Equal Sums of Powers of Four Hexic $(x^6 + y^6 + z^6 + d^6)^k = (x^2 + y^2 + z^2 + d^2)^k(u^2 + v^2 + w^2)$

Abstract

Let x, y, z, d, u, v and w be positive integers. This research develops and introduces the Diophantine equation $(x^6 + y^6 + z^6 + d^6)^k = (x^2 + y^2 + z^2 + d^2)^k(u^2 + v^2 + w^2)$. The equation introduces a power relationship involving sum of four sixth powers and product of sums four and three squares, providing a unique perspective on the interplay between different powers and product of sums of squares.The study involves elementary methodology grounded in integer decomposition and factorization, taking a case-by-case basis alongside generalizations. In this study the case when $k = 1$ has been fully determined.

Keywords: Sum of Sixth Powers, Product of Sums of Squares

Keywords: Diophantine Equation; Sums of Squares, Integer Sequence

1 Introduction

The exploration of Diophantine equations, especially those involving powers and squares, has been a prominent theme in number theory. Various studies have investigated equations similar to the one proposed in this research, albeit [1] with different exponents and arrangements. Notably, works by Nagell [17], [18],[19] and Tijdeman [3],[4], [5] and [6] have laid the groundwork for understanding Diophantine equations involving powers. Nagell's contributions include foundational results on the solvability of specific equations, while Tijdeman's work extends these insights to more general cases. Furthermore, the study by Bennett and Skinner [2] provides a comprehensive analysis of Diophantine equations involving powers, offering insights into the distribution of solutions and the role of heights. The authors present methods grounded in arithmetic geometry and modular forms, showcasing the diversity of approaches to understanding such equations. For recent work on polynomial equations on sums of powers see [7,8,9,10,11] and for detailed recap on integer sums of two square studies the reader may refer to [12,13,14,15,16,17]. While the literature on Diophantine equations is extensive, the specific equation $(x^6+y^6+z^6+d^6)^k = (x^2+y^2+z^2+d^2)^k(u^2+v^2+w^2)$, particularly when $k \geq 1$ appears to be a novel and intriguing topic that warrants detailed exploration. This research builds upon the existing literature, incorporating elementary methodologies to analyze the solutions and relationships unique to this equation. The diophantine equation $(x^6+y^6+z^6+d^6)^k = (x^2+y^2+z^2+d^2)^k(u^2+v^2+w^2)$ stands at the intersection of number theory and algebraic equations, presenting a fascinating exploration into the relationships between sixth powers and squares. This research aims to comprehensively investigate the nature of integer solutions within this equation, with a particular emphasis on the scenario where $k = 1$. By introducing and studying this specific Diophantine equation, the research contributes to the broader understanding of such equations and advances the field of number theory.

2 Main Results

In the following sections, we present our findings in the form of conjecture and proceed to solve particular cases for the exponent k It's important to note that, in this this research, the condition $z > y > x$ is maintained.

Conjecture 2.1. *For any positive integer $k \geq 2$ and any non-zero integers x, y, z, d, u, v and w, the diophantine equation*

$$(x^6+y^6+z^6+d^6)^k = (x^2+y^2+z^2+d^2)^k(u^2+v^2+w^2)$$

admits solutions within the set of integers if $z - y = y - x = d$.

Theorem 2.2. *For any non-zero integers x, y, z, d, u, v and w, the diophantine equation*

$$x^6+y^6+z^6+d^6 = (x^2+y^2+z^2+d^2)(u^2+v^2+w^2)\cdots(1)$$

admits solutions within the set of integers if $z - y = y - x = d$.

Proof. Assume x, y and z are in arithmetic progression.Then, $y = x + d$ and $z = x + 2d$. Expressing the L.H.S of equation (1) in terms of the R.H.S. We have,

$$x^6+y^6+z^6+d^6 = x^6+(x+d)^6+(x+2d)^6+d^6$$

$$= 3x^6+18x^5d+75x^4d^2+180x^3d^3+255x^2d^4+198xd^5+66d^6\cdots(2)$$

Breaking down equation (2) into product of quadratic and quartic polynomial we have,

$$= (3x^2+6d^2+6dx)(x^4+22xd^3+15x^2d^2+4x^3d+11d^4)\cdots(3)$$

Applying algebraic manipulation, the quartic part of equation (3) can be rewritten as

$$= (3x^2+6d^2+6dx)(d^4+(3xd+3d^2)^2+(x+d)^4)\cdots(4)$$

Further simplification of equation (4) and replacing $y = x + d$ into equation (4) yields,

$$= (3x^2+6d^2+6dx)(d^4+9d^2y^2+y^4)\cdots(5).$$

In conclusion, decomposing equation (5) into a product of sums of three and four squares we have,

$$(3x^2+6d^2+6dx)(d^4+9d^2y^2+y^4) = (x^2+(x+d)^2+(x+2d)^2+d^2)((d^2)^2+(3dy)^2+(y^2)^2).$$

Since $y = x + d$ and $z = x + 2d$ and Setting $u = d^2, v = 3dy, w = y^2$. Subsequently,

$$(x^6+y^6+z^6+d^6) = (x^2+y^2+z^2+d^2)(u^2+v^2+w^2).$$

□

2.1 Examples

To argument the above theorem , this research will provide few examples to validate the results:

Example 2.3. *Case* (i) *when* $d = 1$
Let $I = x^6 + y^6 + z^6 + d^6 = (x^2 + y^2 + z^2 + d^2)(u^2 + v^2 + w^2) \cdots (2.3)$. *Assume,* $x = 1, y = 2, z = 3, d = 1, u = d^2 = 1^2 = 1, v = 3dy = 3 * 1 * 2 = 6$ *and* $w = y^2 = 2^2 = 4$. *Replacing this values in equation* $(*)$ *we have the L.H.S as* $I = 1^6 + 2^6 + 3^6 + 1^6 = 795$. *Now, expanding R.H.S we have,* $I = (1^2 + 2^2 + 3^2 + 1^2)(1^2 + 6^2 + 4^2) = 15 * 53 = 795$*. Since the L.H.S is equal to the R.H.S the results easily follows.*

Case (ii) *when* $d = 2$
Let $I = x^6 + y^6 + z^6 + d^6 = (x^2 + y^2 + z^2 + d^2)(u^2 + v^2 + w^2) \cdots (2.3)$. *Assume,* $x = 2, y = 4, z = 6, d = 2, u = d^2 = 2^2 = 4, v = 3dy = 3 * 2 * 4 = 24$ *and* $w = y^2 = 4^2 = 16$. *Replacing this values in equation* $(*)$ *we have the L.H.S as* $I = 2^6 + 4^6 + 6^6 + 2^6 = 50880$. *Now, expanding R.H.S we have,* $I = (2^2 + 4^2 + 4^2 + 2^2)(4^2 + 24^2 + 16^2) = 60 * 848 = 5880$*. Since the L.H.S is equal to the R.H.S the results easily follows*

Case (iii) *when* $d = 3$
Let $I = x^6 + y^6 + z^6 + d^6 = (x^2 + y^2 + z^2 + d^2)(u^2 + v^2 + w^2) \cdots (2.3)$. *Assume,* $x = 1, y = 4, z = 7, d = 3, u = d^2 = 3^2 = 9, v = 3dy = 3 * 3 * 4 = 36$ *and* $w = y^2 = 4^2 = 16$. *Replacing this values in equation* $(*)$ *we have the L.H.S as* $I = 1^6 + 4^6 + 7^6 + 3^6 = 122475$*. Now, expanding R.H.S we have,* $I = (1^2 + 4^2 + 7^2 + 3^2)(9^2 + 36^2 + 16^2) = 75 * 1633 = 122475$*. Since the L.H.S is equal to the R.H.S the results easily follows*

Case (iv) *when* $d = 4$
Let $I = x^6 + y^6 + z^6 + d^6 = (x^2 + y^2 + z^2 + d^2)(u^2 + v^2 + w^2) \cdots (2.3)$. *Assume,* $x = 3, y = 7, z = 11, d = 4, u = d^2 = 4^2 = 16, v = 3dy = 3 * 4 * 7 = 84$ *and* $w = y^2 = 7^2 = 49$. *Replacing this values in equation* $(*)$ *we have the L.H.S as* $I = 3^6 + 7^6 + 11^6 + 4^6 = 1894035$. *Now, expanding R.H.S we have,* $I = (3^2 + 7^2 + 11^2 + 4^2)(16^2 + 84^2 + 49^2) = 195 * 9713 = 1894035$*. Since the L.H.S is equal to the R.H.S the results easily follows*

3 Conclusion

In summary, the of the Diophantine equation, $(x^6+y^6+z^6+d^6)^k = (x^2+y^2+z^2+d^2)^k(u^2+v^2+w^2)$ with $k = 1$, the study has made notable progress in unveiling integer solutions through the application of integer decomposition and factorization. This determination sheds light on the intricate relationships between sums of powers of four hexic and product of sums of powers of four squares and three squares. While the findings are partial, it can serve as a solid starting point for future research. To further advance this research, we recommend extending the analysis to different exponents $k \geq 2$, conducting in-depth case analysis of the given diophantine equation under the investigation.

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