Entropy of the Adjacency Spectrum: A New Graph Invariant

Abstract

We introduce the *spectral entropy topological index* $\mathscr{S}(G)$, defined as the Shannon entropy of the squared adjacency eigenvalues of a simple graph G. This construction converts the adjacency spectrum into a probability distribution and yields a concise measure of global structural complexity. We establish general bounds for $\mathscr{S}(G)$, prove strict positivity for every nonempty simple graph, and compute the index explicitly for some classical families, including stars, complete bipartite graphs, and complete graphs, together with spectral expressions for paths and cycles. These examples illustrate how $\mathscr{S}(G)$ distinguishes between centralized, regular, and irregular structures. We also provide heuristic observations for Erdős–Rényi random graphs and demonstrate that the index has natural applications in computer science, including network anomaly detection, robustness analysis in distributed and wireless sensor networks, graph based data mining, and the analysis of web graphs and blockchain style peer to peer networks. In these domains, $\mathscr{S}(G)$ serves as an interpretable indicator of decentralization, heterogeneity, and structural diversity.

Keywords: Spectral entropy topological index; spectral graph theory; adjacency spectrum; Shannon entropy; graph invariants; network complexity

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1 Introduction

Spectral graph theory provides powerful tools for analyzing graph structure through eigenvalues and eigenvectors of matrices associated with the graph. In many areas of computer science including network security, data mining, machine learning on graphs, and distributed systems, spectral quantities capture both global and local structural features.

Traditional spectral indices such as the spectral radius, energy, Laplacian eigenvalues, algebraic connectivity, and spectral gap have been used extensively. However, these indices often summarize only a limited portion of spectral information. For instance, the spectral radius captures only the largest eigenvalue, and the energy aggregates magnitudes without reflecting their distribution.

Beyond purely spectral quantities, numerous topological indices based on degrees and related graph polynomials have been studied, especially in chemical graph theory and the analysis of nanostructures. In particular, M-polynomials and degree based indices have been applied to graphene and related materials; see for example (10).

In this article, motivated by information theory and graph complexity considerations, we introduce a new measure called the *Spectral Entropy Topological Index*, denoted by $\mathscr{S}(G)$. It quantifies how *spread* the spectrum is and therefore measures structural complexity.

We show that $\mathscr{S}(G)$ satisfies $0 \leq \mathscr{S}(G) \leq \log n$, the complete graph, complete bipartite graph, and star graph exhibit minimal entropy, certain random and scale-free networks exhibit near-maximal entropy, $\mathscr{S}(G)$ behaves predictably under graph operations such as union, join, and complement, the index is highly relevant for detecting anomalies and irregularities in communication and social networks, assessing robustness in distributed systems, and analyzing structural complexity in computer science applications.

2 Spectral Probability Measure

Let G be a simple graph on n vertices with adjacency matrix $A(G)$. Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of $A(G)$.

Definition 2.1 (Spectral Probability Measure)**.** Define the measure

$$\mu_G = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i},$$

where δ_x denotes the Dirac measure at x.

We convert this measure into a probability distribution by assigning weights proportional to the squared eigenvalues.

Definition 2.2 (Squared-Eigenvalue Distribution)**.** Define

$$p_i = \frac{\lambda_i^2}{\sum_{j=1}^{n} \lambda_j^2}, \qquad i = 1, \ldots, n.$$

For any graph with at least one edge we have $\sum_{j=1}^{n} \lambda_j^2 = \mathrm{tr}(A^2) = 2m > 0$, where m is the number of edges; see (4; 2), so $\sum_{i=1}^{n} p_i = 1$, and $(p_1, \ldots, p_n)$ is a probability distribution.

3 The Spectral-Entropy Topological Index

Definition 3.1 (Spectral-Entropy Index)**.** The *Spectral-Entropy Topological Index* of G is defined as

$$\mathscr{S}(G) = -\sum_{i=1}^{n} p_i \log p_i,$$

with the convention $0 \log 0 := 0$.

This resembles Shannon entropy; see (11), measuring the complexity of the eigenvalue distribution.

4 Bounds and Extremal Graphs

Theorem 4.1 (Basic bounds)**.** *Let G be a simple graph on n vertices with at least one edge, and let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of its adjacency matrix. Define*

$$p_i = \frac{\lambda_i^2}{\sum_{j=1}^n \lambda_j^2}, \quad i = 1, \dots, n,$$

and let $r = |\{i : \lambda_i \neq 0\}|$ be the number of nonzero eigenvalues. Then

$$0 \leq \mathscr{S}(G) := -\sum_{i=1}^n p_i \log p_i \leq \log r \leq \log n.$$

Proof. Let A be the adjacency matrix of G and let m denote the number of edges. Since G has at least one edge, A is nonzero and

$$\sum_{j=1}^n \lambda_j^2 = \mathrm{tr}(A^2) = 2m > 0.$$

Thus each $p_i \geq 0$ and

$$\sum_{i=1}^n p_i = \frac{\sum_{i=1}^n \lambda_i^2}{\sum_{j=1}^n \lambda_j^2} = 1,$$

so $(p_1, \dots, p_n)$ is a probability distribution (we adopt the usual convention $0 \log 0 := 0$).

(1) Nonnegativity. For $x \in [0, 1]$ the function $x \mapsto -x \log x$ is nonnegative, with the value 0 at $x = 0$ and $x = 1$. Hence each summand $-p_i \log p_i \geq 0$, and therefore

$$\mathscr{S}(G) = -\sum_{i=1}^n p_i \log p_i \geq 0.$$

(2) Upper bound by $\log r$. Let $r = |\{i : \lambda_i \neq 0\}|$. Since $p_i > 0$ if and only if $\lambda_i \neq 0$, the distribution $(p_1, \dots, p_n)$ has at most r positive entries. Without loss of generality, assume $p_1, \dots, p_r > 0$ and $p_{r+1} = \dots = p_n = 0$. Then

$$\mathscr{S}(G) = -\sum_{i=1}^n p_i \log p_i = -\sum_{i=1}^r p_i \log p_i.$$

Consider the function

$$H(p_1, \dots, p_r) = -\sum_{i=1}^r p_i \log p_i$$

on the simplex

$$\Delta_r := \{(p_1, \dots, p_r) : p_i > 0, \sum_{i=1}^r p_i = 1\}.$$

It is well known that H is concave and invariant under permutations of the coordinates, and therefore attains its maximum at the point where all coordinates are equal, see (11). For completeness, we briefly verify this using Lagrange multipliers.

Define

$$F(p_1, \dots, p_r, \lambda) = -\sum_{i=1}^r p_i \log p_i + \lambda\Big(\sum_{i=1}^r p_i - 1\Big).$$

At a critical point we have

$$\frac{\partial F}{\partial p_i} = -(\log p_i + 1) + \lambda = 0, \qquad i = 1, \dots, r,$$

so $\log p_i$ is constant in i, and hence $p_1 = \cdots = p_r = c$ for some constant $c > 0$. From $\sum_{i=1}^{r} p_i = 1$ we obtain $c = 1/r$, so the only critical point in Δ_r is the uniform distribution $(1/r, \ldots, 1/r)$, and

$$H\Big(\frac{1}{r}, \ldots, \frac{1}{r}\Big) = -\sum_{i=1}^{r} \frac{1}{r} \log \frac{1}{r} = \log r.$$

Since H is concave on Δ_r, this critical point yields the global maximum. It follows that for any probability vector supported on at most r points,

$$-\sum_{i=1}^{r} p_i \log p_i \leq \log r.$$

Applying this to the nonzero entries of (p_i), we get

$$\mathscr{S}(G) \leq \log r.$$

(3) Upper bound by $\log n$. Finally, since $r \leq n$, we obtain

$$\mathscr{S}(G) \leq \log r \leq \log n,$$

which completes the proof. □

Lemma 4.2. *Let G be a simple graph with at least one edge, and let A be its adjacency matrix. Then $\operatorname{rank}(A) \neq 1$.*

Proof. Suppose for a contradiction that $\operatorname{rank}(A) = 1$. Then there exist nonzero vectors $u, v \in \mathbb{R}^n$ such that $A = uv^\top$. Since A is symmetric, we also have $A^\top = A$, hence $uv^\top = (uv^\top)^\top = vu^\top$. In particular, for all i, j,

$$u_i v_j = u_j v_i.$$

If some $u_k \neq 0$, then for every j we obtain

$$v_j = \frac{u_j}{u_k} v_k,$$

so v is a scalar multiple of u, say $v = \alpha u$ with $\alpha \neq 0$. Thus $A = uv^\top = \alpha uu^\top$.

Now, for a simple graph, the diagonal entries of A are zero. On the other hand, the diagonal entries of $uu^\top$ are $(uu^\top)_{ii} = u_i^2$. Hence

$$0 = A_{ii} = \alpha u_i^2 \quad \text{for all } i.$$

Since $\alpha \neq 0$, this implies $u_i = 0$ for all i, so $u = 0$, which contradicts $\operatorname{rank}(A) = 1$.

Therefore $\operatorname{rank}(A) \neq 1$ for any simple graph with at least one edge. □

Remark 4.1. For a simple graph G with at least one edge, the spectral-entropy index $\mathscr{S}(G)$ is in fact *strictly* positive. Indeed, if $\mathscr{S}(G) = 0$, then the underlying probability vector $(p_1, \ldots, p_n)$ would have to be a Dirac mass, i.e. there exists k such that $p_k = 1$ and $p_i = 0$ for all $i \neq k$. In terms of the eigenvalues, this would mean that exactly one λ_k^2 is nonzero, so the adjacency matrix has exactly one nonzero eigenvalue and hence rank 1. This contradicts Lemma 4.2. Therefore $\mathscr{S}(G) > 0$ for every nonempty simple graph.

Proposition 4.1. *Let S_n be the star graph on $n \geq 2$ vertices. Then*

$$\mathscr{S}(S_n) = \log 2.$$

In particular, $r = 2$ and the upper bound $\mathscr{S}(G) \leq \log r$ from Theorem 4.1 is attained.

Proof. It is well known that the adjacency spectrum of S_n; see (2; 4) is

$$\operatorname{spec}(S_n) = \{\sqrt{n-1},\ -\sqrt{n-1},\ 0,\ldots,0\},$$

where 0 occurs with multiplicity $n-2$. Therefore

$$\lambda_1^2 = \lambda_2^2 = n-1, \qquad \lambda_3^2 = \cdots = \lambda_n^2 = 0,$$

and

$$\sum_{j=1}^{n} \lambda_j^2 = (n-1) + (n-1) = 2(n-1).$$

Thus

$$p_1 = p_2 = \frac{n-1}{2(n-1)} = \frac{1}{2}, \qquad p_3 = \cdots = p_n = 0.$$

The spectral-entropy index is therefore

$$\mathscr{S}(S_n) = -\Big(\tfrac{1}{2}\log\tfrac{1}{2} + \tfrac{1}{2}\log\tfrac{1}{2}\Big) = -\log\tfrac{1}{2} = \log 2.$$

Here $r = 2$ and $(p_1, p_2) = (1/2, 1/2)$ is the uniform distribution on two points, so the upper bound $\log r = \log 2$ is attained. □

Proposition 4.2. *Let $K_{p,q}$ be a complete bipartite graph with $p, q \geq 1$ and $p + q \geq 2$. Then*

$$\mathscr{S}(K_{p,q}) = \log 2.$$

Again $r = 2$ and the upper bound $\mathscr{S}(G) \leq \log r$ is attained.

Proof. The adjacency spectrum of $K_{p,q}$; see (2; 4) is

$$\operatorname{spec}(K_{p,q}) = \{\sqrt{pq},\ -\sqrt{pq},\ 0,\ldots,0\},$$

where 0 occurs with multiplicity $p + q - 2$. Hence

$$\lambda_1^2 = \lambda_2^2 = pq, \qquad \lambda_3^2 = \cdots = \lambda_{p+q}^2 = 0,$$

and

$$\sum_{j=1}^{p+q} \lambda_j^2 = pq + pq = 2pq.$$

Thus

$$p_1 = p_2 = \frac{pq}{2pq} = \frac{1}{2}, \qquad p_3 = \cdots = p_{p+q} = 0.$$

Therefore

$$\mathscr{S}(K_{p,q}) = -\Big(\tfrac{1}{2}\log\tfrac{1}{2} + \tfrac{1}{2}\log\tfrac{1}{2}\Big) = \log 2,$$

and, as in Proposition 4.1, the upper bound $\log r$ with $r = 2$ is attained. □

Proposition 4.3. *Let K_n be the complete graph on $n \geq 2$ vertices. Then*

$$\mathscr{S}(K_n) = \log n - \frac{n-2}{n}\log(n-1).$$

In particular, $\mathscr{S}(K_n) < \log n$ for all $n \geq 3$ and $\mathscr{S}(K_n) \to 0$ as $n \to \infty$.

Proof. The adjacency spectrum of K_n; see (2; 4) is

$$\operatorname{spec}(K_n) = \{n-1,\ -1, \dots, -1\},$$

where -1 appears with multiplicity $n-1$. Then

$$\lambda_1^2 = (n-1)^2, \qquad \lambda_2^2 = \cdots = \lambda_n^2 = 1,$$

and

$$\sum_{j=1}^{n} \lambda_j^2 = (n-1)^2 + (n-1)\cdot 1 = n(n-1).$$

Thus

$$p_1 = \frac{(n-1)^2}{n(n-1)} = \frac{n-1}{n}, \qquad p_2 = \cdots = p_n = \frac{1}{n(n-1)}.$$

The spectral-entropy index is

$$\mathscr{S}(K_n) = -\left(\frac{n-1}{n}\log\frac{n-1}{n} + (n-1)\cdot\frac{1}{n(n-1)}\log\frac{1}{n(n-1)}\right).$$

Simplifying,

$$\begin{aligned}
\mathscr{S}(K_n) &= -\frac{1}{n}\left((n-1)\log\frac{n-1}{n} + \log\frac{1}{n(n-1)}\right)\\
&= -\frac{1}{n}\left((n-1)(\log(n-1)-\log n) - \log n - \log(n-1)\right)\\
&= -\frac{1}{n}\Big((n-2)\log(n-1) - n\log n\Big)\\
&= \log n - \frac{n-2}{n}\log(n-1).
\end{aligned}$$

For $n \geq 3$ we have $n-1 > 1$, so $\log(n-1) > 0$ and hence $\frac{n-2}{n}\log(n-1) > 0$, which implies $\mathscr{S}(K_n) < \log n$. Moreover, using asymptotics $\log(n-1) \sim \log n$ as $n \to \infty$, we obtain

$$\mathscr{S}(K_n) = \log n - \frac{n-2}{n}\log(n-1) \longrightarrow 0 \quad \text{as } n \to \infty.$$

□

4.1 Examples and extremal behaviour

In this subsection we collect further examples of the spectral-entropy index for standard graph families. In contrast to the star, complete bipartite, and complete graphs, we do not obtain simple closed forms in all cases, but the explicit spectral representation is still useful for analysis and numerical experiments.

Proposition 4.4. *Let C_n be the cycle graph on $n \geq 3$ vertices. Then the adjacency eigenvalues of C_n are*

$$\lambda_k = 2\cos\Big(\frac{2\pi k}{n}\Big), \qquad k = 0, 1, \dots, n-1,$$

and the spectral-entropy index is given by

$$\mathscr{S}(C_n) = -\sum_{k=0}^{n-1} \frac{\lambda_k^2}{2n}\log\Big(\frac{\lambda_k^2}{2n}\Big).$$

In particular, the nonzero p_i correspond to the nonzero eigenvalues, and the number of nonzero eigenvalues satisfies $r = n$ if n is odd and $r = n-2$ if n is even.

Proof. It is well known; see (2; 4) that the adjacency eigenvalues of the cycle C_n are

$$\lambda_k = 2\cos\Big(\frac{2\pi k}{n}\Big), \qquad k = 0, 1, \ldots, n-1.$$

The cycle has $m = n$ edges, so

$$\sum_{k=0}^{n-1} \lambda_k^2 = \mathrm{tr}(A^2) = 2m = 2n.$$

Hence

$$p_k = \frac{\lambda_k^2}{\sum_{j=0}^{n-1} \lambda_j^2} = \frac{\lambda_k^2}{2n}, \qquad k = 0, \ldots, n-1,$$

and therefore

$$\mathscr{S}(C_n) = -\sum_{k=0}^{n-1} \frac{\lambda_k^2}{2n} \log\Big(\frac{\lambda_k^2}{2n}\Big).$$

The number of nonzero eigenvalues equals the rank of the adjacency matrix of C_n. For odd n the spectrum contains no zero eigenvalues, so $r = n$. For even n, the values $k = n/2$ and $k = 0$ give $\lambda_k = \pm 2$ while $k = n/4$ and $k = 3n/4$ yield $\lambda_k = 0$, so there are exactly two zero eigenvalues and $r = n - 2$. □

Proposition 4.5. *Let P_n be the path graph on $n \geq 2$ vertices. Then the adjacency eigenvalues of P_n are*

$$\lambda_k = 2\cos\Big(\frac{\pi k}{n+1}\Big), \qquad k = 1, 2, \ldots, n,$$

and the spectral-entropy index is

$$\mathscr{S}(P_n) = -\sum_{k=1}^{n} \frac{\lambda_k^2}{2(n-1)} \log\Big(\frac{\lambda_k^2}{2(n-1)}\Big).$$

Moreover, the number of nonzero eigenvalues satisfies $r = n$ when n is even and $r = n - 1$ when n is odd.

Proof. The adjacency eigenvalues of the path P_n are; see (2; 4)

$$\lambda_k = 2\cos\Big(\frac{\pi k}{n+1}\Big), \qquad k = 1, 2, \ldots, n.$$

The path has $m = n - 1$ edges, so

$$\sum_{k=1}^{n} \lambda_k^2 = \mathrm{tr}(A^2) = 2m = 2(n-1).$$

Hence

$$p_k = \frac{\lambda_k^2}{\sum_{j=1}^{n} \lambda_j^2} = \frac{\lambda_k^2}{2(n-1)}, \qquad k = 1, \ldots, n,$$

and thus

$$\mathscr{S}(P_n) = -\sum_{k=1}^{n} \frac{\lambda_k^2}{2(n-1)} \log\Big(\frac{\lambda_k^2}{2(n-1)}\Big).$$

The number of nonzero eigenvalues again equals the rank of the adjacency matrix. The eigenvalue λ_k vanishes exactly when

$$2\cos\Big(\frac{\pi k}{n+1}\Big) = 0 \iff \frac{\pi k}{n+1} = \frac{\pi}{2} \iff k = \frac{n+1}{2}.$$

Thus a zero eigenvalue occurs only when $n + 1$ is even, i.e. when n is odd. In that case exactly one eigenvalue is zero and $r = n - 1$; when n is even, all λ_k are nonzero and $r = n$. □

Remark 4.2. It is of independent interest to understand the typical size of the spectral-ent is the Erdős–Rényi random graph $G(n,p)$, where each edge appears independently with probability $p \in (0,1)$.

For $G(n,p)$, the adjacency matrix A has one leading eigenvalue λ_1 of order np, while the remaining eigenvalues are of order $\sqrt{n}$ and form a bulk described asymptotically by a suitable limiting spectral distribution (after centering), see for example (2; 12) and references therein. The sum of squared eigenvalues satisfies

$$\sum_{i=1}^{n} \lambda_i^2 = \mathrm{tr}(A^2) = 2m,$$

where m is the random number of edges, with $\mathbb{E}[m] = \frac{1}{2}pn(n-1)$. Heuristically, one expects the contribution of the leading eigenvalue λ_1 to account for a nontrivial fraction of $\sum_i \lambda_i^2$, while the remaining eigenvalues share the rest of the mass in a more spread-out fashion. Consequently, the spectral-entropy index $\mathscr{S}(G(n,p))$ is expected to grow on the order of $\log n$ with a prefactor strictly less than 1, reflecting a mixture of a "dominant" direction (encoded by λ_1) and a roughly uniform bulk of smaller eigenvalues.

A precise asymptotic characterization of $\mathscr{S}(G(n,p))$ would require detailed control of the empirical distribution of the squared eigenvalues and lies beyond the scope of this paper. We therefore leave a rigorous analysis of $\mathscr{S}(G(n,p))$ as an interesting topic for future work.

5 Behavior Under Graph Operations

We briefly summarize several qualitative effects of standard graph operations on the spectral-entropy index.

- **Disjoint union:** for disjoint graphs G_1 and G_2, the spectrum of $G_1 \cup G_2$ is the multiset union of the spectra of G_1 and G_2. Consequently, $\mathscr{S}(G_1 \cup G_2)$ can be expressed as a weighted combination of the spectral-entropy contributions of the components.
- **Join:** the join operation tends to create dominant eigenvalues, concentrating spectral mass and often decreasing the entropy, in analogy with the behaviour observed for complete and complete bipartite graphs.
- **Complement:** the complement operation typically redistributes spectral mass but does not drastically increase entropy; in many cases it preserves the order of magnitude of $\mathscr{S}(G)$.

6 Applications to Computer Science

In this section we discuss several domains within computer science in which the spectral-entropy index $\mathscr{S}(G)$ provides meaningful structural information. Many systems in computing like communication networks, data structures, distributed systems, and algorithmic pipelines etc. admit natural graph representations, and their behaviour is often governed by global or local connectivity patterns. The index $\mathscr{S}(G)$ summarizes the dispersion of the adjacency spectrum and therefore captures both regularity and irregularity in such patterns. This makes it a useful tool in settings that require quantifying complexity, heterogeneity, robustness, or anomalous structure.

6.1 Network anomaly detection and intrusion analysis

In network security, suspicious activity often manifests as structural irregularities in the network graph. Examples include sudden increases in connectivity (e.g., botnet formation), local clustering anomalies (e.g., malware-controlled subgraphs), or abrupt degree shifts due to compromise of devices, see (1; 8; 5; 9).

The spectral-entropy index is sensitive to such phenomena. A localized structural change typically introduces new eigenvalues or distorts the existing spectrum, thereby redistributing the spectral weights p_i. Since $\mathscr{S}(G) = -\sum_i p_i \log p_i$ increases when the distribution becomes more spread out, one may detect deviations by monitoring the temporal evolution of $\mathscr{S}(G_t)$ for a dynamic network $\{G_t\}$. The lower bounds of Theorem 4.1 and Remark 4.1 ensure that any nontrivial change must induce a strictly positive increment in entropy.

6.2 Robustness and centralization in distributed systems

Distributed systems including peer to peer overlays, blockchain networks, and distributed hash tables etc. rely heavily on balanced connectivity to avoid bottlenecks and single points of failure. When the underlying communication graph becomes overly centralized, performance and fault tolerance degrade; see (13; 6; 14; 3; 7).

Entropy provides a natural way to detect centralization. For instance, Proposition 4.1 shows that highly centralized architectures (e.g. the star) have low spectral entropy $\mathscr{S}(G) = \log 2$, reflecting dominance by only two spectral components. More distributed structures, which correspond to a wider range of eigenvalues, typically yield higher entropy. Consequently, $\mathscr{S}(G)$ can be used as a quantitative indicator of network balance and structural robustness.

6.3 Wireless sensor networks (WSN)

Wireless sensor networks exhibit dynamic topologies due to mobility, sleep scheduling, failures, and environmental constraints. The spectral entropy offers insight into two key aspects:

- **Coverage and redundancy.** Uniform or near-uniform connectivity across the network leads to a broader spectral distribution and therefore higher entropy, whereas sparse or clustered deployments suppress most eigenvalues and reduce entropy.
- **Fault tolerance.** A drop in $\mathscr{S}(G)$ may indicate that the failure of sensors has induced centralized routing patterns or fragmented connectivity, making the network more vulnerable. This observation aligns with the behaviour of $\mathscr{S}(K_n)$ in Proposition 4.3, which decreases as the graph becomes denser but less diverse in structure.

Entropy-based monitoring can thus complement traditional WSN metrics such as node degree, residual energy, or coverage ratio.

6.4 Graph-based machine learning and data mining

Many clustering, classification, and ranking algorithms in data mining operate on graphs—social networks, web graphs, citation networks, and knowledge graphs. These methods often rely on spectral information, e.g. in spectral clustering, graph Laplacians, or message-passing neural networks.

The spectral-entropy index provides a summary statistic indicating the complexity of a dataset's connectivity pattern:

- **High entropy** indicates diversified structural features, useful for detecting heterogeneous communities or multi-scale clusters.
- **Low entropy** suggests the presence of dominant clusters, strongly connected hubs, or a few high-energy eigenmodes.

Since $\mathscr{S}(G)$ is easy to compute from the spectrum (and may be approximated efficiently via Krylov-subspace methods), it serves as a practical feature for large-scale mining tasks.

6.5 Algorithmic complexity and structural randomness

The adjacency spectrum of a graph is closely related to various notions of algorithmic and combinatorial complexity. Random-like structures, as in Erdős–Rényi graphs, tend to distribute their eigenvalues in a broad range. As discussed in Remark 4.2, this results in spectral entropy that typically grows on the order of $\log n$, reflecting high structural diversity.

Conversely, highly structured graphs such as cliques, complete bipartite graphs, or stars have spectra concentrated in few eigenvalues and therefore low entropy (see Propositions 4.1, 4.2, and 4.3).

Thus, $\mathscr{S}(G)$ serves as a compact descriptor of structural randomness and may be used to characterize complexity in algorithmic graph models, randomized algorithms, or graph compression schemes.

6.6 Web graphs and blockchain style peer to peer networks

Large-scale web graphs and blockchain-style peer-to-peer (P2P) networks are canonical examples of complex networks arising in computer science. In both settings, nodes represent logical entities (web pages, user accounts, or network peers), while edges represent hyperlinks, transactions, or communication channels. The resulting graphs are often highly irregular, with heavy-tailed degree distributions and dynamically evolving structures.

Web graphs

A web graph typically exhibits a mixture of hub like pages (e.g. search engines, portals) and numerous low degree pages. This heterogeneity is reflected in the adjacency spectrum: a few large eigenvalues associated with hub-like connectivity and a bulk of smaller eigenvalues that encode local and meso-scale structure such as communities or topic clusters.

The spectral-entropy index $\mathscr{S}(G)$ naturally captures this mixture. If the web graph becomes more centralized, for example due to the domination of a few very popular sites or the collapse of peripheral regions, the squared eigenvalue distribution (p_i) becomes more concentrated, leading to a decrease in $\mathscr{S}(G)$. Conversely,when the web graph becomes more diverse, for instance, through the emergence of multiple comparable hubs or the growth of distinct communities, the spectrum tends to spread and $\mathscr{S}(G)$ increases.

From a practical viewpoint, $\mathscr{S}(G)$ can be used as a global descriptor of *structural diversity* in web graphs. Monitoring the temporal evolution of $\mathscr{S}(G_t)$ for a sequence of crawled web snapshots $\{G_t\}$ may reveal major shifts in the ecosystem, such as the appearance of new large platforms, fragmentation of existing communities, or abnormal link-farming behaviour.

Blockchain and transaction networks

Transaction graphs and blockchain-style P2P overlays also admit graph-theoretic representations. In a transaction graph, vertices represent accounts or addresses and edges represent transfers of digital assets. In a P2P overlay, vertices represent peers and edges represent direct communication or neighbour relations.

Two structural aspects are particularly relevant in this context:

- **Decentralization.** A highly decentralized network should avoid a small set of nodes controlling a disproportionate share of connectivity. If a transaction or P2P graph becomes effectively star-like or core-periphery in structure, the spectral mass concentrates in a few eigenvalues, and the spectral entropy decreases toward values comparable to those of S_n or $K_{p,q}$, cf. Propositions 4.1 and 4.2.

- **Anomalous substructures.** Illicit activity (e.g. mixing services, wash trading, or Sybil attacks) often generates subgraphs with atypical connectivity patterns. Such subgraphs perturb the spectrum, redistributing the weights p_i and thus changing $\mathscr{S}(G)$. A sudden increase or decrease in $\mathscr{S}(G_t)$ over time may serve as a signal for the onset of anomalous behaviour requiring further investigation.

In both web and blockchain scenarios, the spectral-entropy index $\mathscr{S}(G)$ can therefore be interpreted as a quantitative indicator of decentralization and structural complexity. Its definition in terms of the eigenvalues of the adjacency matrix makes it compatible with existing spectral algorithms used for ranking, community detection, and robustness analysis, and allows efficient approximation on large graphs via standard numerical linear algebra techniques.

7 CONCLUSIONS

We introduced the spectral-entropy topological index $\mathscr{S}(G)$, defined as the Shannon entropy of the squared adjacency eigenvalues, and showed that it provides a compact and interpretable measure of global structural complexity in graphs. We established general bounds, proved that $\mathscr{S}(G)$ is strictly positive for every nonempty simple graph, and computed its value for several canonical graph families, including stars, complete bipartite graphs, and complete graphs, along with spectral representations for cycles and paths. These examples highlight the ability of $\mathscr{S}(G)$ to differentiate highly centralized, highly regular, and more irregular structures.

We briefly commented on the behaviour of $\mathscr{S}(G)$ in Erdős–Rényi random graphs, offering heuristic insight based on their well-known spectral structure and demonstrated that the index has natural applications in multiple areas of computer science, such as network anomaly detection, distributed-system robustness, wireless sensor networks, graph-based data mining, and the study of web and blockchain-style peer-to-peer networks.

Possible directions for future work include sharper extremal characterizations, a rigorous analysis for random graph models, empirical evaluation in machine-learning pipelines and anomaly-detection tasks, and extensions of the index to directed, weighted, and temporal graphs.

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