Original Research Article

# Bounds and Tree Structures for the Hyperbolic Sombor Index

## Abstract

For a simple graph $G$ with order $n$ and size $m$. We study the Hyperbolic Sombor index $HSO(G) = \sum_{uv \in E} \frac{\sqrt{d_u^2+d_v^2}}{\min\{d_u,d_v\}}$, where $d_u$ and $d_v$ are the degrees of vertices $u$ and $v$ respectively, a recently introduced degree-based topological index. We first establish several new bounds for $HSO(G)$ in terms of fundamental graph parameters, including the maximum and minimum degrees, the number of pendent vertices, vertex deletion processes and the relationship between a graph and its complement. In addition, we compute exact values of $HSO(G)$ for several well-known families, including Kragujevac trees, perfect binary trees, binary caterpillar trees and dendrimer trees.

*Keywords: Hyperbolic Sombor index; Kragujevac trees; Binary caterpillar trees; dendrimer trees.*
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# 1 Introduction

Chemical graph theory has become an essential tool for understanding the structural features of chemical compounds through graph theoretic descriptors known as topological indices. These indices translate molecular architecture into numerical values that correlate with physicochemical properties, biological activity or reactivity patterns. Among degree based indices, the recently introduced Sombor index have gained significant attention due to their strong discriminating power and simple analytical structure.

The *Sombor index* was introduced by Gutman (1) as a geometric degree based topological index defined for a simple graph $G$ as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2},$$

where $d_u$ and $d_v$ denote the degrees of the adjacent vertices $u$ and $v$. Since its introduction, the Sombor index has inspired various modifications and analytical investigations due to its strong correlation with molecular properties.

To incorporate nonlinear growth more effectively, Barman et al. (2) introduced the *Hyperbolic Sombor index*, which is defined for a graph $G$ as

$$HSO(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{\min\{d_u, d_v\}}.$$

In which it established its basic properties, derived initial bounds and demonstrated potential chemical applicability. Subsequent studies have strengthened this foundation through deeper analytical investigation. Albalahi et al. revisited the index by identifying and correcting earlier inaccuracies and introducing a complementary Diminished Sombor Index (3). In parallel, Das and Ahmad further corrected and refined earlier results while providing sharper and more general bounds for trees, unicyclic and bicyclic graphs (4). Collectively, these works underscore the mathematical richness of the Hyperbolic Sombor index and highlight the need for additional structural results, bounds and closed analytical formulas for broader graph families.

Throughout this work, we adopt the standard terminology, notation and structural conventions of graph theory as formulated in (5).

# 2 Bounds of Hyperbolic Sombor index

In this section, we derive basic bounds for the Hyperbolic Sombor index $HSO(G)$. The derived results include bounds expressed in terms of the maximum and minimum degrees, the number of pendent vertices and the effect of vertex deletion.

**Theorem 2.1.** *Let $G(V, E)$ be a simple conneced graph of order $n$, size $m$, minimum degree $\delta$ and maximum degree $\Delta$. Then*

$$HSO(G) \leq m\sqrt{1 + \frac{\Delta^2}{\delta^2}}.$$

*Proof.* Let $uv \in E(G)$ be any edge and assume without loss of generality that $d_u \leq d_v$,
so $min\{d_u, d_v\} = d_u$.
Define

$$a_i = 1 \quad \text{and} \quad b_i = \frac{\sqrt{d_u^2 + d_v^2}}{\min\{d_u, d_v\}} = \frac{\sqrt{d_u^2 + d_v^2}}{d_u}.$$

By the Cauchy Schwarz inequality, we have

$$\left( \sum_{uv \in E} \frac{\sqrt{d_u^2 + d_v^2}}{\min\{d_u, d_v\}} \right)^2 \leq \left( \sum_{uv \in E} 1^2 \right) \left( \sum_{uv \in E} \frac{d_u^2 + d_v^2}{d_u^2} \right) = m \sum_{uv \in E} \left( 1 + \frac{d_v^2}{d_u^2} \right),$$

Since $\delta \leq d_u \leq d_v \leq \Delta$ for every edge $uv \in E$, it follows that

$$\frac{d_v^2}{d_u^2} \leq \frac{\Delta^2}{\delta^2}.$$

Hence,

$$\sum_{uv \in E} \left( 1 + \frac{d_v^2}{d_u^2} \right) \leq \sum_{uv \in E} \left( 1 + \frac{\Delta^2}{\delta^2} \right) = m \left( 1 + \frac{\Delta^2}{\delta^2} \right).$$

Combining the above inequalities yields

$$HSO(G) = \sum_{uv \in E} \frac{\sqrt{d_u^2 + d_v^2}}{\min\{d_u, d_v\}} \leq m\sqrt{1 + \frac{\Delta^2}{\delta^2}}.$$

Hence, the theorem is proved. □

**Theorem 2.2.** *Let $G(V, E)$ be a simple connected graph of order $n$, size $m$ and $l$ pendant vertices. Then*

$$HSO(G) \geq \sqrt{2}m + (\sqrt{5} - \sqrt{2})l.$$

*Proof.* Let $G = (V, E)$ be simple connected graph with $l$ pendant vertices. The contribution of edges to the Hyperbolic Sombor index $HSO(G)$ can be classified into two categories:

- Edges incident to a pendant vertex.
- Edges whose both endpoints have degree at least two.

**Case 1:** Let $uv \in E(G)$ be an edge such that $d_u = 1$ and $d_v \geq 2$. Then the corresponding term in $HSO(G)$ is

$$\frac{\sqrt{1^2 + d_v^2}}{\min\{1, d_v\}} = \sqrt{1 + d_v^2} \geq \sqrt{1 + 2^2} = \sqrt{5}.$$

Since there are exactly $l$ pendant vertices, so $l$ such edges. Hence their total contribution to $HSO(G)$ is at least $l\sqrt{5}$.

**Case 2:** Let $uv \in E(G)$ be an edge such that $d_u, d_v \geq 2$. Let $m' = \min\{d_u, d_v\}$, so that $d_u, d_v \geq m'$. Then

$$\sqrt{d_u^2 + d_v^2} \geq m'\sqrt{2}.$$

Thus, for each such edge,

$$\frac{\sqrt{d_u^2 + d_v^2}}{\min\{d_u, d_v\}} \geq \frac{m'\sqrt{2}}{m'} = \sqrt{2}.$$

Since there are $m - l$ edges of this type, their total contribution to $HSO(G)$ is at least $(m - l)\sqrt{2}$. Combining the contributions from both cases, we obtain

$$HSO(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{\min\{d_u, d_v\}} \geq l\sqrt{5} + (m - l)\sqrt{2} = \sqrt{2}\,m + (\sqrt{5} - \sqrt{2})\,l.$$

Thus, we arrive at the required result. □

This provides a stronger lower bound than the classical bound $m\sqrt{2}$, expressed in terms of the number of pendant vertices.

**Theorem 2.3.** *Let $G$ be a simple connected graph and let $v \in V(G)$ be a vertex of degree $d_v = k$. Then the Hyperbolic Sombor index satisfies*

$$k\sqrt{2} \leq HSO(G) - HSO(G - v) \leq k \cdot \frac{\sqrt{\Delta^2 + k^2}}{\min\{\Delta, k\}},$$

*Proof.* From the definition of the Hyperbolic Sombor index,

$$HSO(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{\min\{d_u, d_v\}}.$$

Deleting the vertex $v$ removes precisely the $k$ edges incident with $v$. Hence,

$$HSO(G) - HSO(G - v) = \sum_{u \in N_G(v)} \frac{\sqrt{d_u^2 + d_v^2}}{\min\{d_u, d_v\}}. \tag{1}$$

**Lower bound.** For every neighbor $u \in N_G(v)$,

$$d_u \geq 1, \qquad d_v = k \geq 1.$$

Therefore

$$\sqrt{d_u^2 + k^2} \ \geq \ \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Also,

$$\min\{d_u, k\} \leq k.$$

The expression

$$\frac{\sqrt{d_u^2 + k^2}}{\min\{d_u, k\}}$$

attains its minimum value when $d_u = k = 1$, yielding $\sqrt{2}$. Thus each removed edge contributes at least $\sqrt{2}$, and since there are $k$ such edges,

$$HSO(G) - HSO(G - v) \ \geq \ k\sqrt{2}.$$

**Upper bound.** Since $d_u \leq \Delta$ for all $u \in N_G(v)$,

$$\sqrt{d_u^2 + k^2} \ \leq \ \sqrt{\Delta^2 + k^2}.$$

Also,

$$\min\{d_u, k\} \geq \min\{\Delta, k\}.$$

Hence, for each term in (1),

$$\frac{\sqrt{d_u^2 + k^2}}{\min\{d_u, k\}} \ \leq \ \frac{\sqrt{\Delta^2 + k^2}}{\min\{\Delta, k\}}.$$

Summing over all $k$ neighbors of $v$ gives the desired upper bound:

$$HSO(G) - HSO(G - v) \ \leq \ k \cdot \frac{\sqrt{\Delta^2 + k^2}}{\min\{\Delta,\, k\}}.$$

This completes the proof. □

**Theorem 2.4.** *Let $G$ be a simple graph on $n$ vertices with degree sequence $d_1, d_2, \ldots, d_n$, and let $\overline{G}$ denote its complement. Then the sum of the Hyperbolic Sombor indices of $G$ and $\overline{G}$ satisfies*

$$HSO(G) + HSO(\overline{G}) \ \geq \ \binom{n}{2}\sqrt{2},$$

*Proof.* The Hyperbolic Sombor index of a graph $G$ is defined as

$$HSO(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{\min\{d_u, d_v\}}.$$

For every edge $uv \in E(G)$,

$$\frac{\sqrt{d_u^2 + d_v^2}}{\min\{d_u, d_v\}} \ \geq \ \frac{\sqrt{1^2 + 1^2}}{1} = \sqrt{2}.$$

Thus,

$$HSO(G) \geq m\sqrt{2}. \tag{1}$$

Now consider the complement graph $\overline{G}$. For each vertex $v$,

$$d_{\overline{G}}(v) = n - 1 - d_v.$$

Applying the definition,

$$HSO(\overline{G}) = \sum_{uv \in E(\overline{G})} \frac{\sqrt{d_{\overline{G}}(u)^2 + d_{\overline{G}}(v)^2}}{\min\{d_{\overline{G}}(u), d_{\overline{G}}(v)\}}.$$

Since every vertex in a simple graph has degree at least $0$ and at most $n-1$, for each edge of the complement we still have:

$$\frac{\sqrt{d_{\overline{G}}(u)^2 + d_{\overline{G}}(v)^2}}{\min\{d_{\overline{G}}(u), d_{\overline{G}}(v)\}} \;\geq\; \sqrt{1^2+1^2} = \sqrt{2},$$

because $\overline{G}$ has no isolated edges unless $G$ has universal vertices. Hence,

$$HSO(\overline{G}) \geq m_{\overline{G}}\sqrt{2}. \tag{2}$$

Adding inequalities (1) and (2) gives

$$HSO(G) + HSO(\overline{G}) \;\geq\; m\sqrt{2} + m_{\overline{G}}\sqrt{2}.$$

Since

$$m + m_{\overline{G}} = \binom{n}{2},$$

we obtain

$$HSO(G) + HSO(\overline{G}) \;\geq\; \binom{n}{2}\sqrt{2}.$$

This completes the proof. □

# 3 Hyperbolic Sombor index of Trees

In this section, we investigate the explicit computation of the Hyperbolic Sombor index for several classical families of trees. In particular, we establish closed formulas for Kragujevac trees, perfect binary trees, binary caterpillar trees, dendrimer trees and related structures. Throughout this section, we adopt the notation $(u, v)$-edge to denote an edge whose end vertices have degree $u$ and $v$, respectively.

**Definition 3.1.** (6) Let $k_1, k_2, \ldots, k_n$ be integers satisfying $0 \leq k_1 \leq k_2 \leq \cdots \leq k_n$. The *Kragujevac tree*, denoted by $Kg(k_1, k_2, \ldots, k_n)$, is defined as follows:

1. For each $k_i$, construct the rooted tree $B_{k_i}$ with $2k_i+1$ vertices, obtained by attaching $k_i$ copies of a two-vertex branch to the root.
2. Introduce a new central vertex and join it with the root of each $B_{k_i}$.

The resulting graph is a rooted tree with the new central vertex as the main root.

**Theorem 3.1.** *Let $Kg(k_1, k_2, \ldots, k_n)$ is the Kragujevac tree. Then*

$$HSO(Kg(k_1, k_2, \ldots, k_n)) = \sqrt{5}\,k + \sum_{i=1}^{n}\left(\frac{k_i}{2}\sqrt{(k_i+1)^2+4} + \frac{\sqrt{(k_i+1)^2+n^2}}{\min\{k_i+1,\, n\}}\right),$$

*where $k = k_1 + k_2 + \cdots + k_n$.*

*Proof.* Let $Kg = Kg(k_1, k_2, \ldots, k_n)$ denote the Kragujevac tree. If $k = k_1 + k_2 + \cdots + k_n$, then the graph $Kg$ contains $2k+n+1$ vertices. Furthermore, it has $k$ edges of type $(1, 2)$, exactly $k_i$ edges of type $(k_i+1,2)$ for each $i=1,2,\ldots,n$, and one edge of type $(k_i+1,n)$ for every $i=1,2,\ldots,n$. Hence, the Hyperbolic Sombor index of $Kg(k_1,k_2,\ldots,k_n)$ is

$$\begin{aligned}
HSO(Kg) &= \sum_{(u,v)-edge} \frac{\sqrt{d_u^2+d_v^2}}{\min\{d_u,d_v\}} \\
&= \sum_{(1,2)-edge} \sqrt{1^2+2^2} + \sum_{i=1}^{n} \sum_{(k_i+1,2)-edges} \frac{\sqrt{(k_i+1)^2+2^2}}{2} \\
&\quad + \sum_{i=1}^{n} \sum_{(k_i+1,2)-edges} \frac{\sqrt{(k_i+1)^2+n^2}}{\min\{k_i+1,n\}} \\
&= \sqrt{5}k + \sum_{i=1}^{n} \frac{k_i}{2}\sqrt{(k_i+1)^2+4} + \sum_{i=1}^{n} \frac{\sqrt{(k_i+1)^2+n^2}}{\min\{k_i+1,n\}} \\
&= \sqrt{5}k + \sum_{i=1}^{n} \Big(\frac{k_i}{2}\sqrt{(k_i+1)^2+4} + \frac{\sqrt{(k_i+1)^2+n^2}}{\min\{k_i+1,n\}}\Big).
\end{aligned}$$

This completes the proof. □

**Definition 3.2.** (5) A *binary tree* is a rooted tree with at least three vertices in which exactly one vertex, called the root, has degree two, while all other vertices have degree either one (leaves) or three (internal nodes). The root is considered to be at level $0$. A vertex $v_i$ is said to be at level $k_i$ if the distance between $v_i$ and the root is $k_i$.

**Definition 3.3.** (5) A *perfect binary tree* $T_k$ of height $k$ is a binary tree in which all vertices occur up to level $k$, and every level contains the maximum possible number of vertices. In particular, all internal vertices have exactly two children and all leaf vertices are at the same level $k$.

**Theorem 3.2.** *Let $T_k$ be a perfect binary tree of height $k$. Then*

$$HSO(T_k) = \sqrt{13} - 4\sqrt{2} + 2^k\big(\sqrt{2}+\sqrt{10}\big).$$

*Proof.* Let $T_k$ be a perfect binary tree of height $k$ with root vertex $r$. The tree $T_k$ contains $2^{k+1}-1$ vertices and consequently, $2^{k+1}-2$ edges. It has $2$ edges of type $(2,3)$, $2^k-4$ edges of type $(3,3)$ and $2^k$ edges of type $(3,1)$. Hence, the Hyperbolic Sombor index of $T_k$ is

$$\begin{aligned}
HSO(T_k) &= \sum_{(u,v)-edge} \frac{\sqrt{d_u^2+d_v^2}}{\min\{d_u,d_v\}} \\
&= \sum_{(2,3)-edges} \frac{\sqrt{2^2+3^2}}{2} + \sum_{(3,3)-edges} \frac{\sqrt{3^2+3^2}}{3} \\
&\quad + \sum_{(3,1)-edges} \sqrt{3^2+1} \\
&= \sqrt{13} + (2^k-4)\sqrt{2} + 2^k\sqrt{10} \\
&= \sqrt{13} - 4\sqrt{2} + 2^k(\sqrt{2}+\sqrt{10}).
\end{aligned}$$

Thus, the desired result is established. □

**Definition 3.4.** (7) A *binary caterpillar tree* $T$ is a specialized form of a caterpillar tree. In a caterpillar tree, all vertices are either on a central path, called the *spine* or are directly connected to a vertex on this spine. In a binary caterpillar tree, every internal vertex on the spine has at most two children, so the tree satisfies the properties of a binary tree. Consequently, each internal vertex is either a leaf or has exactly two children.

**Theorem 3.3.** *Let $T$ be a binary caterpillar tree on $n$ vertices with height $k$. Then*

$$HSO(T) = \sqrt{5} + \tfrac{\sqrt{13}}{2} - 2\sqrt{2} + k\big(\sqrt{2} + \sqrt{10}\big).$$

*Proof.* Let $T$ be a binary caterpillar tree of $n$ vertices and height $k$. Then $T$ has $2k+1$ vertices and $2k$ edges, consisting of one edge of type $(1,2)$, one edge of type $(2,3)$, $k$ edges of type $(1,3)$ and $k-2$ edges of type $(3,3)$.
Hence, the Hyperbolic Sombor index of $T$ is

$$\begin{aligned}
HSO(T) &= \sum_{(u,v)-edge} \frac{\sqrt{d_u^2 + d_v^2}}{\min\{d_u, d_v\}} \\
&= \sum_{(1,2)-edge} \sqrt{1+2^2} + \sum_{(2,3)-edge} \frac{\sqrt{2^2+3^2}}{2} \\
&\quad + \sum_{(3,1)-edges} \sqrt{3^2+1} + \sum_{(3,3)-edges} \frac{\sqrt{3^2+3^2}}{3} \\
&= \sqrt{5} + \frac{\sqrt{13}}{2} + k\sqrt{10} + (k-2)\sqrt{2} \\
&= \sqrt{5} + \frac{\sqrt{13}}{2} - 2\sqrt{2} + k(\sqrt{2} + \sqrt{10}).
\end{aligned}$$

Hence, the desired result follows. □

**Definition 3.5.** (8) For a positive integer $k$, two special families of trees, denoted by $A_{2k}$ and $A_{2k+1}$, are constructed recursively. The tree $A_{2k+1}$ is derived from $A_{2k-1}$, while $A_{2k}$ is derived from $A_{2k-2}$, according to the following rule: three pendant vertices (one in the horizontal direction and two in the vertical direction) are attached to the extreme leftmost and rightmost pendant vertices of the preceding tree. Furthermore, each of the remaining pendant vertices of $A_{2k-1}$ (or $A_{2k-2}$) receives one additional pendant vertex attached vertically.

Note that:

1. In $A_{2k}$, there are $2(2k-1)$ pendant vertices and $(2k-2)$ vertices of degree $4$.
2. In $A_{2k+1}$, there are $4k$ pendant vertices and $(2k-1)$ vertices of degree $4$.

**Theorem 3.4.** *Let $A_{2k}$ be the tree. Then*

$$HSO(A_{2k}) = 6\sqrt{17} + (8k-16)\sqrt{5} + (2k^2 - 8k + 9)\sqrt{2}.$$

*Proof.* Let $k$ be any positive integer then, the tree $A_{2k}$ contains $6$ edges of type $(1,4)$, $4k-8$ edges of type $(1,2)$, $2k-3$ edges of type $(4,4)$, $2(2k-4)$ edges of type $(2,4)$ and $2(k-2)(k-3)$ edges of type $(2,2)$.

Hence, the Hyperbolic Sombor index of tree $A_{2k}$ is

$$\begin{aligned} HSO(A_{2k}) &= \sum_{(u,v)-edge} \frac{\sqrt{d_u^2+d_v^2}}{\min\{d_u,d_v\}} \\ &= \sum_{(1,4)-edges} \sqrt{1+4^2} + \sum_{(1,2)-edges} \sqrt{1+2^2} \\ &\quad + \sum_{(4,4)-edges} \frac{\sqrt{4^2+4^2}}{4} + \sum_{(2,4)-edges} \frac{\sqrt{2^2+4^2}}{2} \\ &\quad + \sum_{(2,2)-edges} \frac{\sqrt{2^2+2^2}}{2} \\ &= 6\sqrt{17} + (4k-8)\sqrt{5} + (2k-3)\sqrt{2} + (2k-4)\sqrt{20} \\ &\quad + (k-2)(k-3)\sqrt{8} \\ &= 6\sqrt{17} + (8k-16)\sqrt{5} + (2k^2-8k+9)\sqrt{2}. \end{aligned}$$

This concludes the proof. □

**Theorem 3.5.** *Let $A_{2k+1}$ be the tree. Then*

$$HSO(A_{2k+1}) = 6\sqrt{17} + (8k-12)\sqrt{5} + (2k^2-6k+6)\sqrt{2}.$$

*Proof.* Let $k$ be any positive integer then, the tree $A_{2k+1}$ contains $6$ edges of type $(1,4)$, $4k-6$ edges of type $(1,2)$, $2k-2$ edges of type $(4,4)$, $2(2k-3)$ edges of type $(2,4)$ and $2(k-2)^2$ edges of type $(2,2)$.
Hence, the Hyperbolic Sombor index of tree $A_{2k+1}$ is

$$\begin{aligned} HSO(A_{2k+1}) &= \sum_{(u,v)-edge} \frac{\sqrt{d_u^2+d_v^2}}{\min\{d_u,d_v\}} \\ &= \sum_{(1,4)-edges} \sqrt{1+4^2} + \sum_{(1,2)-edges} \sqrt{1+2^2} \\ &\quad + \sum_{(4,4)-edges} \frac{\sqrt{4^2+4^2}}{4} + \sum_{(2,4)-edges} \frac{\sqrt{2^2+4^2}}{2} \\ &\quad + \sum_{(2,2)-edges} \frac{\sqrt{2^2+2^2}}{2} \\ &= 6\sqrt{17} + (4k-6)\sqrt{5} + (2k-2)\sqrt{2} + (2k-3)\sqrt{20} \\ &\quad + 2(k-2)^2\sqrt{2} \\ &= 6\sqrt{17} + (8k-12)\sqrt{5} + (2k^2-6k+6)\sqrt{2}. \end{aligned}$$

Thus, we arrive at the required result. □

**Definition 3.6.** (9) A *dendrimer tree* $T_{k,d}$ is a highly regular branched structure with parameters $k \geq 0$ and $d \geq 3$. For any $d \geq 3$, $T_{0,d}$ is the one-vertex tree and $T_{1,d}$ is the star graph $S_d$. For $k = 2, 3, \ldots$, the tree $T_{k,d}$ is obtained from $T_{k-1,d}$ by attaching $d-1$ new vertices of degree one to each pendant vertex of $T_{k-1,d}$.

**Theorem 3.6.** *Let $T_{k,d}$ be a dendrimer tree. Then*

$$HSO(T_{k,d}) = d(d-1)^k \left( \frac{\sqrt{2}}{d-2} + \sqrt{d^2+1} \right) - \frac{\sqrt{2}d}{d-2}.$$

*Proof.* Let $T_{k,d}$ be a dendrimer tree with $k \geq 0$ and $d \geq 3$. The number of vertices in $T_{k,d}$ is

$$1 + d + d(d-1) + d(d-1)^2 + \ldots + d(d-1)^{k-1} = 1 + \frac{d}{d-2}\big((d-1)^k - 1\big)$$

and the number of edges is $\frac{d}{d-2}\big((d-1)^k - 1\big)$.
In $T_{k,d}$, there are $\frac{d}{d-2}((d-1)^{k-1}-1)$ edges of type $(d,d)$ and $d(d-1)^{k-1}$ edges of type $(1,d)$. Hence, the Hyperbolic Sombor index of tree $T_{k,d}$ is

$$\begin{aligned} HSO(T_{k,d}) &= \sum_{(u,v)-edge} \frac{\sqrt{d_u^2 + d_v^2}}{\min\{d_u, d_v\}} \\ &= \sum_{(d,d)-edges} \frac{\sqrt{d^2 + d^2}}{d} + \sum_{(1,d)-edges} \sqrt{1 + d^2} \\ &= \frac{d}{d-2}\big((d-1)^{k-1} - 1\big)\sqrt{2} + d(d-1)^{k-1}\sqrt{1+d^2} \\ &= d(d-1)^{k-1}\Big(\frac{\sqrt{2}}{d-2} + \sqrt{1+d^2}\Big) - \frac{\sqrt{2}d}{d-2}. \end{aligned}$$

Therefore, the obtained result confirms the statement. □

# 4 CONCLUSIONS

In this paper, we investigated several important bounds for the Hyperbolic Sombor index $HSO(G)$ that describe its behavior under different graph parameters. Bounds based on the maximum and minimum degrees, the number of pendent vertices, vertex deletion from graph and the sum of $HSO$ of graph with its complement. Furthermore, explicit formulas for $HSO(G)$ were obtained for several fundamental families of trees, including Kragujevac trees, perfect binary trees, binary caterpillar trees and dendrimer trees.

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