we derive Binet-type formulas, construct both ordinary and exponential generating functions, and establish generalized Simson-type identities. Our analysis further encompasses closed-form summation expressions, algebraic characterizations, recurrence dynamics, and matrix-based representations.

We present these fourth-order generalizations as a natural progression within the broader landscape of sequence theory, offering fresh insights and versatile tools for modeling, analysis, and optimization across both pure and applied mathematical domains.

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