Convergence Analysis of Temporally Distributed Nonlinear Systems

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Abstract

Aims: To analyze four-compartment dynamical systems combining delay distributions with saturating nonlinearities, establishing threshold conditions for convergence to equilibrium states.
Study Design: Mathematical analysis integrating spectral methods, Lyapunov functional techniques, and computational simulations.
Methodology: The study formulates functional differential equations with gamma-distributed delay kernels and saturating transfer functions. Spectral analysis identifies a threshold parameter $\mathcal{R}_0$ governing system behavior. Lyapunov techniques establish convergence properties for subcritical and supercritical regimes, while numerical simulations explore parameter sensitivity and transient dynamics across different kernel distributions.
Findings: The threshold parameter $\mathcal{R}_0$ characterizes long-term dynamics: when $\mathcal{R}_0 \leq 1$, solutions converge to trivial equilibrium; when $\mathcal{R}_0 > 1$, a unique nontrivial equilibrium exists and attracts solutions from the interior of the feasible region. Kernel variance modulates transient dynamics, while saturation stabilizes the system by preventing oscillatory instabilities common in purely linear models. The survival probability ϕ during transitions influences the threshold through $\mathcal{R}_0 = \frac{\beta\Lambda\phi}{\mu(\mu+\gamma)}$.
Conclusion: Delay distributions and saturating nonlinearities jointly determine system outcomes through a single threshold parameter. This framework applies to computational systems, networked algorithms, multi-agent coordination, and optimization dynamics where heterogeneous processing times and resource limitations matter. The Lyapunov techniques extend to broader classes of functional differential equations with nonlinear coupling.

Keywords: Delay distributions; dynamical systems; convergence analysis; nonlinear dynamics; saturating functions; Lyapunov functionals; threshold phenomena.

2020 Mathematics Subject Classification: 34K20; 37N99; 34D23; 37C75; 34K18

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1 Introduction

Temporal delays profoundly affect interconnected system behavior. When agents coordinate or computational nodes exchange information, dynamics depend critically on communication latencies and processing durations. While early models assumed instantaneous transitions [Kermack and McKendrick, 1927], realistic models require explicit delay representation [Cooke, 1967, Driver, 1977].

Gamma-distributed delay kernels provide more realistic representations than fixed-lag models when transition times vary across system components [MacDonald, 1978, Cushing, 1977]. Beretta and Takeuchi [Beretta and Takeuchi, 1995] established foundational treatments using Lyapunov functionals, demonstrating convergence despite history-dependence introducing infinite-dimensional state spaces. Kuang [Kuang, 1993] showed how functional differential equations retain many structural properties of ordinary differential equations under suitable conditions.

1.1 Nonlinear Coupling and Saturation

Response saturation—where increasing inputs yield diminishing returns—appears across domains. Neural networks employ sigmoid functions to bound activations, optimization algorithms adapt step sizes to prevent divergence, and physical systems face throughput limits from finite resources. Mathematical models incorporating boundedness require nonlinear coupling terms that flatten at extreme values [Capasso and Serio, 1978, Korobeinikov and Maini, 2004, Korobeinikov, 2005].

Recent work combined delay distributions with nonlinear coupling separately [Li et al., 2014, Guo and Ma, 2021], typically focusing on local stability. The combined effects on global system behavior remain incompletely understood. Standard Lyapunov constructions fail when both features appear simultaneously [Huang et al., 2012, Wang and Zhao, 2012, Nakata et al., 2011].

1.2 Objectives and Novel Contributions

This paper examines four-compartment systems where one transition incorporates gamma-distributed delays and another features saturating coupling. While McCluskey [McCluskey, 2010] analyzed delay distributions and Korobeinikov [Korobeinikov, 2005] treated saturating incidence separately, the key contribution is the unified Lyapunov framework handling both simultaneously, with explicit formulas relating kernel variance to transient dynamics. Main contributions include:

(a) Explicit threshold parameter $\mathcal{R}_0$ characterizing system behavior

(b) Global convergence analysis via Lyapunov functionals accommodating both delay distributions and saturation

(c) Systematic examination of kernel variance influence on transient dynamics

(d) Framework applicability to computational and distributed systems

2 Model Formulation

The analysis considers state variables $X_1(t), X_2(t), X_3(t), X_4(t)$ evolving according to

$$\frac{dX_1}{dt} = \Lambda - \frac{\beta X_1 X_3}{1+\alpha X_3} - \mu X_1, \tag{2.1}$$

$$\frac{dX_2}{dt} = \frac{\beta X_1 X_3}{1+\alpha X_3} - \mu X_2 - \int_0^\infty f(\tau)\frac{\beta X_1(t-\tau)X_3(t-\tau)}{1+\alpha X_3(t-\tau)}e^{-\mu\tau}d\tau, \tag{2.2}$$

$$\frac{dX_3}{dt} = \int_0^\infty f(\tau)\frac{\beta X_1(t-\tau)X_3(t-\tau)}{1+\alpha X_3(t-\tau)}e^{-\mu\tau}d\tau - (\mu+\gamma)X_3, \tag{2.3}$$

$$\frac{dX_4}{dt} = \gamma X_3 - \mu X_4. \tag{2.4}$$

Parameter $\Lambda > 0$ represents constant input flux; $\mu > 0$ denotes linear dissipation rate. Transfer from X_1 to X_2 depends on both compartments through saturating function $\frac{\beta X_1 X_3}{1+\alpha X_3}$, where $\beta > 0$ sets maximum transfer rate and $\alpha \geq 0$ controls saturation strength.

2.1 System Architecture

Figure 1 illustrates the four-compartment system architecture.

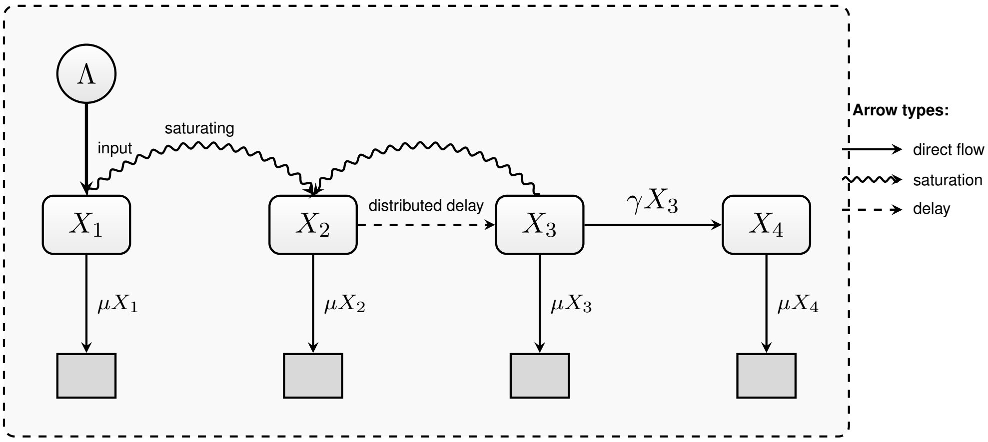


Figure 1: System flow diagram. Constant input Λ feeds X_1; saturating coupling connects $(X_1, X_3) \to X_2$; delay distribution governs $X_2 \to X_3$; linear transfer moves $X_3 \to X_4$. All compartments dissipate at rate μ.

2.2 Kernel Distribution

Transition from X_2 to X_3 incorporates temporal heterogeneity through kernel $f(\tau)$: $\int_0^\infty f(\tau)d\tau = 1$, $f(\tau) \geq 0$, and $\int_0^\infty \tau f(\tau)d\tau < \infty$. The exponential factor $e^{-\mu\tau}$ accounts for dissipation during transition [Beretta and Takeuchi, 1995, Thieme, 1992].

Following McCluskey [McCluskey, 2010], we use gamma-distributed kernels:

$$f(\tau) = \frac{a^n \tau^{n-1} e^{-a\tau}}{\Gamma(n)}, \quad \tau \geq 0, \tag{2.5}$$

where shape parameter $n \geq 1$ and rate parameter $a > 0$ determine characteristics. Mean duration equals n/a while coefficient of variation equals $1/\sqrt{n}$.

Table 1 summarizes system parameters.

Table 1: System parameters and interpretations

Parameter	Interpretation	Dimension
Λ	Input flux rate	units $\cdot$ s^{-1}
β	Maximum transfer coefficient	s^{-1}
α	Saturation parameter	units^{-1}
μ	Dissipation rate	s^{-1}
γ	Linear transfer rate	s^{-1}
$f(\tau)$	Kernel distribution	s^{-1}

3 Well-Posedness and Boundedness

Standard theory guarantees existence and uniqueness [Hale, 1977, Smith, 2011].

Lemma 3.1. *For continuous bounded initial function $\phi : [-\tau_{\max}, 0] \to \mathbb{R}^4_+$, system (2.1)–(2.4) possesses a unique nonnegative solution for all $t \geq 0$.*

Proof. Local Lipschitz continuity follows from standard calculations [Smith, 2011]. Positivity preservation occurs because the vector field points inward along the boundary of $\mathbb{R}^4_+$ [Kuang, 1993]. ☐

Ultimate boundedness follows from examining total quantity $N(t) = X_1(t) + X_2(t) + X_3(t) + X_4(t)$.

Lemma 3.2. *Solutions of (2.1)–(2.4) satisfy*

$$\limsup_{t\to\infty} \left(X_1(t) + X_2(t) + X_3(t) + X_4(t)\right) \leq \frac{\Lambda}{\mu}.$$

Proof. Summing equations (2.1)–(2.4) gives $\frac{dN}{dt} = \Lambda - \mu N$, yielding the claimed limit. ☐

4 Threshold Parameter and Equilibrium Analysis

System (2.1)–(2.4) admits trivial equilibrium $E_0 = (\Lambda/\mu, 0, 0, 0)$. Linearizing around this equilibrium yields

$$\frac{dX_2}{dt} = \frac{\beta\Lambda}{\mu}X_3 - \mu X_2 - \phi\frac{\beta\Lambda}{\mu}X_3, \tag{4.1}$$

$$\frac{dX_3}{dt} = \phi\frac{\beta\Lambda}{\mu}X_3 - (\mu+\gamma)X_3, \tag{4.2}$$

where

$$\phi = \int_0^\infty f(\tau)e^{-\mu\tau}d\tau \tag{4.3}$$

represents survival probability during transition [Diekmann et al., 1990, van den Driessche and Watmough, 2002].

Definition 4.1. The threshold parameter $\mathcal{R}_0$ equals the spectral radius of the next-generation operator for linearized system (4.1)–(4.2).

Theorem 4.1. *The threshold parameter for system (2.1)–(2.4) is*

$$\mathcal{R}_0 = \frac{\beta\Lambda\phi}{\mu(\mu+\gamma)}, \tag{4.4}$$

where ϕ is defined by (4.3).

Proof. Following van den Driessche and Watmough [van den Driessche and Watmough, 2002], the next-generation matrix analysis yields the expression in (4.4). ☐

For gamma distribution (2.5), survival probability is

$$\phi = \left(\frac{a}{a+\mu} \right)^n . \tag{4.5}$$

4.1 Equilibrium Existence

Theorem 4.2. *Trivial equilibrium $E_0 = (\Lambda/\mu, 0, 0, 0)$ is locally asymptotically stable when $\mathcal{R}_0 < 1$ and unstable when $\mathcal{R}_0 > 1$.*

Proof. Stability follows from eigenvalue analysis as in Smith [Smith, 2011] and Beretta and Takeuchi [Beretta and Takeuchi, 1995]. ☐

Theorem 4.3. *If $\mathcal{R}_0 > 1$, system (2.1)–(2.4) possesses unique equilibrium $E^* = (X_1^*, X_2^*, X_3^*, X_4^*)$ with all components strictly positive.*

Proof. At equilibrium, from (2.3), $\phi \frac{\beta X_1^* X_3^*}{1+\alpha X_3^*} = (\mu+\gamma) X_3^*$, which rearranges to

$$X_1^* = \frac{(\mu+\gamma)(1+\alpha X_3^*)}{\phi\beta} . \tag{4.6}$$

Substituting into (2.1) yields unique positive solution $X_3^* > 0$ when $\mathcal{R}_0 > 1$ [Korobeinikov and Maini, 2004]. ☐

5 Global Convergence Analysis

Lyapunov techniques establish convergence from arbitrary initial conditions.

Theorem 5.1. *When $\mathcal{R}_0 \le 1$, every solution of (2.1)–(2.4) satisfies*

$$\lim_{t\to\infty} (X_1(t), X_2(t), X_3(t), X_4(t)) = E_0 .$$

Proof. Consider functional

$$V_1(t) = X_2(t) + X_3(t) + \frac{\beta\Lambda}{\mu} \int_0^\infty f(\tau) e^{-\mu\tau} \int_{t-\tau}^t \frac{X_1(\theta)X_3(\theta)}{1+\alpha X_3(\theta)} d\theta \, d\tau . \tag{5.1}$$

Computing $\frac{dV_1}{dt}$ along solutions and using $X_1(t) \le \Lambda/\mu$ yields

$$\frac{dV_1}{dt} \le -\mu X_2 - (\mu+\gamma)(1-\mathcal{R}_0) X_3 .$$

When $\mathcal{R}_0 \le 1$, LaSalle's principle [LaSalle, 1976] implies convergence to E_0. ☐

Theorem 5.2. *When $\mathcal{R}_0 > 1$, every solution from any interior point satisfies*

$$\lim_{t\to\infty} (X_1(t), X_2(t), X_3(t), X_4(t)) = E^* .$$

Proof. Following Wang and Zhao [Wang and Zhao, 2012] and Nakata et al. [Nakata et al., 2011], construct Lyapunov functional with logarithmic terms accommodating saturating nonlinearity. Careful calculation shows $V_2' \le 0$ with equality only at E^*. ☐

6 Numerical Experiments

Computational experiments illustrate analytical results.

6.1 Parameter Selection

Table 2 lists baseline values.

Table 2: Baseline parameters for numerical experiments

Parameter	Value	Variation range	Notes
Λ	0.02	fixed	Input flux
β	0.4	[0.1, 1.0]	Transfer coefficient
α	0.001	[10^{-4}, 10^{-1}]	Saturation strength
μ	0.02	fixed	Dissipation rate
γ	0.1	[0.05, 0.2]	Linear transfer
n	3	[1, 5]	Gamma shape
a	0.5	[0.2, 1.0]	Gamma rate

6.2 Threshold Phenomenon

Figure 2 demonstrates the dichotomy predicted by Theorems 5.1 and 5.2. When $\mathcal{R}_0 = 0.74 < 1$, variables decay toward trivial equilibrium. For $\mathcal{R}_0 = 2.96 > 1$, convergence occurs to nontrivial equilibrium.

6.3 Phase Space Structure

Figure 3 displays trajectories in (X_1, X_3) projection. Different initial conditions produce distinct transient responses, but all solutions converge to the same nontrivial equilibrium when $\mathcal{R}_0 > 1$, confirming Theorem 5.2.

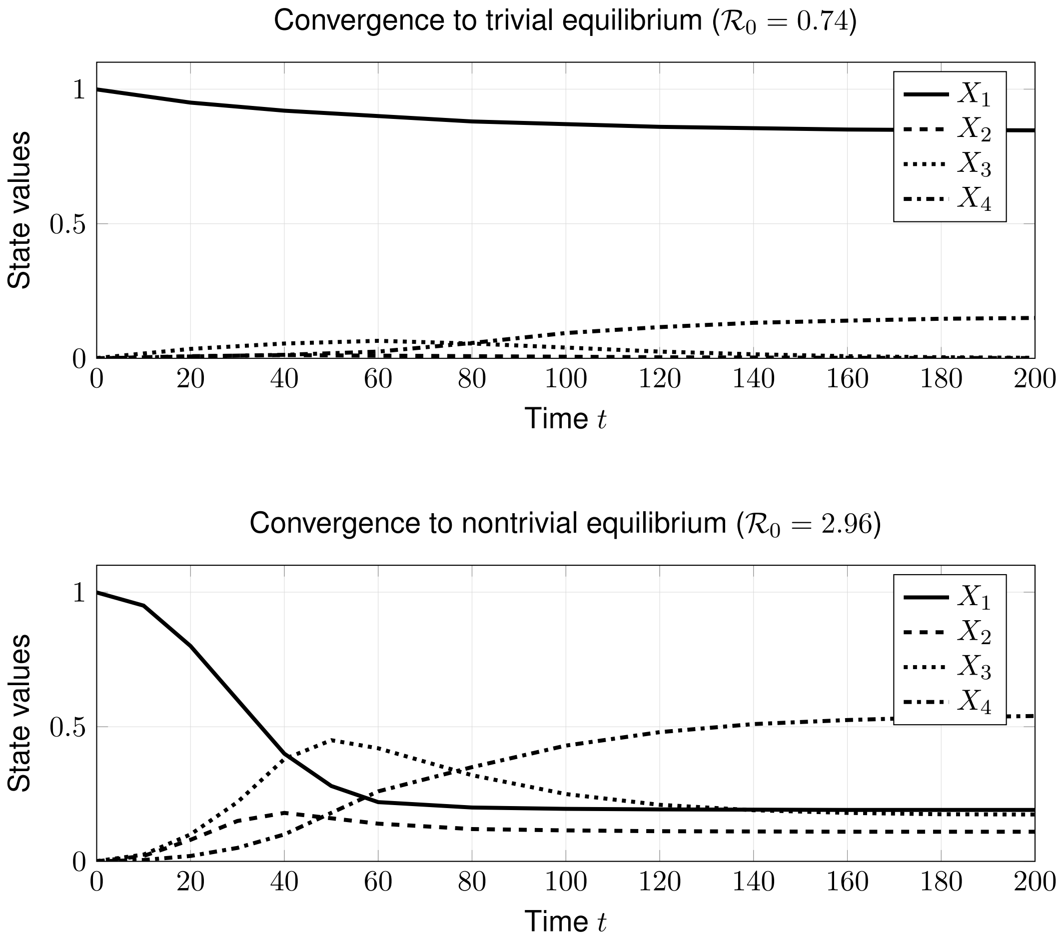
6.4 Influence of Kernel Distribution

Kernel variance affects transient dynamics even when mean duration remains fixed. Figure 4 compares three gamma distributions with identical means but different variances. Lower shape parameter n (higher variance) produces earlier and sharper peaks in X_3. Higher n (lower variance) leads to delayed but more sustained peaks [Feng and Thieme, 2000, McCluskey, 2010].

7 Discussion

This paper analyzed four-compartment dynamical systems featuring delay distributions and saturating nonlinearities. A single threshold parameter $\mathcal{R}_0$ determines long-term behavior: systems with $\mathcal{R}_0 \leq 1$ decay toward trivial states, while $\mathcal{R}_0 > 1$ produces convergence to nontrivial equilibria.

The unified Lyapunov framework advances prior work treating delays [McCluskey, 2010] and saturation [Korobeinikov, 2005] separately. Logarithmic Lyapunov terms simultaneously accommodate both features, enabling global convergence proofs neither approach achieves independently.


Convergence to trivial equilibrium ($\mathcal{R}_0 = 0.74$)

Convergence to nontrivial equilibrium ($\mathcal{R}_0 = 2.96$)

Figure 2: System evolution illustrating threshold phenomenon. Top: decay to trivial equilibrium when $\mathcal{R}_0 < 1$. Bottom: convergence to nontrivial equilibrium when $\mathcal{R}_0 > 1$.

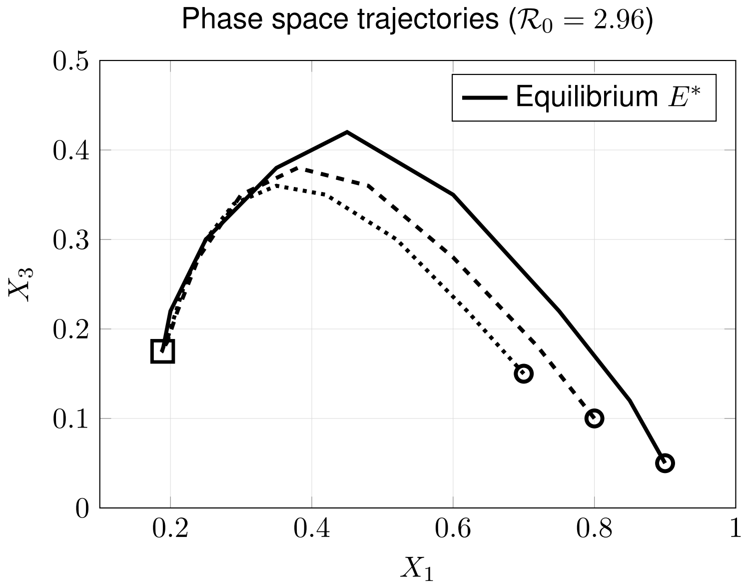


Figure 3: Phase space structure showing convergence from multiple initial conditions to unique nontrivial equilibrium.

Saturation eliminates oscillatory instabilities common in linear delay-differential equations by bounding feedback strength. When X_3 grows large, coupling rate $\beta X_1 X_3/(1 + \alpha X_3)$ asymptotes to $\beta X_1/\alpha$, preventing runaway dynamics. This boundedness ensures Lyapunov functional derivatives remain negative definite regardless of state magnitude or history dependence.

The framework applies to machine learning (gradient descent with momentum [Bottou et al., 2018]), distributed computing (variable communication patterns [Lynch, 1996]), and multi-agent systems (heterogeneous response durations [Olfati-Saber et al., 2007]). Threshold $\mathcal{R}_0$ provides tractable design criterion for predicting system behavior.

Limitations include deterministic assumptions (extensions to stochastic systems [Mao, 2007]), spatial homogeneity (ignoring reaction-diffusion [Ruan, 2007]), and fixed parameters (adaptive parameters for learning systems).

8 Conclusion

This analysis established threshold conditions governing four-compartment systems with distributed delays and saturating coupling. Key results include:

- **(a)** Threshold $\mathcal{R}_0 = \frac{\beta\Lambda\phi}{\mu(\mu+\gamma)}$ determines outcomes: decay to trivial equilibria when $\mathcal{R}_0 \leq 1$, convergence to nontrivial equilibria when $\mathcal{R}_0 > 1$.
- **(b)** Global convergence established through Lyapunov functionals handling both delay distributions and nonlinear coupling.
- **(c)** Survival factor $\phi = \int_0^\infty f(\tau)e^{-\mu\tau}d\tau$ links kernel characteristics to outcomes, with $\phi = (a/(a+\mu))^n$ for gamma kernels.
- **(d)** Kernel variance influences transient dynamics: high-variance kernels produce rapid responses, low-variance kernels generate smoother convergence.

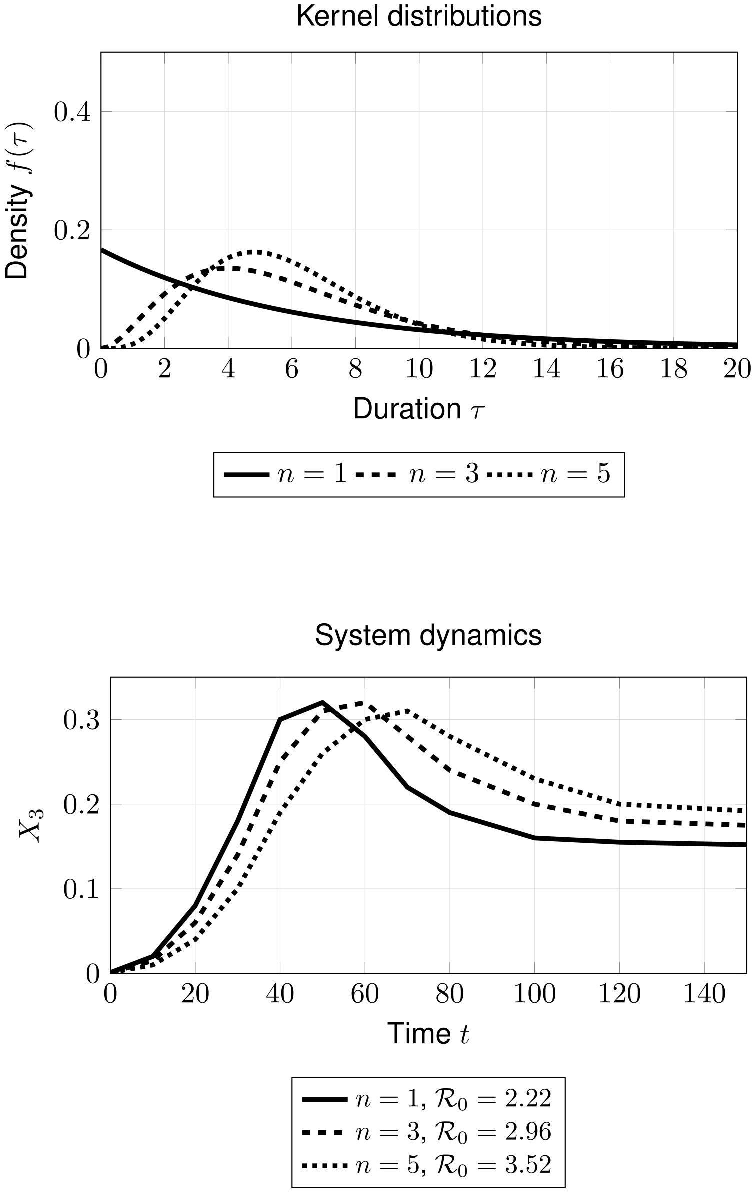


Figure 4: Effect of kernel variance on system response. Top: gamma distributions with mean duration 6 units. Bottom: corresponding X_3 dynamics. Higher variance (lower n) produces earlier peaks; lower variance (higher n) yields delayed but sustained responses.

(e) Saturation stabilizes systems by preventing oscillatory instabilities.

The framework applies to AI systems, distributed algorithms, networked control, and multi-agent coordination where resource limitations and heterogeneous communication latencies matter.

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