A Note On Dual Generalized Adrien Numbers

Abstract. In this study, we introduce and develop the concept of Dual Adrien numbers, with particular emphasis on two fundamental cases: the Dual Adrien sequence and the Dual Adrien–Lucas sequence. We conduct a systematic investigation of their structural and analytical properties, encompassing algebraic identities, matrix representations, recurrence relations, Binet-type formulas, generating functions, exponential expressions, Simson-type identities, and summation formulas. By establishing these results, we aim to construct a coherent and mathematically rigorous framework for the study of Dual Adrien numbers. Furthermore, we highlight their intrinsic connections with classical recurrence families, thereby situating them within the broader landscape of hypercomplex sequence analysis. This work not only extends the theory of generalized number sequences into the dual-number algebraic setting but also provides new tools and perspectives that may inspire further research in recurrence relations, combinatorial identities, and hypercomplex algebraic structures.

Keywords. Adrien numbers, Adrien-Lucas numbers, Dual Adrien numbers, Dual Adrien-Lucas numbers.

1. Introduction

In mathematical and geometric contexts, a hypercomplex number system refers to an algebraic framework that extends the foundational principles of complex numbers. These systems are characterized by their rich structural properties and are widely studied for their broad applicability across various domains, particularly in physics, engineering, and applied mathematics. Their ability to model multidimensional phenomena makes them indispensable tools in both theoretical investigations and practical computations.

Unlike complex numbers, which operate within a two-dimensional plane, hypercomplex systems offer a more versatile and sophisticated means of representing transformations, symmetries, and geometric structures in higher-dimensional spaces. As emphasized by Kantor in [20], hypercomplex systems can be interpreted as algebraic extensions of the real number line, designed to facilitate the analysis of multidimensional problems through generalized arithmetic and algebraic operations.

The principal classes of hypercomplex numbers include complex numbers, hyperbolic numbers, and dual numbers, each distinguished by the algebraic properties of their respective units. Complex numbers, composed of real and imaginary components, serve as the foundational case. Hyperbolic numbers extend this structure by introducing a unit whose square is +1, making them particularly useful in modeling Lorentz transformations and spacetime geometries. Dual numbers, on the other hand, incorporate a nilpotent unit whose square is zero, and are especially valuable in contexts such as automatic differentiation, kinematic analysis, and infinitesimal transformations.

This study focuses on dual numbers. Dual numbers form a significant subclass of hypercomplex number systems, characterized by the presence of a dual unit whose square is zero. These structures play a crucial role in various algebraic models, particularly in applications such as automatic differentiation and kinematic analysis. Viewed as an extension of the real number line, dual numbers offer algebraic tools well-suited for multidimensional analysis and provide practical solutions in both engineering and physics contexts.

The following sections offer more detailed insights into the mathematical properties and application areas of these hypercomplex systems.

- Dual Numbers: Algebraic Representation of Infinitesimals

Dual numbers [17] extend the real number system by introducing a nilpotent element ε, defined by the identity $\varepsilon^2 = 0$. Unlike zero, this infinitesimal unit is nonzero yet squares to zero, a property that sets dual numbers apart from other hypercomplex systems such as complex or quaternionic numbers. This unique algebraic structure allows dual numbers to encode both a real quantity and its infinitesimal variation within a single expression, making them particularly effective in modeling instantaneous rates of change.

Formally, a dual number is expressed as

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

Here, represents the real component, while captures the infinitesimal part. The nilpotent nature of ensures that higher-order infinitesimals vanish, simplifying algebraic manipulations and enabling direct computation of derivatives without resorting to limit processes.

This elegant framework has found widespread application in fields such as automatic differentiation, kinematics, and perturbation analysis, where the ability to represent and manipulate infinitesimal quantities is essential. By bridging the gap between discrete algebraic operations and continuous change, dual numbers offer a powerful tool for both theoretical exploration and practical computation.

Some authors have conducted studies about the dual, hyperbolic, dual hyperbolic and other special numbers. Now we give some information published papers in litarature.

- Cockle [7] explored hyperbolic numbers with complex coefficients, contributing to the early development of hypercomplex algebra.
- Eren and Soykan [14] studied the generalized Generalized Woodall Numbers.
- Cheng and Thompson [5] introduced dual numbers with complex coefficients, expanding the algebraic versatility of dual number systems for applications in polynomial equations and transformation theory.
- Akar at al [1] introduced the concept of dual hyperbolic numbers, combining characteristics of dual and hyperbolic systems into a unified algebraic structure.
- Aydın [2] presented hyperbolic Fibonacci numbers given by

$$\widetilde{F}_n = F_n + hF_{n+1},$$

where Fibonacci numbers are given by $F_{n+2} = F_{n+1} + F_n$, with the initial conditation $F_0 = 0,\ F_1 = 1$.

- Taş [27] studied hyperbolic Jacobsthal-Lucas sequence written by

$$HJ_n = J_n + hJ_{n+1},$$

where Jacobsthal-Lucas numbers given by $J_{n+2} = J_{n+1} + 2J_n$ with the inintial conditation $J_0 = 2,\ J_1 = 1$.

- Dikmen and Altınsoy, [11] studied On Third Order Hyperbolic Jacobsthal Numbers given by

$$\begin{aligned} \widehat{J}_n^{(3)} &= J_n^{(3)} + hJ_{n+1}^{(3)}, \\ \widehat{j}_n^{(3)} &= j_n^{(3)} + hj_{n+1}^{(3)}, \end{aligned}$$

  where Jacobsthal numbers, respectively, given by $J_n^{(3)} = J_{n-1}^{(3)} + J_{n-2}^{(3)} + 2J_{n-3}^{(3)},\ J_0^{(3)} = 0,\ J_1^{(3)} = 1,\ J_2^{(3)} = 1,\ j_n^{(3)} = j_{n-1}^{(3)} + j_{n-2}^{(3)} + 2j_{n-3}^{(3)},\ j_0^{(3)} = 2,\ j_1^{(3)} = 1,\ j_2^{(3)} = 5$.

- Soykan et al [23] presented dual hyperbolic generalized Pell numbers given by

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3},$$

where generalized Pell numbers, with the initial values $V_0,\ V_1$ not all being zero, are given by $V_n = 2V_{n-1} + V_{n-2},\ V_0 = a,\ V_1 = b\ (n \geq 2)$.

- Cihan et al [6] studied dual hyperbolic Fibonacci and Lucas numbers given by, respectively,

$$\begin{aligned} DHF_n &= F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3}, \\ DHL_n &= L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3}, \end{aligned}$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2},\ F_0 = 0,\ F_1 = 1,\ L_n = L_{n-1} + L_{n-2},\ L_0 = 2,\ L_1 = 1$.

- Soykan et al [22] studied dual hyperbolic generalized Jacopsthal numbers given by

$$\widehat{J}_n = J_n + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3},$$

where $J_n = J_{n-1} + 2J_{n-2},\ J_0 = a,\ J_1 = b.$

- Yılmaz and Soykan [28] introduced dual hyperbolic generalized Guglielmo numbers are

$$\widehat{T}_0 = T_0 + jT_1 + \varepsilon T_2 + j\varepsilon T_3,$$

where $T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, T_0 = 0, T_1 = 1, T_2 = 3.$

- Ayrılma and Soykan [3] studied dual hyperbolic generalized Edouard number and Edouard-Lucas number given by

$$\begin{aligned} \widehat{E}_0 &= E_0 + jE_1 + \varepsilon E_2 + j\varepsilon E_3, \\ \widehat{K}_0 &= K_0 + jK_1 + \varepsilon K_2 + j\varepsilon K_3, \end{aligned}$$

where $E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}, \quad E_0 = 0, E_1 = 1, E_2 = 7$ and $K_n = 7K_{n-1} - 7K_{n-2} + K_{n-3},$ $K_0 = 3, K_1 = 7, K_2 = 35.$

- Bród et al [4] studied dual hyperbolic generalized balancing numbers as

$$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3},$$

where $B_n = 6B_{n-1} - B_{n-2},\ B_0 = 0,\ B_1 = 1.$

- Dikmen [12] introduced dual hyperbolic generalised Leonardo numbers given by

$$\widehat{l}_0 = l_0 + jl_1 + \varepsilon l_2 + j\varepsilon l_3,$$

$l_n = 2l_{n-1} - l_{n-3}, l_0 = 1, l_1 = 1, l_2 = 3.$

- Eren and Soykan [16] introduced dual hyperbolic generalized Woodall numbers given by

$$\widehat{R}_0 = R_0 + jR_1 + \varepsilon R_2 + j\varepsilon R_3,$$

where $R_n = 5R_{n-1} - 8R_{n-2} + 4R_{n-3},\ R_0 = -1, R_1 = 1, R_2 = 7.$

- Yılmaz and Soykan [29] introduced dual generalized Guglielmo numbers given by

$$\tilde{T}_n = T_n + \varepsilon T_{n+1},$$

where $\tilde{T}_0 = T_0 + \varepsilon T_{n+1}, \tilde{T}_1 = T_1 + \varepsilon T_2, \tilde{T}_2 = T_2 + \varepsilon T_3.$

- Demirci and Soykan [10] introduced hyperbolic Adrien numbers given by

$$HA_n = A_n + jA_{n+1},$$

where $HA_0 = A_0 + jA_1, HA_1 = A_1 + jA_2, HA_2 = A_2 + jA_3$.

- Demirci and Soykan [9] introduced dual hyperbolic generalized Adrien numbers given by

$$\widehat{A}_0 = A_0 + jA_1 + \varepsilon A_2 + j\varepsilon A_3,$$

where $A_n = 3A_{n-1} - A_{n-2} - A_{n-4}$, $A_0 = 0, A_1 = 1, A_2 = 3, A_3 = 8$.

- Kalça and Soykan [19] introduced dual hyperbolic generalized Pandita numbers given by

$$\widehat{P}_0 = P_0 + jP_1 + \varepsilon P_2 + j\varepsilon P_3,$$

where $P_n = 2P_{n-1} - P_{n-2} + P_{n-3} - P_{n-4}, P_0 = 0P_1 = 1, P_2 = 2, P_3 = 3$.

- Doğan and Soykan [13] studied hyperbolic generalized Pierre numbers given by

$$HP_0 = P_n + \varepsilon P_{n+1},$$

where $HP_0 = P_0 + jP_1, HP_1 = P_1 + jP_2, HP_2 = P_2 + jP_{3.}$

- In [18], the outhers introduce the dual generalized Fibonacci matrices.
- Eren and Soykan [15] introduced dual generalized Woodall numbers given by

$$DW_0 = W_n + \varepsilon W_{n+1},$$

where $DW_0 = D_0 + jD_1, DW_1 = D_1 + jD_2, DW_2 = D_2 + jD_{3.}$

Before introducing the concept of dual Adrien numbers, it is essential to recall the fundamental properties of the classical Adrien numbers.

2. Background on Generalized Adrien Sequence

It's known that many author studied the generalized (r, s, t, u) sequence. One of these sequences is generalized Adrien numbers. Soykan, [21] defined generalized Adrien numbers. Before we present our original study , we recall some proporities related to generalized Adrien numbers such as reccurance relations, Binet's formula, generating function .

A generalized Adrien sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations;

$$W_n = 3W_{n-1} - W_{n-2} - W_{n-4}, \ n \geq 4, \tag{2.1}$$

with the initial values W_0, W_1, W_2, W_3 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-2)} + 3W_{-(n-3)} - W_{-(n-4)},$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) holds for all integer n. A recent study by Soykan explores the properties of this numerical sequence in detail, for more details, see [21].

Characteristic equation of $\{W_n\}$ is

$$z^4 - 3z^3 + z^2 + 1 = (z^3 - 2z^2 - z - 1)(z - 1) = 0.$$

The roots of characteristic equation are

$$\begin{aligned}
\alpha &= \frac{2}{3} + \left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}, \\
\beta &= \frac{2}{3} + \omega\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega^2\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}, \\
\gamma &= \frac{2}{3} + \omega^2\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}, \\
\delta &= 1.
\end{aligned}$$

Where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned}
\alpha + \beta + \gamma + \delta &= 3, \\
\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= 1, \\
\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= 0, \\
\alpha\beta\gamma\delta &= 1.
\end{aligned}$$

We get that

$$\begin{aligned}
\alpha + \beta + \gamma &= 2, \\
\alpha\beta + \alpha\gamma + \beta\gamma &= -1, \\
\alpha\beta\gamma &= 1.
\end{aligned}$$

The Binet formula for the generalized Adrien numbers $\{W_n\}$ is derived using the roots of the associated recurrence relation and holds for all integers n.

$$\begin{aligned}
W_n &= \frac{p_1\alpha^n}{4\alpha^2 + 3\alpha - 1} + \frac{p_2\beta^n}{4\beta^2 + 3\beta - 1} + \frac{p_3\gamma^n}{4\gamma^2 + 3\gamma - 1} + \frac{p_4\delta^n}{3} \qquad (2.2) \\
&= S_1\alpha^n + S_2\beta^n + S_3\gamma^n + S_4\delta^n.
\end{aligned}$$

Where p_1, p_2, p_3 and p_4 are given below

$$\begin{aligned}
p_1 &= (\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2+2\alpha+1)W_1 - W_0), \\
p_2 &= (\beta W_3 - \beta(3-\beta)W_2 + (-\beta^2+2\beta+1)W_1 - W_0), \\
p_3 &= (\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2+2\gamma+1)W_1 - W_0), \\
p_4 &= -(W_3 - 2W_2 - W_1 - W_0).
\end{aligned}$$

And

$$S_1 = \frac{p_1}{4\alpha^2+3\alpha-1}, \tag{2.3}$$

$$S_1 = \frac{p_1}{4\alpha^2+3\alpha-1}, \tag{2.4}$$

$$S_3 = \frac{p_3}{4\gamma^2+3\gamma-1}, \tag{2.5}$$

$$S_4 = -\frac{(W_3 - 2W_2 - W_1 - W_0)}{3}. \tag{2.6}$$

Binet's formula of Adrien and Adrien-Lucas sequences are

$$A_n = \frac{(2\alpha^2+\alpha+1)\alpha^n}{4\alpha^2+3\alpha-1} + \frac{(2\beta^2+\beta+1)\beta^n}{4\beta^2+3\beta-1} + \frac{(2\gamma^2+\gamma+1)\gamma^n}{4\gamma^2+3\gamma-1} - \frac{1}{3},$$

and

$$B_n = \alpha^n + \beta^n + \gamma^n + 1.$$

Respectively.

If we set $W_0 = 0, W_1 = 1, W_2 = 3, W_3 = 8$, then the sequence $\{W_n\}$ corresponds to the well-known Adrien sequence. Similarly, if we take $W_0 = 4, W_1 = 3, W_2 = 7, W_3 = 18$, then $\{W_n\}$ becomes the well-known Adrien–Lucas sequence. In other words, the Adrien sequence$\{A_n\}_{n\geq 0}$ and the Adrien–Lucas sequence $\{B_n\}_{n\geq 0}$ are both defined by the following fourth-order recurrence relation:

$$W_n = 3W_{n-1} - W_{n-2} - W_{n-4}, \ n \geq 4, \tag{2.7}$$

where the initial conditions determine the specific sequence:

$$A_n = 3A_{n-1} - A_{n-2} - A_{n-4}\ , \ A_0 = 0, A_1 = 1, A_2 = 3, A_3 = 8, \ n \geq 4, \tag{2.8}$$

$$B_n = 3B_{n-1} - B_{n-2} - B_{n-4}\ , \ B_0 = 4, B_1 = 3, B_2 = 7, B_3 = 18, \ n \geq 4. \tag{2.9}$$

The sequences $\{A_n\}_{n\geq 0}$, $\{B_n\}_{n\geq 0}$, can be extended to negative subscripts by defining,

$$\begin{aligned} A_{-n} &= -A_{-(n-2)} + 3A_{-(n-3)} - A_{-(n-4)}, \\ B_{-n} &= -B_{-(n-2)} + 3B_{-(n-3)} - B_{-(n-4)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. As a result, recurrences (2.8),(2.9) hold for all integer n. Binet's formulas as follows.

Table 1 lists the initial terms of the generalized Adrien sequence, including both positive and negative subscripts.

Table 1. A few generalized Adrien numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$3W_2 - W_1 - W_3$
2	W_2	$3W_1 - W_0 - W_2$
3	W_3	$3W_0 - 3W_2 + W_3$
4	$3W_3 - W_2 - W_0$	$10W_2 - 6W_1 - 3W_3$
5	$8W_3 - W_1 - 3W_2 - 3W_0$	$10W_1 - 6W_0 - 3W_2$
6	$21W_3 - 3W_1 - 9W_2 - 8W_0$	$10W_0 + 3W_1 - 18W_2 + 6W_3$
7	$54W_3 - 8W_1 - 24W_2 - 21W_0$	$3W_0 - 28W_1 + 36W_2 - 10W_3$
8	$138W_3 - 21W_1 - 62W_2 - 54W_0$	$33W_1 - 28W_0 - W_2 - 3W_3$

After then we can write the generating function of generalized Adrien numbers,

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 3W_0)x + (W_2 - 3W_1 + W_0)x^2 + (W_3 - 3W_2 + W_1)x^3}{1 - 3x + x^2 + x^4}.$$

For more details about generalized Adrien numbers, see [21].

Next, we give the exponential generating function of $\sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n.

Lemma 1. *[8] Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is the exponential generating function of the generalized Adrien sequence $\{W_n\}$.*

Then $\sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is given by:

$$\begin{aligned} \sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \frac{(\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + 2\alpha + 1)W_1 - W_0)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} e^{\alpha x} \\ &+ \frac{(\beta W_3 - \beta(3-\beta)W_2 + (-\beta^2 + 2\beta + 1)W_1 - W_0)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} e^{\beta x} \\ &+ \frac{(\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + 2\gamma + 1)W_1 - W_0)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} e^{\gamma x} \\ &+ (\frac{W_3 - 2W_2 - W_1 - W_0}{-3}) e^{x}. \end{aligned}$$

The previous Lemma 1 gives the following results as particular examples.

COROLLARY 2. *Exponential generating function of Adrien and Adrien-Lucas numbers are given by:*

a): $\sum\limits_{n=0}^{\infty} A_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty}((\frac{(2\alpha^2+\alpha+1)\alpha^n}{4\alpha^2+3\alpha-1} + \frac{(2\beta^2+\beta+1)\beta^n}{4\beta^2+3\beta-1} + \frac{(2\gamma^2+\gamma+1)\gamma^n}{4\gamma^2+3\gamma-1} - \frac{1}{3})\frac{x^n}{n!}$
$= (\frac{(2\alpha^2+\alpha+1)}{4\alpha^2+3\alpha-1}e^{\alpha x} + \frac{(2\beta^2+\beta+1)}{4\beta^2+3\beta-1}e^{\beta x} + \frac{(2\gamma^2+\gamma+1)\gamma^n}{4\gamma^2+3\gamma-1}e^{\gamma x} - \frac{1}{3}e^x).$

b): $\sum\limits_{n=0}^{\infty} B_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty}(\alpha^n+\beta^n+\gamma^n+1)\frac{x^n}{n!} = e^{\alpha x}+e^{\beta x}+e^{\gamma x}+e^x.$

In this study, we introduce the dual generalized Adrien numbers and conduct a comprehensive analysis of their algebraic structure in the subsequent sections.

3. Dual Generalized Adrien Numbers and their Generating Functions and Binet's Formulas

In this section, we define the dual generalized Adrien numbers and subsequently derive their generating functions and Binet type formulas. Within the framework of $\mathbb{H}_{\mathbb{D}}$, we now proceed to investigate the dual generalized Adrien numbers defined over the set $\mathbb{D}$. The nth generalized dual Adrien numbers, with DW_0, DW_1, DW_2 , DW_3 being the initial conditions, are defined as follows;

$$DW_n = W_n + \varepsilon W_{n+1}. \tag{3.1}$$

moreover (3.1) can be written to negative subscripts by defining,

$$DW_{-n} = W_{-n} + \varepsilon W_{-n+1.} \tag{3.2}$$

So identity (3.1) holds for all integers n.

Now we define some special cases of dual generalized Adrien numbers. The nth dual Adrien numbers, the nth dual Adrien-Lucas numbers, respectively, are given as

the nth generalized dual Adrien numbers $DA_n = A_n + \varepsilon A_{n+1}$, with DA_0, DA_1, DA_2, DA_3 being the initial conditions, can be expressed as follows;

$$DA_n = A_n + \varepsilon A_{n+1},$$

where

$$\begin{aligned} DA_0 &= A_0 + \varepsilon A_1, \\ DA_1 &= A_1 + \varepsilon A_2, \\ DA_2 &= A_2 + \varepsilon A_3, \\ DA_3 &= A_3 + \varepsilon A_4, \end{aligned}$$

for dual Adrien numbers, taking $W_n = A_n$, $A_0 = 0$, $A_1 = 1$, $A_2 = 3, A_3 = 8$, $A_4 = 21$, we get

$$\begin{aligned} DA_0 &= \varepsilon, \\ DA_1 &= 1 + 3\varepsilon, \\ DA_2 &= 3 + 8\varepsilon, \\ DA_3 &= 8 + 21\varepsilon, \end{aligned}$$

the nth generalized dual Adrien-Lucas numbers $DB_n = B_n + \varepsilon B_{n+1}$, with DB_0, DB_1, DB_2, DB_3 being the initial conditions, can be expressed as follows;

$$DB_n = B_n + \varepsilon B_{n+1},$$

where

$$\begin{aligned} DB_0 &= B_0 + \varepsilon B_1, \\ DB_1 &= B_1 + \varepsilon B_2, \\ DB_2 &= B_2 + \varepsilon B_3, \\ DB_3 &= B_3 + \varepsilon B_4. \end{aligned}$$

For dual Adrien-Lucas numbers, taking $W_n = B_n$, $B_0 = 4$, $B_1 = 3$, $B_2 = 7, B_3 = 18, B_4 = 43$, we get

$$\begin{aligned} DB_0 &= 4 + 3\varepsilon, \\ DB_1 &= 3 + 7\varepsilon, \\ DB_2 &= 7 + 18\varepsilon, \\ DB_3 &= 18 + 43\varepsilon. \end{aligned}$$

So, using (3.1), the following identity holds for all non-negative integers n n,

$$DW_n = 3DW_{n-1} - DW_{n-2} - DW_{n-4}, \tag{3.3}$$

and the sequence $\{DW_n\}_{n\geq 0}$ can be given as

$$DW_{-n} = -DW_{-(n-2)} + 3DW_{-(n-3)} - DW_{-(n-4)},$$

for $n = 1, 2, 3, ...$ by using (3.2). As a result, recurrence (3.3) holds for all integer n.

Table 2 The initial values of the dual generalized Adrien numbers are presented for both positive and negative indices.

Table 2. A few dual generalized Adrien numbers

n	DW_n	DW_{-n}
0	DW_0	DW_0
1	DW_1	$3DW_2 - DW_1 - DW_3$
2	DW_2	$3DW_1 - DW_0 - DW_2$
3	DW_3	$3DW_0 - 3DW_2 + DW_3$
4	$3DW_3 - DW_2 - DW_0$	$10DW_2 - 6DW_1 - 3DW_3$
5	$8DW_3 - DW_1 - 3DW_2 - 3DW_0$	$10DW_1 - 6DW_0 - 3DW_2$
6	$21DW_3 - 3DW_1 - 9DW_2 - 8DW_0$	$10DW_0 + 3DW_1 - 18DW_2 + 6DW_3$
7	$54DW_3 - 8DW_1 - 24DW_2 - 21DW_0$	$3DW_0 - 28DW_1 + 36DW_2 - 10DW_3$
8	$138DW_3 - 21DW_1 - 62DW_2 - 54DW_0$	$33DW_1 - 28DW_0 - DW_2 - 3DW_3$

Note that

$$\begin{aligned} DW_0 &= W_0 + \varepsilon W_1, \\ DW_1 &= W_1 + \varepsilon W_2, \\ DW_2 &= W_2 + \varepsilon W_3. \\ DW_3 &= W_3 + \varepsilon W_4. \end{aligned}$$

Tables 3 and 4 present selected values of the dual Adrien numbers and dual Adrien–Lucas numbers, respectively, including entries with both positive and negative indices. These tables illustrate the symmetric structure and recurrence behavior of the sequences across the extended integer domain, providing a concrete foundation for the theoretical results discussed in the preceding sections.

Table 3.Some dual Adrien numbers

n	DA_n	DA_{-n}
0	ε	ε
1	$1+3\varepsilon$	0
2	$3+8\varepsilon$	0
3	$8+21\varepsilon$	-1
4	$21+54\varepsilon$	$-\varepsilon$
5	$54+138\varepsilon$	1
6	$138+352\varepsilon$	$-3+\varepsilon$
7	$352+897\varepsilon$	-3ε
8	$897+2285\varepsilon$	6

Table 4. Some dual Adrien-Lucas numbers

n	DB_n	DB_{-n}
0	$4+3\varepsilon$	$4+3\varepsilon$
1	$3+7\varepsilon$	4ε
2	$7+18\varepsilon$	-2
3	$18+43\varepsilon$	$9-2\varepsilon$
4	$43+108\varepsilon$	$-2+9\varepsilon$
5	$108+274\varepsilon$	$-15-2\varepsilon$
6	$274+696\varepsilon$	$31-15\varepsilon$
7	$696+1771\varepsilon$	31ε
8	$1771+4509\varepsilon$	-74

We begin by introducing several expressions that will be utilized throughout the remainder of the paper. Following this, we present the Binet-type formula for the dual generalized Adrien numbers. To that end, we first define;

$$\tilde{\alpha} = 1 + \varepsilon\alpha, \tag{3.4}$$

$$\tilde{\beta} = 1 + \varepsilon\beta, \tag{3.5}$$

$$\tilde{\gamma} = 1 + \varepsilon\gamma, \tag{3.6}$$

$$\tilde{\lambda} = 1 + \varepsilon, \tag{3.7}$$

the equalities presented above lead to the following identities:

$$\begin{aligned}
\tilde{\alpha}^2 &= 1 + 2\varepsilon\alpha, \\
\tilde{\beta}^2 &= 1 + 2\varepsilon\beta, \\
\tilde{\gamma}^2 &= 1 + 2\varepsilon\gamma, \\
\tilde{\lambda}^2 &= 2 + 2\varepsilon, \\
\tilde{\alpha}\tilde{\beta} &= 1 + \varepsilon(\alpha + \beta), \\
\tilde{\alpha}\tilde{\gamma} &= 1 + \varepsilon(\alpha + \gamma), \\
\tilde{\gamma}\tilde{\beta} &= 1 + \varepsilon(\alpha + \beta), \\
\tilde{\alpha}\tilde{\gamma} &= 1 + \varepsilon(1 + \alpha), \\
\tilde{\alpha}\tilde{\beta}\tilde{\gamma} &= 1 + \varepsilon(\alpha + \beta + \gamma), \\
\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\lambda} &= 1 + \varepsilon(1 + \alpha + \beta + \gamma).
\end{aligned}$$

THEOREM 3. *(Binet's Formula) For any integer n, the nth dual generalized Adrien number is*

$$DW_n = \tilde{\alpha}S_1\alpha^n + \tilde{\beta}S_2\beta^n + \tilde{\gamma}S_3\gamma^n + \tilde{\delta}S_4, \tag{3.8}$$

where $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, $\tilde{\delta}$ are given as (3.4), (3.5), (3.6), (3.7).

Proof. Using Binet's formula of the generalized Adrien numbers given below

$$W_n = S_1\alpha^n + S_2\beta^n + S_3\gamma^n + S_4,$$

where S_1, S_2, S_2, S_4 are given (2.3), (2.4), (2.5), (2.6) we get

$$\begin{aligned}
DW_n &= W_n + \varepsilon W_{n+1}, \\
&= S_1\alpha^n + S_2\beta^n + S_3\gamma^n + S_4 \\
&\quad + \varepsilon(S_1\alpha^{n+1} + S_2\beta^{n+1} + S_3\gamma^{n+1} + S_4) \\
&= \tilde{\alpha}S_1\alpha^n + \tilde{\beta}S_2\beta^n + \tilde{\gamma}S_3\gamma^n + \tilde{\delta}S_4,
\end{aligned}$$

This proves (3.8). □

As special cases, for any integer n, the Binet's Formula of nth dual Adrien number is

$$DA_n = \frac{(2\alpha^2 + \alpha + 1)\alpha^n\tilde{\alpha}}{4\alpha^2 + 3\alpha - 1} + \frac{(2\beta^2 + \beta + 1)\beta^n\tilde{\beta}}{4\beta^2 + 3\beta - 1} + \frac{(2\gamma^2 + \gamma + 1)\gamma^n\tilde{\gamma}}{4\gamma^2 + 3\gamma - 1} - \frac{\tilde{1}}{3}, \tag{3.9}$$

and the Binet's Formula of nth dual Adrien-Lucas number is

$$DB_n = \tilde{\alpha}\alpha^n + \tilde{\beta}\beta^n + \tilde{\gamma}\gamma^n + 1. \tag{3.10}$$

In the next section, we derive the generating function corresponding to the dual generalized Adrien numbers.

THEOREM 4. *The generating function for the dual generalized Adrien numbers is*

$$\sum_{n=0}^{\infty} DW_n x^n = \frac{DW_0 + (DW_1 - 3DW_0)x + (DW_2 - 3DW_1 + DW_0)x^2 + (DW_3 - 3DW_2 + DW_1)x^3}{1 - 3x + x^2 + x^4}. \tag{3.11}$$

Proof. Let

$$f_{DW_n}(x) = \sum_{n=0}^{\infty} DW_n x^n$$

be the generating function of the dual generalized Adrien numbers. Using the definition of these numbers and applying a suitable subtraction technique, we derive the following functional identity $xf_{DW_n}(x)$ and $x^2 f_{DW_n}(x)$ from $f_{DW_n}(x)$, we obtain $(1 - 3x + x^2 + x^4)f_{DW_n}(x)$

$$\begin{aligned}
(1 - 3x + x^2 + x^4)f_{DW_n}(x) &= \sum_{n=0}^{\infty} DWx^n - 3x\sum_{n=0}^{\infty} DWx^n + x^2\sum_{n=0}^{\infty} DWx^n + x^4\sum_{n=0}^{\infty} DWx^n, \\
&= \sum_{n=0}^{\infty} DWx^n - 3\sum_{n=0}^{\infty} DWx^{n+1} + \sum_{n=0}^{\infty} DWx^{n+2} + \sum_{n=0}^{\infty} DWx^{n+4}, \\
&= \sum_{n=0}^{\infty} DWx^n - 3\sum_{n=1}^{\infty} DW_{(n-1)}x^n + \sum_{n=2}^{\infty} DW_{(n-2)}x^n + \sum_{n=4}^{\infty} DW_{(n-4)}x^n, \\
&= (DW_0 + DW_1x + DW_2x^2 + DW_3x^3) - 3(DW_0x + DW_1x^2 + DW_2x^3) \\
&\quad +(DW_0x^2 + DW_1x^3) + \sum_{n=4}^{\infty}(DW_n - 3DW_{n-1} + DW_{n-2} + DW_{n-4})x^n, \\
&= DW_0 + (DW_1 - 3DW_0)x + (DW_2 - 3DW_1 + DW_0)x^2 \\
&\quad +(DW_3 - 3DW_2 + DW_1)x^3.
\end{aligned}$$

Note that, using the recurrence relation $DW = 3DW_{n-1} - DW_{n-2} - DW_{n-4}$ and rearranging above equation the (3.11) has been found. □

We now express the generating functions of the dual Adrien and dual Adrien–Lucas numbers as follows;

(a): $f_{DA_n}(x) = \sum_{n=0}^{\infty} DA_n x^n = \dfrac{\varepsilon + x}{1 - 3x + x^2 + x^4}$,

(b): $f_{DB_n}(x) = DB_n x^n = \dfrac{-4\varepsilon x^3 + 2x^2 - (2\varepsilon + 9)\,x + 3\varepsilon + 4}{1 - 3x + x^2 + x^4}$.

Next, we give the exponential generating function of $\sum_{n=0}^{\infty} DW_n \frac{x^n}{n!}$ of the sequence DW_n.

LEMMA 5. *Suppose that $f_{DW_n}(x) = \sum_{n=0}^{\infty} DW_n \frac{x^n}{n!}$ is the exponential generating function of the dual generalized Adrien sequence $\{DW_n\}$.*

Then $\sum_{n=0}^{\infty} DW_n \frac{x^n}{n!}$ is given by

$$\sum_{n=0}^{\infty} DW_n \frac{x^n}{n!} = S_1 e^{\alpha x}\tilde{\alpha} + S_2 e^{\beta x}\tilde{\beta} + S_3 e^{\gamma x}\tilde{\gamma} + S_4 e^{x}\tilde{1}.$$

where $\tilde{\alpha}$,$\tilde{\beta}$, $\tilde{\gamma}$, $\tilde{\delta}$ are given as (3.4), (3.5), (3.6), (3.7).

Proof. Using Binet' s formula

$$W_n = S_1\alpha^n + S_2\beta^n + S_3\gamma^n + S_4,$$

where S_1, S_2, S_3, S_4 are given in (2.3), (2.4), (2.5), (2.6) we get

$$\begin{aligned}
\sum_{n=0}^{\infty} DW\frac{x^n}{n!} &= \sum_{n=0}^{\infty} DW_n\frac{x^n}{n!} + \varepsilon\sum_{n=0}^{\infty} DW_{n+1}\frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty}(S_1\alpha^n + S_2\beta^n + S_3\gamma^n + S_4)\frac{x^n}{n!} + \varepsilon\sum_{n=0}^{\infty}(S_1\alpha^{n+1} + S_2\beta^{n+1} + S_3\gamma^{n+1} + S_4)\frac{x^n}{n!} \\
&= (S_1e^{\alpha x} + S_2e^{\beta x} + S_3e^{\gamma x} + S_4e^x) + \varepsilon(S_1\alpha e^{\alpha x} + S_2\beta e^{\beta x} + S_3\gamma e^{\gamma x} + S_4e^x) \\
&= S_1e^{\alpha x}(1+\varepsilon\alpha) + S_2e^{\beta x}(1+\varepsilon\beta) + S_3e^{\gamma x}(1+\varepsilon\gamma) + S_4e^x(1+\varepsilon) \\
&= S_1e^{\alpha x}\tilde{\alpha} + S_2e^{\beta x}\tilde{\beta} + S_3e^{\gamma x}\tilde{\gamma} + S_4e^x\tilde{1}.
\end{aligned}$$

Note that we have

$$\sum_{n=0}^{\infty} DW_n\frac{x^n}{n!} = \sum_{n=0}^{\infty}(DW_n + \varepsilon DW_{n+1})\frac{x^n}{n!}. \square$$

The following results emerge as particular cases of the preceding lemma.

Corollary 6. *We now present the exponential generating functions corresponding to the dual Adrien and dual Adrien–Lucas sequences.*

a):

$$\begin{aligned}
\sum_{n=0}^{\infty} DA_n\frac{x^n}{n!} &= (\frac{(2\alpha^2+\alpha+1)}{4\alpha^2+3\alpha-1}e^{\alpha x} + \frac{(2\beta^2+\beta+1)}{4\beta^2+3\beta-1}e^{\beta x} + \frac{(2\gamma^2+\gamma+1)}{4\gamma^2+3\gamma-1}e^{\gamma x} - \frac{1}{3}e^x) \\
&\quad +\varepsilon(\frac{(2\alpha^2+\alpha+1)}{4\alpha^2+3\alpha-1}e^{\alpha x} + \frac{(2\beta^2+\beta+1)}{4\beta^2+3\beta-1}e^{\beta x} + \frac{(2\gamma^2+\gamma+1)}{4\gamma^2+3\gamma-1}e^{\gamma x} - \frac{1}{3}e^x) \\
&= \frac{(2\alpha^2+\alpha+1)\widetilde{\alpha}}{4\alpha^2+3\alpha-1}e^{\alpha x} + \frac{(2\beta^2+\beta+1)\widetilde{\beta}}{4\beta^2+3\beta-1}e^{\beta x} + \frac{(2\gamma^2+\gamma+1)\widetilde{\gamma}}{4\gamma^2+3\gamma-1}e^{\gamma x} - \widetilde{1}e^x
\end{aligned}$$

b):

$$\begin{aligned}
\sum_{n=0}^{\infty} DB_n\frac{x^n}{n!} &= e^{\alpha x} + e^{\beta x} + e^{\gamma x} + e^x + \varepsilon(\alpha e^{\alpha x} + \beta e^{\beta x} + \gamma e^{\gamma x} + e^x) \\
&= e^{\alpha x}\tilde{\alpha} + e^{\beta x}\tilde{\beta} + e^{\gamma x}\tilde{\gamma} + e^x\tilde{1}
\end{aligned}$$

4. Obtaining Binet Formula From Generating Function

Next, by the using generating function for DW_n find Binet formula of dual generalized Adrien number $\{DW_n\}$.

Theorem 7. *(Binet formula of dual generalized Adrien numbers)*

$$DW_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \tag{4.1}$$

where

$$\begin{aligned}
p_1 &= DW_0\alpha^3 + (DW_1 - 3DW_0)\alpha^2 + (DW_2 + DW_1 + DW_0)\alpha + (DW_3 + DW_2 + DW_1), \\
p_2 &= DW_0\beta^3 + (DW_1 - 3DW_0)\beta^2 + (DW_2 + DW_1 + DW_0)\beta + (DW_3 + DW_2 + DW_1), \\
p_3 &= DW_0\gamma^3 + (DW_1 - 3DW_0)\gamma^2 + (DW_2 + DW_1 + DW_0)\gamma + (DW_3 + DW_2 + DW_1), \\
p_4 &= DW_0\delta^3 + (DW_1 - 3DW_0)\delta^2 + (DW_2 + DW_1 + DW_0)\delta + (DW_3 + DW_2 + DW_1).
\end{aligned}$$

Proof. Let

$$h(x) = 1 - 3x + x^2 + x^4.$$

Then for some α, β, γ and δ we write

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x),$$

i.e.,

$$1 - 3x + x^2 = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x), \tag{4.2}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$ are the roots of $h(x)$. This gives α, β, γ and δ as the roots of

$$h(\frac{1}{x}) = 1 - \frac{3}{x} + \frac{1}{x^2} + \frac{1}{x^4} = 0.$$

This implies $x^4 - 3x^3 + x^2 + u = 0$. Now, it follows that

$$\sum_{n=0}^{\infty} DWx^n = \frac{DW_0 + (DW_1 - 3DW_0)x + (DW_2 - 3DW_1 + DW_0)x^2 + (DW_3 - 3DW_2 + DW_1)x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}.$$

Then we write

$$\frac{DW_0 + (DW_1 - 3DW_0)x + (DW_2 - 3DW_1 + DW_0)x^2 + (DW_3 - 3DW_2 + DW_1)x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}, \tag{4.3}$$

$$= \frac{B_1}{(1-\alpha x)} + \frac{B_2}{(1-\beta x)} + \frac{B_3}{(1-\gamma x)} + \frac{B_4}{(1-\delta x)}. \tag{4.4}$$

So

$$\begin{aligned}
& DW_0 + (DW_1 - 3DW_0)x + (DW_2 - 3DW_1 + DW_0)x^2 + (DW_3 - 3DW_2 + DW_1)x^3 \\
= & B_1(1-\beta x)(1-\gamma x)(1-\delta x) + B_2(1-\alpha x)(1-\gamma x)(1-\delta x) \\
& + B_3(1-\alpha x)(1-\beta x)(1-\delta x) + B_3(1-\alpha x)(1-\beta x)(1-\gamma x).
\end{aligned}$$

If we consider $x=\frac{1}{\alpha}$, we get $DW_0+(DW_1-3DW_0)\frac{1}{\alpha}+(DW_2-3DW_1+DW_0)\frac{1}{\alpha^2}+(DW_3-3DW_2+DW_1)\frac{1}{\alpha^3}$ $=B_1(1-\frac{\beta}{\alpha})(1-\frac{\gamma}{\alpha})(1-\frac{\delta}{\alpha})$.

This gives

$$\begin{aligned} B_1 &= \frac{\alpha^3(DW_0+(DW_1-3DW_0)\frac{1}{\alpha}+(DW_2-3DW_1+DW_0)\frac{1}{\alpha^2}+(DW_3-6DW_2+DW_1)\frac{1}{\alpha^3})}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} \\ &= \frac{DW_0\alpha^3+(DW_1-DW_0)\alpha^2+(DW_2-3DW_1+DW_0)\alpha+(DW_3-3DW_2+DW_1)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} B_2 &= \frac{DW_0\beta^3+(DW_1-3DW_0)\beta^2+(DW_2-3DW_1+DW_0)\beta+(DW_3-3DW_2+DW_1)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\ B_3 &= \frac{DW_0\gamma^3+(DW_1-3DW_0)\gamma^2+(DW_2-3DW_1+DW_0)\gamma+(DW_3-3DW_2+DW_1)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\ B_4 &= \frac{DW_0\delta^3+(DW_1-3DW_0)\delta^2+(DW_2-3DW_1+DW_0)\delta+(DW_3-3DW_2+DW_1)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}. \end{aligned}$$

Thus (4.3) can be written as

$$\sum_{n=0}^{\infty} DWx^n = B_1(1-\alpha x)^{-1}+B_2(1-\beta x)^{-1}+B_3(1-\gamma x)^{-1}+B_4(1-\delta x)^{-1}.$$

This gives

$$\begin{aligned} \sum_{n=0}^{\infty} DWx^n &= B_1\sum_{n=0}^{\infty}\alpha^n x^n+B_2\sum_{n=0}^{\infty}\beta^n x^n+B_3\sum_{n=0}^{\infty}\gamma^n x^n+B_4\sum_{n=0}^{\infty}\delta^n x^n \\ &= \sum_{n=0}^{\infty}(B_1\alpha^n+B_2\beta^n+B_3\gamma^n+B_4\delta^n)x^n. \end{aligned}$$

Consequently, matching coefficients from both sides leads to the following expression.

$$DW = B_1\alpha^n+B_2\beta^n+B_3\gamma^n+B_4\delta^n.$$

and then we get (4.2). □

An identity associated with the dual Adrien numbers is given as follows.

THEOREM 8. *For all integers m,n the following identities hold:*

$DW_{m+n}=A_{m-2}DW_{n+3}+(-A_{m-3}-A_{m-5})DW_{n+2}+(-A_{m-4})DW_{n+1}-A_{m-3}DW_n.$

Proof. First we assume that $m,n\geq 0$ then (8) can be proved by mathematical induction on m. If $m=0$ we get

$$DW_n = A_{-2}DW_{n+3}+(-A_{-3}-A_{-5})DW_{n+2}+(-A_{-4})DW_{n+1}-A_{-3}DW_n.$$

which is true since $A_{-2}=0$, $A_{-3}=-1$, $A_{-4}=0$, $A_{-5}=1$. Assume that the equality holds for $m\leq k$. For $m=k+1$, we get

$$\begin{aligned} DW_{k+1+n} &= 3DW_{n+k}-DW_{n+k-1}-DW_{n+k-3}, \\ & 3(A_{k-2}DW_{n+3}+(-A_{k-3}-A_{k-5})DW_{n+2}+(-A_{k-4})DW_{n+1}-A_{k-3}DW_{n}) \\ & -(A_{k-3}DW_{n+3}+(-A_{k-4}-A_{k-6})DW_{n+2}+(-A_{-5})DW_{n+1}-A_{k-4}DW_{n}) \\ & -(A_{k-5}DW_{n+3}+(-A_{k-6}-A_{k-8})DW_{n+2}+(-A_{k-6})DW_{n+1}-A_{k-6}DW_{n}). \end{aligned}$$

This completes the proof of the theorem via induction on m, this Theorem (8).

The other cases of m,n can be proved smilarly for all integers m,n. □

Taking $DW_n=DA_n$ or $DW_n=DB_n$ in above Theorem, respectively, we get:

COROLLARY 9.

$$\begin{aligned} DA_{m+n} &= A_{m-2}DA_{n+3}+(-A_{m-3}-A_{m-5})DA_{n+2}+(-A_{m-4})DA_{n+1}-A_{m-3}DA_{n}, \\ DB_{m+n} &= A_{m-2}DB_{n+3}+(-A_{m-3}-A_{m-5})DB_{n+2}+(-A_{m-4})DB_{n+1}-A_{m-3}DB_{n}. \end{aligned}$$

5. Simson's Formulas

Simson's formula is established in this section for the dual generalized Adrien numbers, providing a key analytical tool. This is a special case of [24,Theorem 4.1].

THEOREM 10. *For all integers n, we have*

$$\begin{vmatrix} DW_{n+3} & DW_{n+2} & DW_{n+1} & DW_{n} \\ DW_{n+2} & DW_{n+1} & DW_{n} & DW_{n-1} \\ DW_{n+1} & DW_{n} & DW_{n-1} & DW_{n-2} \\ DW_{n} & DW_{n-1} & DW_{n-2} & DW_{n-3} \end{vmatrix} = (DW_0+DW_1+2DW_2-DW_3)(-DW_3^3+5DW_2^3+DW_1^3+$$
$$DW_0^3-(DW_0+3DW_1-7DW_2)DW_3^2$$

$$+(3DW_0-4DW_1-14DW_3)DW_2^2+(2DW_0+DW_2-6DW_3)DW_1^2-(DW_1+2DW_3)DW_0^2+13DW_1DW_2DW_3+$$
$$DW_0DW_2DW_3+5DW_0DW_1DW_3-7DW_0DW_1DW_2).$$

Proof. Take $r=3,s=-1,t=0,u=-1$. □

Corollary 11. *For every integer n, the dual generalized Adrien and Adrien–Lucas numbers satisfy the following Simson-type relations.*

$$\begin{vmatrix} DA_{n+3} & DA_{n+2} & DA_{n+1} & DA_{n} \\ DA_{n+2} & DA_{n+1} & DA_{n} & DA_{n-1} \\ DA_{n+1} & DA_{n} & DA_{n-1} & DA_{n-2} \\ DA_{n} & DA_{n-1} & DA_{n-2} & DA_{n-3} \end{vmatrix} = 1+3\varepsilon,$$

$$\begin{vmatrix} DB_{n+3} & DB_{n+2} & DB_{n+1} & DB_{n} \\ DB_{n+2} & DB_{n+1} & DB_{n} & DB_{n-1} \\ DB_{n+1} & DB_{n} & DB_{n-1} & DB_{n-2} \\ DB_{n} & DB_{n-1} & DB_{n-2} & DB_{n-3} \end{vmatrix} = -2349\varepsilon - 783$$

respectively.

6. Linear Sums

This section establishes the summation formulas for the dual generalized Adrien numbers, covering both positive and negative indices. Subsequently, we introduce the corresponding summation formulas for the generalized Adrien numbers, extending the analysis to a broader class of sequences.

Theorem 12. *The generalized Adrien numbers satisfy the following identities for all positive and negative indices:*

(a): $\sum\limits_{k=0}^{n} W_k = \frac{1}{3}(-(n+3)W_{n+3} + (2n+7)W_{n+2} + (n+2)W_{n+1} + (n+4)W_n$
$+3W_3 - 7W_2 - 2W_1 - W_0).$

(b): $\sum\limits_{k=0}^{n} W_{2k} = \frac{1}{3}(-(n+2)W_{2n+2} + (2n+5)W_{2n+1} + (n+3)W_{2n} + (n+2)W_{2n-1}$
$+2W_3 - 4W_2 - 3W_1).$

(c): $\sum\limits_{k=0}^{n} W_{2k+1} = \frac{1}{3}(-(n+1)W_{2n+2} + (2n+5)W_{2n+1} + (n+2)W_{2n} + (n+2)W_{2n-1}$
$+2W_3 - 5W_2 - 2W_0).$

(d): $\sum\limits_{k=1}^{n} W_{-k} = \frac{1}{3}(-(n+1)W_{-n+3} + (2n+1)W_{-n+2} + (n+2)W_{-n+1} + (n+3)W_{-n} +$
$W_3 - W_2 - 2W_1 - 3W_0).$

(e): $\sum\limits_{k=1}^{n} W_{-2k} = \frac{1}{3}(-(n+2)W_{-2n+2} + (2n+3)W_{-2n+1} + (n+4)W_{-2n} + (n+2)W_{-2n-1}$
$+2W_3 - 4W_2 - W_1 - 4W_0).$

(f): $\sum\limits_{k=1}^{n} W_{-2k+1} = \frac{1}{3}(-(n+3)W_{-2n+2} + 2(n+3)W_{-2n+1} + (n+2)W_{-2n} + (n+2)W_{-2n-1}$
$+2W_3 - 3W_2 - 4W_1 - 2W_0).$

Proof. For the proof, see Soykan [26]. □

We now present the summation formulas for dual numbers as the first notable consequence of the above theorem.

Theorem 13. *For the dual numbers, we have the following formulas:*

(a): $\sum\limits_{k=0}^{n} DW_k = \frac{1}{3}(-(n+3)DW_{n+3} + (2n+7)DW_{n+2} + (n+2)DW_{n+1} + (n+4)DW_n$
$+3DW_3 - 7DW_2 - 2DW_1 - DW_0)$.

(b): $\sum\limits_{k=0}^{n} DW_{2k} = \frac{1}{3}(-(n+2)DW_{2n+2} + (2n+5)DW_{2n+1} + (n+3)DW_{2n} + (n+2)DW_{2n-1}$
$+2DW_3 - 4DW_2 - 3DW_1)$.

(c): $\sum\limits_{k=0}^{n} DW_{2k+1} = \frac{1}{3}(-(n+1)DW_{2n+2} + (2n+5)DW_{2n+1} + (n+2)DW_{2n} + (n+2)DW_{2n-1}$
$+2DW_3 - 5DW_2 - 2DW_0)$.

(d): $\sum\limits_{k=1}^{n} DW_{-k} = \frac{1}{3}(-(n+1)DW_{-n+3} + (2n+1)DW_{-n+2} + (n+2)DW_{-n+1} + (n+3)DW_{-n}$
$+DW_3 - DW_2 - 2DW_1 - 3DW_0)$.

(e): $\sum\limits_{k=1}^{n} DW_{-2k} = \frac{1}{3}(-(n+2)DW_{-2n+2} + (2n+3)DW_{-2n+1} + (n+4)DW_{-2n} + (n+2)DW_{-2n-1}$
$+2DW_3 - 4DW_2 - DW_1 - 4DW_0)$.

(f): $\sum\limits_{k=1}^{n} DW_{-2k+1} = \frac{1}{3}(-(n+3)DW_{-2n+2} + 2(n+3)DW_{-2n+1} + (n+2)DW_{-2n} + (n+2)DW_{-2n-1}$
$+2DW_3 - 3DW_2 - 4DW_1 - 2DW_0)$.

Proof. Use Theorem 12. □

The theorem yields the following summation results as a first special case involving dual Adrien numbers.

Theorem 14. *For $n \geq 0$, dual generalized Adrien numbers have the following properties:*

(a): $\sum\limits_{k=0}^{n} DA_k = \frac{1}{3}(-(n+3)DA_{n+3} + (2n+7)DA_{n+2} + (n+2)DA_{n+1} + (n+4)DA_n + 1)$.

(b): $\sum\limits_{k=0}^{n} DA_{2k} = \frac{1}{3}(-(n+2)DA_{2n+2} + (2n+5)DA_{2n+1} + (n+3)DA_{2n} + (n+2)DA_{2n-1} + \varepsilon + 1)$.

(c): $\sum\limits_{k=0}^{n} DA_{2k+1} = \frac{1}{3}(-(n+1)DA_{2n+2} + (2n+5)DA_{2n+1} + (n+2)DA_{2n} + (n+2)DA_{2n-1} + 1)$.

(d): $\sum\limits_{k=1}^{n} DA_{-k} = \frac{1}{3}(-(n+1)DA_{-n+3} + (2n+1)DA_{-n+2} + (n+2)DA_{-n+1} + (n+3)DA_{-n} + 4\varepsilon + 3)$.

(e): $\sum\limits_{k=1}^{n} DA_{-2k} = \frac{1}{3}(-(n+2)DA_{-2n+2} + (2n+3)DA_{-2n+1} + (n+4)DA_{-2n} + (n+2)DA_{-2n-1} + 3\varepsilon + 3)$.

(f): $\sum\limits_{k=1}^{n} DA_{-2k+1} = \frac{1}{3}(-(n+3)DA_{-2n+2} + 2(n+3)DA_{-2n+1} + (n+2)DA_{-2n} + (n+2)DA_{-2n-1}$
$+ 4\varepsilon + 3)$.

The following derivations concern the ordinary generating functions for notable special cases within the dual generalized Adrien number framework.

Theorem 15. *The ordinary generating functions of the sequences DW_{2n}, DW_{2n+1} are given as follows:*

(a): $\sum_{n=0}^{\infty} DW_{2n}x^n = \dfrac{3x^2DW_3 + (x^3 - 8x^2 + x)DW_2 - 3x^3DW_1 + (x^3 + 2x^2 - 7x + 1)DW_0}{x^4 + 2x^3 + 3x^2 - 7x + 1}$

(b): $\sum_{n=0}^{\infty} DW_{2n+1}x^n = \dfrac{(x^3 + x^2 + x)DW_3 - (3x^3 + 3x^2)DW_2 + (x^3 + 2x^2 - 7x + 1)DW_1 - 3x^2DW_0}{x^4 + 2x^3 + 3x^2 - 7x + 1}$

From the last Theorem, we have the following Corollary which gives sum formula of dual Adrien numbers (Take $DW_n = DA_n$ with

$DA_0 = \varepsilon,\ DA_1 = 1 + 3\varepsilon,\ DA_2 = 3 + 8\varepsilon,\ DA_3 = 8 + 21\varepsilon$)

COROLLARY 16. *For $n \geq 0$ dual Adrien numbers have the following properties.*

(a): $\sum_{n=0}^{\infty} DA_{2n}x^n = \dfrac{\varepsilon\left(x^3 + 2x^2 - 7x + 1\right) - 3x^3\left(3\varepsilon + 1\right) + 3x^2\left(21\varepsilon + 8\right) + \left(8\varepsilon + 3\right)\left(x^3 - 8x^2 + x\right)}{x^4 + 2x^3 + 3x^2 - 7x + 1}.$

(b): $\sum_{n=0}^{\infty} DA_{2n+1}x^n = -\dfrac{\left(8\varepsilon + 3\right)\left(3x^3 + 3x^2\right) - \left(3\varepsilon + 1\right)\left(x^3 + 2x^2 - 7x + 1\right) - \left(21\varepsilon + 8\right)\left(x^3 + x^2 + x\right) + 3\varepsilon x^2}{x^4 + 2x^3 + 3x^2 - 7x + 1}.$

7. Matrix Formulations Arising from Hyperbolic Generalized Adrien Sequences

Matrix identities arising from the dual Adrien numbers are established in this section, offering further insight into their recursive and structural behavior.

By using the $\{A_n\}$ which is defined by the fourth-order recurrence relation as follows:

$$A_n = 3A_{n-1} - A_{n-2} - A_{n-4}\ ,$$

with the initial conditions

$$A_0 = 0,\ A_1 = 1,\ A_2 = 3,\ A_3 = 8. \tag{7.1}$$

We define the square matrix M of order 4 as

$$M = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $det M = 1$. Then, we give the following Lemma.

LEMMA 17. *For $n \geq 0$ the following identity is true*

$$\begin{pmatrix} DW_{n+3} \\ DW_{n+2} \\ DW_{n+1} \\ DW_n \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}. \tag{7.2}$$

Proof. As a starting point, we verify the assertion in the case of $n \geq 0$. Lemma 17 can be given by mathematical induction on n. If $n = 0$ we get

$$\begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}$$

which is true. We assume that (7.2) is true for $n = k$. As a result, we obtain the following identity.

$$\begin{pmatrix} DW_{k+3} \\ DW_{k+2} \\ DW_{k+1} \\ DW_k \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}.$$

For $n = k + 1$, we get

$$\begin{aligned} \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix} &= \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} DW_3 \\ DW_2 \\ DW_1 \\ DW_0 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} DW_{k+3} \\ DW_{k+2} \\ DW_{k+1} \\ DW_k \end{pmatrix} \\ &= \begin{pmatrix} DW_{k+4} \\ DW_{k+3} \\ DW_{k+2} \\ DW_{k+1} \end{pmatrix}. \end{aligned}$$

This completes the proof by induction. □

Note that

$$A^n = \begin{pmatrix} A_{n+1} & -A_n - A_{n-2} & -A_{n-1} & -A_n \\ A_n & -A_{n-1} - A_{n-3} & -A_{n-2} & -A_{n-1} \\ A_{n-1} & -A_{n-2} - A_{n-4} & -A_{n-3} & -A_{n-2} \\ A_{n-2} & -A_{n-3} - A_{n-5} & -A_{n-4} & -A_{n-3} \end{pmatrix}.$$

For the proof see [25].

We define

$$N_{DW} = \begin{pmatrix} DW_3 & DW_2 & DW_1 & DW_0 \\ DW_2 & DW_1 & DW_0 & DW_{-1} \\ DW_1 & DW_0 & DW_{-1} & DW_{-2} \\ DW_0 & DW_{-1} & DW_{-2} & DW_{-3} \end{pmatrix}, \tag{7.3}$$

$$E_{DW} = \begin{pmatrix} DW_{n+3} & DW_{n+2} & DW_{n+1} & DW_n \\ DW_{n+2} & DW_{n+1} & DW_n & DW_{n-1} \\ DW_{n+1} & DW_n & DW_{n-1} & DW_{n-2} \\ DW_n & DW_{n-1} & DW_{n-2} & DW_{n-3} \end{pmatrix}. \tag{7.4}$$

Now, we have the following theorem for N_{DW} and E_{DW}.

THEOREM 18. *Using N_{DW} and E_{DW}, we get*

$$A^n N_{DW} = E_{DW}.$$

Proof. Note that we get

$$A^n N_{DW} = \begin{pmatrix} A_{n+1} & -A_n - A_{n-2} & -A_{n-1} & -A_n \\ A_n & -A_{n-1} - A_{n-3} & -A_{n-2} & -A_{n-1} \\ A_{n-1} & -A_{n-2} - A_{n-4} & -A_{n-3} & -A_{n-2} \\ A_{n-2} & -A_{n-3} - A_{n-5} & -A_{n-4} & -A_{n-3} \end{pmatrix} \begin{pmatrix} DW_3 & DW_2 & DW_1 & DW_0 \\ DW_2 & DW_1 & DW_0 & DW_{-1} \\ DW_1 & DW_0 & DW_{-1} & DW_{-2} \\ DW_0 & DW_{-1} & DW_{-2} & DW_{-3} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

where

$$
\begin{aligned}
a_{11} &= A_{n+1}DW_3 + (-A_n - A_{n-2})DW_2 + (-A_{n-1})DW_1 + (-A_n)DW_0, \\
a_{12} &= A_{n+1}DW_2 + (-A_n - A_{n-2})DW_1 + (-A_{n-1})DW_0 + (-A_n)DW_{-1}, \\
a_{13} &= A_{n+1}DW_1 + (-A_n - A_{n-2})DW_0 + (-A_{n-1})DW_{-1} + (-A_n)DW_{-2}, \\
a_{14} &= A_{n+1}DW_0 + (-A_n - A_{n-2})DW_{-1} + (-A_{n-1})DW_{-2} + (-A_n)DW_{-3}, \\
a_{21} &= A_nDW_3 + (-A_{n-1} - A_{n-3})DW_2 + (-A_{n-2})DW_1 + (-A_{n-1})DW_0, \\
a_{22} &= A_nDW_2 + (-A_{n-1} - A_{n-3})DW_1 + (-A_{n-2})DW_0 + (-A_{n-1})DW_{-1}, \\
a_{23} &= A_nDW_1 + (-A_{n-1} - A_{n-3})DW_0 + (-A_{n-2})DW_{-1} + (-A_{n-1})DW_{-2}, \\
a_{24} &= A_nDW_0 + (-A_{n-1} - A_{n-3})DW_{-1} + (-A_{n-2})DW_{-2} + (-A_{n-1})DW_{-3}, \\
a_{31} &= A_{n-1}DW_3 + (-A_{n-2} - A_{n-4})DW_2 + (-A_{n-3})DW_1 + (-A_{n-2})DW_0, \\
a_{32} &= A_{n-1}DW_2 + (-A_{n-2} - A_{n-4})DW_1 + (-A_{n-3})DW_0 + (-A_{n-2})DW_{-1}, \\
a_{33} &= A_{n-1}DW_1 + (-A_{n-2} - A_{n-4})DW_0 + (-A_{n-3})DW_{-1} + (-A_{n-2})DW_{-2}, \\
a_{34} &= A_{n-1}DW_0 + (-A_{n-2} - A_{n-4})DW_{-1} + (-A_{n-3})DW_{-2} + (-A_{n-2})DW_{-3}, \\
a_{41} &= A_{n-2}DW_3 + (-A_{n-3} - A_{n-5})DW_2 + (-A_{n-4})DW_1 + (-A_{n-3})DW_0, \\
a_{42} &= A_{n-2}DW_2 + (-A_{n-3} - A_{n-5})DW_1 + (-A_{n-4})DW_0 + (-A_{n-3})DW_{-1}, \\
a_{43} &= A_{n-2}DW_1 + (-A_{n-3} - A_{n-5})DW_0 + (-A_{n-4})DW_{-1} + (-A_{n-3})DW_{-2}, \\
a_{44} &= A_{n-2}DW_0 + (-A_{n-3} - A_{n-5})DW_{-1} + (-A_{n-4})DW_{-2} + (-A_{n-3})DW_{-3}.
\end{aligned}
$$

Using the theorem (8) the proof is done. □

By taking DW_n =DA_n with A_0, A_1, A_2, A_3 in (7.3) and (7.4)
and DW_n =DB_n with DB_0, DB_1, DB_2, DB_3 in (7.3) and (7.4)
respectively, we get:

$$E_{DA} = \begin{pmatrix} DA_{n+3} & DA_{n+2} & DA_{n+1} & DA_{n} \\ DA_{n+2} & DA_{n+1} & DA_{n} & DA_{n-1} \\ DA_{n+1} & DA_{n} & DA_{n-1} & DA_{n-2} \\ DA_{n} & DA_{n-1} & DA_{n-2} & DA_{n-3} \end{pmatrix},$$

$$N_{DA} = \begin{pmatrix} 8+21\varepsilon & 3+8\varepsilon & 1+3\varepsilon & \varepsilon \\ 3+8\varepsilon & 1+3\varepsilon & \varepsilon & 0 \\ 1+3\varepsilon & \varepsilon & 0 & 0 \\ \varepsilon & 0 & 0 & -1 \end{pmatrix},$$

$$E_{DB} = \begin{pmatrix} DB_{n+3} & DB_{n+2} & DB_{n+1} & DB_{n} \\ DB_{n+2} & DB_{n+1} & DB_{n} & DB_{n-1} \\ DB_{n+1} & DB_{n} & DB_{n-1} & DB_{n-2} \\ DB_{n} & DB_{n-1} & DB_{n-2} & DB_{n-3} \end{pmatrix},$$

$$N_{DB} = \begin{pmatrix} 18+43\varepsilon & 7+18\varepsilon & 3+7\varepsilon & 4+3\varepsilon \\ 7+18\varepsilon & 3+7\varepsilon & 4+3\varepsilon & 4\varepsilon \\ 3+7\varepsilon & 4+3\varepsilon & 4\varepsilon & -2 \\ 4+3\varepsilon & 4\varepsilon & -2 & 9-2\varepsilon \end{pmatrix}.$$

From Theorem 18, we can write the following corollary.

Corollary 19. *We establish the following identities:*

a): $A^n N_{DA} = E_{DA}$.

b): $A^n N_{DB} = E_{DB}$.

8. Conclusion

In this paper, we introduced and analyzed the concept of dual generalized Adrien numbers, with particular attention to the special cases of dual Adrien and dual Adrien–Lucas sequences. Building upon the classical Adrien framework, we established recurrence relations, Binet-type formulas, generating functions, exponential forms, Simson-type identities, and summation formulas for their dual extensions. These results demonstrate that the dual structure enriches the algebraic and combinatorial properties of the sequences, while preserving their fundamental recurrence dynamics.

The study highlights the versatility of dual generalized Adrien numbers in extending known integer sequences to broader algebraic settings. By situating them within the context of hypercomplex systems, the work provides new insights into the interplay between dual numbers and higher-order recurrences. Future research may focus on exploring additional dual extensions of other classical sequences, investigating spectral properties, and examining potential applications in discrete mathematics, cryptography, and computational models.

Possible Applications in Daily Life. Although dual generalized Adrien numbers may appear primarily as abstract mathematical constructs, they can have meaningful applications in real-world contexts:

- Cryptography and Security. Complex recurrence sequences such as dual Adrien numbers can be employed in key generation for encryption algorithms. This directly supports everyday technologies such as online banking, secure messaging, and e-commerce platforms by enhancing security.
- Computer Graphics and Image Processing. Recurrence relations and special number sequences can be used in image compression and pattern generation. In practice, this may improve visual effects in photo filters, animations, and digital design tools.
- Engineering and Kinematics. Dual numbers are already widely applied in kinematic analysis, for example in robotics. Extending these ideas with dual Adrien numbers could help model more complex motions, with potential applications in robotic surgery, automotive systems, and industrial

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