Original Research Article

ON THE STABILITY OF EQUI NEIGHBOR POLYNOMIAL OF GRAPHS

Abstract

Let $G(V, E)$ be a simple graph of order n with vertex set V and edge set E. Let (u, v) denotes an unordered vertex pair of distinct vertices of G. For a vertex $u \in G$, let $N(u)$ be the set of all vertices of G which are adjacent to u in G. Then for $0 \leq i \leq n-1$, the i-equi neighbor set of G is defined as: $N_e(G, i) = \{(u, v) : u, v \in V, u \neq v$ and $|N(u)| = |N(v)| = i\}$. The equi-neighbor polynomial $N_e[G; x]$ of G is defined as $N_e[G; x] = \sum_{i=0}^{(n-1)} |N_e(G, i)| x^i$. A root of the polynomial $N_e[G; x]$ is defined as the equi neighbor root of the graph G. In this paper we study the nature of roots of equi neighbor polynomial of graphs and discuss the stability of equi neighbor polynomial of some graph classes.

Keywords: $i-$ *equi neighbor set; equi neighbor polynomial.*

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1 Introduction

Let $G(V, E)$ be a simple graph of order n. Let (u, v) denotes an unordered vertex pair of distant vertices of G. The i-equi neighbour set of G is defined as $N_e(G, i) = \{(u, v) : u, v \in V, u \neq v$ and $|N(u)| = |N(v)| = i\}$, for $0 \leq i \leq n-1$. The polynomial $N_e[G; x] = \sum_{i=0}^{(n-1)} |N_e(G, i)| x^i$ is defined as the equi neighbour polynomial of G. The present authors derived the equi neighbor polynomial of some well known graphs. The equi neighbor polynomial of some graph operations were discussed. We say that two graphs G and H are said to be ENP-equivalent if and only if $N_e[G; x] = N_e[H; x]$. A zero of the polynomial $N_e[G; x]$ is defined as the equi neighbor root of the graph G.

A polynomial $f(x_1,\ldots,x_n)$ is said to be stable (7) with respect to a region $\Omega \in \mathbb{C}^n$ if no root of f lies in Ω. Polynomials which are stable with respect to the closed right half plane and with respect to the open unit disk are called Hurwitz polynomial and Schur polynomial respectively. Hurwitz polynomials are important in control systems theory, because they represent the characteristic equations of stable linear systems(2). A graph polynomial is worthwhile to study only if it models some stable physical systems. In this article we study the stability of common neighbor polynomial of graphs with respect to the closed right half plane and thus identify the conditions under which the common neighbor polynomial of certain graph classes become a Hurwitz polynomial.

2 Main results

Definition 2.1. *A polynomial $f(x_1,x_2,\ldots,x_n)$ is said to be stable with respect to the closed right half plane if and only if all of its non zero roots lie in the open left half plane.*

Lemma 2.1. *For a cycle graph C_n with n vertices, we have*

$$N_e[C_n;x] = \binom{n}{2}x^2, n \geq 3.$$

Theorem 2.2. *If C_n is a cycle graph, then $N_e[C_n;x]$ is stable; $n \geq 3$.*

Proof. From Lemma, we have

$$N_e[C_n;x] = \binom{n}{2}x^2, n \geq 3.$$

Since zero is the only root of $N_e[C_n;x]$, it is stable for $n \geq 3$. ☐

Lemma 2.3. *For $n \geq 2$, we have $N_e[F_n;x] = n(2n-1)x^2$.*

Theorem 2.4. *If F_n is a friendship graph, then $N_e[F_n;x]$ is stable; $n \geq 2$.*

Proof. From Lemma, we have $N_e[F_n;x] = n(2n-1)x^2$; $n \geq 2$.
Since zero is the only root of $N_e[F_n;x]$, it is stable for $n \geq 2$. ☐

Lemma 2.5. *For a bipartite cocktail party graph B_n, we have $N_e[B_n;x] = n(2n-1)x^{n-1}$; $n \geq 1$.*

Theorem 2.6. *If B_n is a bipartite cocktail party graph, then $N_e[B_n;x]$ is stable; $n \geq 1$.*

Proof. From Lemma, we have $N_e[B_n;x] = n(2n-1)x^{n-1}$; $n \geq 1$.
Since zero is the only root of $N_e[B_n;x]$, it is stable for $n \geq 1$. ☐

Lemma 2.7. *For a windmill graph $W_n^{(m)}$;$m \geq 2$; $n \geq 2$ we have*
$N_e[W_n^{(m)};x] = \binom{m(n-1)}{2}x^{n-1}$.

Theorem 2.8. *If $W_n^{(m)}$, is a windmill graph, then $N_e[W_n^{(m)};x]$ is stable; $m \geq 2$; $n \geq 2$.*

Proof. From Lemma, we have $N_e[W_n^{(m)};x] = \binom{m(n-1)}{2}x^{n-1}$;$m \geq 2$; $n \geq 2$.
Since zero is the only root of $N_e[W_n^{(m)};x]$, it is stable for $m \geq 2$; $n \geq 2$. ☐

Lemma 2.9. *For a dutch windmill graph $N_e[D_n^{(m)};x] = m(2m-1)x^2$.; $m \geq 2$.*

Theorem 2.10. *If $D_n^{(m)}$, is a dutch windmill graph, then $N_e[D_n^{(m)};x]$ is stable; $m \geq 2$.*

Proof. From Lemma, we have $N_e[D_n^{(m)}; x] = m(2m-1)x^2; m \geq 2$.

Since zero is the only root of $N_e[D_n^{(m)}; x]$, it is stable for $m \geq 2$. ☐

Lemma 2.11. *For n- barbell graph,* $N_e[B_{n,1}; x] = x^n + (n-1)(2n-3)x^{n-1}$*;* $n \geq 1$*.*

Theorem 2.12. *If* $B_{n,1}$ *is n- barbell graph, then* $N_e[B_{n,1}; x]$ *is stable;* $n \geq 1$*.*

Proof. From Lemma, we have

$$\begin{aligned} N_e[B_{n,1}; x] &=^n + (n-1)(2n-3)x^{n-1} \\ &= x^{n-1}[x + (n-1)(2n-3)] \end{aligned}$$

When $n = 1$, $N_e[B_{1,1}; x] = x$, which has only one root namely zero. When $n > 1$, the equi neighbor root of $B_{n,1}$ is , $x = -(n-1)(2n-3)$; which lies in the open left half plane. Hence the result follows. ☐

Theorem 2.13. *Let* G *be a graph with common neighbor polynomial* $N_e[G; x]$ *of degree 2. Then the following hold:*

1. *If* $N_e(G, 0) = \phi$ *and* $N_e(G, 1) \neq \phi$*, then* $N_e[G; x]$ *is a stable polynomial.*
2. *If* $N_e(G, 0) \neq \phi$ *and* $N_e(G, 1) = \phi$*, then* $N_e[G; x]$ *is not a stable polynomial.*

Proof. Since $N_e[G; x]$ is of degree 2, $|N_e(G, 2)| \neq 0$. We consider the two cases:

1. Let $N_e(G, 0) = \phi$ and $N_e(G, 1) \neq \phi$. In this case, the roots of $N_e[G; x]$ are given by $x = 0$ and $x = -\dfrac{|N_e(G, 1)|}{|N_e(G, 2)|}$. It follows that $N[G; x]$ is stable.
2. Let $N_e(G, 0) \neq \phi$ and $N_e(G, 1) = \phi$. Then the roots of $N_e[G; x]$ are given by $x = \pm\sqrt{\dfrac{|N_e(G, 0)|}{|N_e(G, 2)|}}\, i$.

   Since $N_e[G; x]$ has non zero roots in the closed right half plane, $N_e[G; x]$ is not stable.

This completes the proof. ☐

Lemma 2.14. *For a path graph* P_n*, we have*

$$N_e[P_n; x] = \ x + \binom{n-2}{2} x^2; n \geq 2.$$

Theorem 2.15. *If* $P_n,$ *is a path graph, then* $N_e[P_n; x]$ *is stable;* $n \geq 2$*.*

Proof. From Lemma, we have $N_e[P_n; x] = \ x + \binom{n-2}{2} x^2; n \geq 2$.

$N_e[P_n; x]$ is of degree 2 with $|N_e(P_n, 0)| = 0$ and $|N_e(P_n, 1)| \neq 0$. So the result follows from Theorem 2.13 ☐

Theorem 2.16. *(Routh-Hurwitz Criteria) (8) Given a polynomial,* $P(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_0$*, where the coefficients* a_i *are real constants,* $i = 1, 2, \ldots, n$ *define the* n *Hurwitz matrices using the coefficients* a_i *of the above polynomial as*

$$H_1 = [a_1] \qquad H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix} \quad \cdots \quad H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & a_{2n-4} & \cdots & a_n \end{bmatrix}$$

where $a_j = 0$ if $j > n$. All the roots of the polynomial $P(x)$ are negative or have negative real part if and only if the determinants of all Hurwitz matrices are positive: det $H_j > 0$, $j = 1, 2, \ldots, n$.

Lemma 2.17. *For a helm grah H_n, $n > 3$, we have*

$$N_e[H_n; x] = \begin{cases} \binom{n-1}{2}(x^4 + x), & \text{if } n \neq 5 \\ 10x^4 + 6x, & \text{if } n = 5. \end{cases}$$

Theorem 2.18. *If H_n, $n > 3$ is a helm graph with $2n - 1$ vertices, then $N_e[H_n; x]$ is not stable.*

Proof. We consider two cases.

Case(i) Let $n \neq 5$. From Lemma, we have $N_e[H_n; x] = \binom{n-1}{2}(x^4 + x)$. So considering the polynomial $x^3 + 1$ the determinant of the first Hurwitz matrix is given by $|H_1| = 0$. Hence the result follows from Theorem 2.16.

Case(ii) Let $n = 5$. From Lemma, we have $N_e[H_n; x] = 10x^4 + 6x$. So considering the polynomial $10x^3 + 6$ the determinant of the first Hurwitz matrix is given by $|H_1| = 0$. Hence the result follows from Theorem 2.16.

□

Lemma 2.19. *For a lollipop graph $L_{m,n}$, we have*

$$N_e[L_{m,n}; x] = \begin{cases} \binom{m-1}{2}x^{m-1} + \binom{n-1}{2}x^2, & \text{if } m \geq 4 \\ \binom{n-1}{2}x^2 + x, & \text{if } m = 1 \\ \binom{n}{2}x^2 + x, & \text{if } m = 2 \\ \binom{n+1}{2}x^2, & \text{if } m = 3. \end{cases}$$

Theorem 2.20. *If $L_{m,n}$ is a lollipop graph, then the following results hold.*

1. *$N_e[L_{m,n}; x]$ is not stable for $m \geq 4$.*
2. *$N_e[L_{m,n}; x]$ is stable for $m = 1$ and $n > 2$.*
3. *$N_e[L_{m,n}; x]$ is stable for $m = 2$ and $n > 1$.*
4. *$N_e[L_{m,n}; x]$ is stable for $m = 3$.*

Proof. We consider the following four cases.

Case(i) Let $m \geq 4$. From Lemma, we have $N_e[L_{m,n}; x] = \binom{m-1}{2}x^{m-1} + \binom{n-1}{2}x^2$. Then the determinant of the first Hurwitz matrix itself is zero. Hence $N_e[L_{m,n}; x]$ is not stable under this case, by Theorem 2.16.

Case(ii) Let $m = 1, n > 2$. From Lemma, we have $N_e[L_{m,n}; x] = \binom{n-1}{2}x^2 + x$. So considering the polynomial $x + \frac{1}{\binom{n-1}{2}}$; the determinant of the first Hurwitz matrix is given by $|H_1| = \frac{1}{\binom{n-1}{2}}$. Hence the result follows by Theorem 2.16.

Case(iii) Let $m = 2, n > 1$. From Lemma, we have $N_e[L_{m,n}; x] = \binom{n}{2}x^2 + x$ So considering the polynomial $x + \frac{1}{\binom{n}{2}}$; the determinant of the first Hurwitz matrix is given by $|H_1| = \frac{1}{\binom{n}{2}}$. Hence the result follows by Theorem 2.16.

Case(iv) Let $m = 2$. From Lemma, we have $N_e[L_{m,n}; x] = \binom{n+1}{2}x^2$. Since zero is the only root of $N_e[L_{m,n}; x]$, it is stable for $m = 2$.

This completes the proof. □

Lemma 2.21. *For a web graph WB_n where $n > 3$, we have*

$$N_e[WB_n; x] = \begin{cases} (n-1)(2n-3)x^4 + \binom{n-1}{2}x, & \text{if } n \neq 5 \\ 36x^4 + 6x, & \text{if } n = 5. \end{cases}$$

Theorem 2.22. *If $WB_n; n > 3$ is a web graph with $3(n-1)$ vertices, then $N_e[WB_n; x]$ is not stable.*

Proof. We consider two cases.

Case(i) Let $n \neq 5$. From Lemma, we have $N_e[WB_n; x] = (n-1)(2n-3)x^4 + \binom{n-1}{2}x$. So considering the polynomial $(4n-6)x^3 + (n-2)$, the determinant of the first Hurwitz matrix is given by $|H_1| = 0$. Hence the result follows from Theorem 2.16.

Case(ii) Let $n = 5$. From Lemma, we have $N_e[WB_n; x] = 36x^4 + 6x$. So considering the polynomial $6x^3 + 1$, the determinant of the first Hurwitz matrix is given by $|H_1| = 0$. Hence the result follows from Theorem 2.16.

□

Lemma 2.23. *For a shell graph $S_n; n \geq 3$, we have*

$$N_e[S_n; x] = \begin{cases} \binom{n-3}{2}x^3 + x^2, & \text{if } n \neq 3, 4 \\ 3x^2, & \text{if } n = 3 \\ x^3 + x^2, & \text{if } n = 4. \end{cases}$$

Theorem 2.24. *If S_n is a shell graph with $n \geq 3$ vertices, then $N_e[S_n; x]$ is stable.*

Proof. We consider three cases.

Case(i) Let $n \neq 3, 4$. From Lemma, we have $N_e[S_n; x] = \binom{n-3}{2}x^3 + x^2$.

So considering the polynomial $\binom{n-3}{2}x + 1$, the determinant of the first Hurwitz matrix is given by $|H_1| = 1$. Hence the result follows from Theorem 2.16.

Case(ii) Let $n = 3$. From Lemma, we have $N_e[S_3; x] = 3x^2$. Since zero is the only root of $N_e[S_n; x]$, it is stable for $n = 3$.

Case(iii) Let $n = 4$. From Lemma, we have $N_e[S_4; x] = x^3 + x^2$. So considering the polynomial $x+1$, the determinant of the first Hurwitz matrix is given by $|H_1| = 1$. Hence the result follows from Theorem 2.16.

□

Lemma 2.25. *If BF is a butterfly graph with $N \geq 7$ vertices, then*

$$N_e[BF; x] = \binom{N-7}{2}x^3 + 6x^2 + x.$$

Theorem 2.26. *If BF is a butterfly graph with $N \geq 7$ vertices, then $N_e[BF; x]$ is stable.*

Proof. We consider two cases.

Case(i) Let $N > 8$. From Lemma, we have $N_e[BF; x] = \binom{N-7}{2}x^3 + 6x^2 + x$. So considering the polynomial $x^2 + \dfrac{6}{k}x + \frac{1}{k}$; where $k = \binom{N-7}{2}$, the determinants of the Hurwitz matrices are given by $|H_1| = \frac{6}{k}$. and $|H_2| = \begin{vmatrix} \frac{6}{k} & 1 \\ 0 & \frac{1}{k} \end{vmatrix} = \frac{6}{k^2}$. Since both the determinants are positive, by Theorem 2.16, $N_e[BF; x]$ is stable for $N > 8$.

Case(ii) Let $N = 7, 8$. From lemma we have $N_e[BF; x] = 6x^2 + x$.
So considering the polynomial $x + \frac{1}{6}$; the determinant of the first Hurwitz matrix is given by $|H_1| = \frac{1}{6}$. Hence the result follows by Theorem 2.16.

□

Lemma 2.27. *For a complete bipartite graph $K_{m,n}$; where $m, n \geq 2$, we have the following.*

$$N_e[K_{m,n}; x] = \begin{cases} \binom{m}{2}x^n + \binom{n}{2}x^m, & m \neq n, \\ n(2n-1)x^n, & m = n. \end{cases}$$

Theorem 2.28. *If $K_{m,n}$; where $m > n; m, n \geq 2$ is a complete bipartite graph with $m + n$ vertices, then $N_e[K_{m,n}; x]$ is stable if and only if $m = n$ or $n - m = 1$.*

Proof. Assume that $m = n$. From Lemma, we have
$N_e[K_{m,n}; x] = n(2n-1)x^n$. Since zero is the only root of $N_e[K_{m,n}; x]$, it is stable for $m = n$.
Now assume that $n - m = 1$. From Lemma, we have
$N_e[K_{m,n}; x] = x^m[\binom{m}{2}x + \binom{n}{2}]$. So considering the polynomial $x + \dfrac{\binom{n}{2}}{\binom{m}{2}}$; the determinant of the first Hurwitz matrix is given by $|H_1| = \dfrac{\binom{n}{2}}{\binom{m}{2}}$. Since the determinant of the Hurwitz matrix is positive, by Theorem 2.16 $N_e[K_{m,n}; x]$ is stable.
Assume that $n - m > 1$. Then the determinant of the first Hurwitz matrix itself is zero. Hence $N_e[K_{m,n}; x]$ is not stable, by Theorem 2.16.

This completes the proof. □

Theorem 2.29. *Let G be a graph with common neighbor polynomial $N_e[G; x]$ of degree 2 where $|N_e(G, i)| > 0$ for $i = 0, 1, 2$. Then $N_e[G; x]$ is stable. Moreover, $N_e[G; x]$ has two negative real roots if $|N_e(G, 1)|^2 \geq 4|N_e(G, 2)||N_e(G, 0)|$ and $N_e[G; x]$ has two complex roots with negative real parts otherwise.*

Proof. Since $N_e[G; x]$ is a polynomial of degree 2, $N_e[G; x]$ can be represented in the form $N_e[G; x] = |N_e(G, 2)|x^2 + |N_e(G, 1)|x + |N_e(G, 0)|$. The Hurwitz matrices of $N_e[G; x]$ are given by $H_1 = \left[\frac{|N_e(G,1)|}{|N_e(G,2)|}\right]$ and $H_2 = \begin{bmatrix} \frac{|N_e(G,1)|}{|N_e(G,2)|} & 1 \\ 0 & \frac{|N_e(G,0)|}{|N_e(G,2)|} \end{bmatrix}$. Since $|N_e(G, i)| > 0$ for $i = 0, 1, 2$; it follows that $\det H_1 > 0$ and $\det H_2 > 0$. Hence by Theorem 2.16, $N_e[G; x]$ is stable so that all the roots of $N_e[G; x]$ lie in the open left half plane. Moreover, the discriminant of $N_e[G; x]$ is given by $\Delta = |N_e(G, 1)|^2 - 4|N_e(G, 2)||N_e(G, 0)|$. It follows that $N_e[G; x]$ has 2 real roots if $\Delta \geq 0$ and has two complex roots if $\Delta < 0$.
This completes the proof. □

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3 Conclusions

In this article we study the nature of roots of equi neighbor polynomial of graphs and discuss the stability of equi neighbor polynomial of some graph classes. Also identify the conditions under which the common neighbor polynomial of certain graph classes become a Hurwitz polynomial.

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