MODELLING WITHDRAWAL RISK USING CONVOLUTIONS APPROACH

Abstract

Insurance companies are required to have reliable estimates of withdrawal probabilities or lapse rates for planning purposes. Some insurers are not able to adopt advanced methodologies given the misconceptions of the best techniques in modelling. Advancing the work of previous researchers, this manuscript applies the negative-binomial-geometric convolutions mechanism to carry out withdrawal risk modelling. We generate the predicted withdrawal distribution for a variety of cases using convolutions approach. Further, we have examined various cases to check the models behaviour. The results shows that the model explains various effects of withdrawal probability on customer size and as well as withdrawal size.

Keywords: Withdrawal risk management; convolutions approach; withdrawal probability; Bayesian criterion; negative binomial distribution.
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1 Introduction

Withdrawal risk management for investments in insurance companies in Kenya has led to misconception about its usefulness. For instance, the best methodologies to apply, socio-economic benefits of activities in Kenya and usefulness to insurance companies. In this context, insurance industry representatives, regulators and academics worldwide have given new stimulating efforts to improve withdrawal risk modelling for corporations, insurance institutions and financial instruments (Changki (2005)). The misconception of techniques to apply in minimizing risks involved with promises to pay both on the insurance companies and the policyholder is in question.

Withdrawal risk refers to the risk that the financial holder of a contract will be unable to fulfill his/her contractual promises resulting in financial loss to the authority (Shumway (2001)). It is therefore a function of; the withdrawals peril exposure, the withdrawal amount and the withdrawal probability. According to Chen & Pan (2012), withdrawal is the level of unsteadiness in the worth of debt and derivatives which are gotten from the changes made in the investors and counter-parties qualities. Withdrawal risk is therefore a main cause of uncertainty about the financial situation in the future.

The standard convolutions parametric distribution is the Poisson, but this distribution assumes the mean and variance of withdrawals to be equal. We propose the negative-binomial and geometric distributions to carry out withdrawals management in companies, which represents variance that exceeds its mean and better explains withdrawals variability. Insurance companies issues peril protection for the needy members of the society and is among the growing international micro-finance industries that emerged in 1970's (Churchill (2007); Roth *et al.,* (2007); Matul *et al.,* (2010);Onchere (2022)). The approximately number of insurers worldwide is around 135 million with forecast growth rate estimated at 10% p.a (Lloyds' of London 2009). Through safeguarding the needy from the financial losses and unexpected shocks, insurance is considered to be a formal loss protection solution to worldwide poverty and a major economic growth impetus (Churchill *et al.,* (2011)).

There have been various studies that have been carried out about the topic using various models. They include;

Structural models

These models assume that a withdrawal can be explained by a specific trigger point. The value of investment itself is modeled as a stochastic process. An example of such a model is the Merton model.

Reduced form models

These models assume that the withdrawals are driven by withdrawal intensity. No specific trigger event is assumed, but the withdrawal rate might depend on changes in external factors such as inflation, unemployment, etc. This relationship is estimated using historical data and economic methodologies.

1.1 Common features of existing models

It is possible to find similarities between various withdrawal risk models such as;

Joint withdrawal behavior

Withdrawal rates vary over time due to changes in the micro-economic variables therefore each investors rate of withdrawal is driven by states of nature.

Conditional distribution of portfolio withdrawal rate

It's calculated for each state assuming independence of the policyholders.

Aggregation

The marginal withdrawals density is derived by linking homogeneous sub portfolio's densities of withdrawal in each level.

1.2 The model variables

In-order to account for and report the risk of withdrawal we apply a mathematical model intended to quantify the withdrawal loss for an insurance company. This is obtained from the withdrawals of its policyholders. This model requires input data in terms of; withdrawal risk exposure of every policyholder, the rate of withdrawal, the chances of recovering a loss from a policyholder.

The remainder of the manuscript is structured as follows: firstly, we present the convolutions model where the geometric and negative binomial distributions are applied as the Bayes criteria, then the data analysis and interpretation of the study are presented and end with the conclusion.

2 Methodology

2.1 Convolution Model

Convolution is a mathematical operation that allows one to derive the distribution of W random variables from the distribution of w commands.

If M is a discrete random variable with support R_m and pmf $P_M(m)$. Let N be another discrete random variable, independent of M with support R_n and pmf $P_N(n)$. The pmf $P_W(w)$ of the sum $W = M + N$ can be derived using the following formula

$$P_W\left(w\right) = \sum_{n \epsilon Rn} P_M(w-n)P_N(n)$$

In a case where there is a sum of j independent random variables and W is their sum. i.e. $W = M_1 + M_2 + M_3 + \cdots + M_j$. The density of W can be generated repeatedly applying the result from the sums of W variables.

First define $N_2 = M_1 + M_2$ and compute the density of N_2

Then define $N_3 = N_2 + M_3$ and compute the density of N_3

And so on, until the distribution of W can be computed from $W = N_j = N_{j-1} + M_j$

This research utilizes the convolutions methodology to obtain the Withdrawal Probabilities (WPS).

2.2 Bayesian Criterion

The Bayesian theorem states

$$P(\Phi|\psi) = \frac{P(\psi|\Phi)P(\Phi)}{P(\psi)},$$

where Φ is the percentage of policyholders and ψ is an event of withdrawal. We describe below the distributions that will be applied in the study.

The Geometric distribution

The PMF of the geometric distribution is given by

$$f(x) = \lambda(1-\lambda)^{x-1}; x = 1, 2, 3, \ldots$$

where λ represents the success probability. The moments generating function is derived as

$$E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} f(x) = \sum_{x=1}^{\infty} e^{tx} \lambda(1-\lambda)^{x-1}$$
$$= \lambda(1-\lambda)^{-1} \sum_{x=1}^{\infty} ((1-\lambda)e^{t})^{x}$$
$$E(e^{tX}) = \frac{\lambda e^{t}}{1-(1-\lambda)e^{t}}$$

The MGF is applied to compute the mean and variance as $\frac{1}{\lambda}$ and $\frac{1-\lambda}{\lambda^2}$ respectively.

Negative binomial distribution

The NBD has the PMF

$$f(x) = \binom{x-1}{r-1} p^{r}(1-p)^{x-r}; x = r, r+1, r+2, \ldots\ldots\ldots$$

where, r = number of withdrawals in a particular policy.
p = probability as estimated by Bayes' theorem.
x = total number of withdrawals in portfolio.

The moments generating function is derived as

$$\begin{aligned}
E(e^{tx}) &= \sum_{x=1}^{\infty} e^{tx} f(x) = \sum_{x=1}^{\infty} e^{tx} \binom{x-1}{r-1} p^r (1-p)^{x-r} \\
&= (pe^t)^r \sum_{x=r}^{\infty} \binom{x-1}{r-1} [(1-p)e^t]^{x-r} \\
&= (pe^t)^r \sum_{k=0}^{\infty} \binom{k+r-1}{r-1} [(1-p)e^t]^{k} \\
&= (pe^t)^r [1-(1-p)e^t]^{-r} \\
E(e^{tx}) &= \frac{(pe^t)^r}{[1-(1-p)e^t]^r}
\end{aligned}$$

The MGF is applied to compute the mean and variance as $\frac{r}{p}$ and $\frac{r(1-p)}{p^2}$ respectively.

3 Data Analysis and Interpretation

This section deals exclusively with the analysis of the determinants of withdrawal risk. Our model will utilize withdrawal/lapse rates obtained from a major Kenyan insurer for policyholders with indexation date 2018. The policies considered for the new policyholders are; A (pure endowment), B (education plans), C (life annuity), D (whole life), E (term assurance), F (last survivor), G (endowment plans) & H (differed annuity) . The model uses only the total number of new policyholders with indexation date 2018 and the total number of withdrawals in each policy.

3.1 Withdrawal probability estimation

Referring to the above table; firstly, we compute the success probability (λ) for the density. Which will account for the effect of the policyholders numbers and withdrawals in each policy.

$$\lambda = \frac{\texttt{number of withdrawals}}{\texttt{number of policyholders}}$$

Table 1: Number of policyholders and withdrawals in each policy.

Policy	**Number of policyholders**	**Number of Withdrawals**
A	4473	165
B	8140	97
C	13004	125
D	20119	85
E	28416	104
F	13511	27
G	2291	5
H	285	1
Total	**90239**	**609**

For Policy A: $\lambda = (165/4473) * 100 = 3.69\%$
For Policy B: $\lambda = (97/8140) * 100 = 1.19\%$
For Policy C: $\lambda = (125/13004) * 100 = 0.96\%$
For Policy D: $\lambda = (85/20119) * 100 = 0.42\%$
For Policy E: $\lambda = (104/28416) * 100 = 0.37\%$
For Policy F: $\lambda = (27/13511) * 100 = 0.20\%$
For Policy G: $\lambda = (5/2291) * 100 = 0.22\%$
For Policy H: $\lambda = (1/285) * 100 = 0.35\%$

Table 2: The values of lambda for each policy.

Policy	**Lambda (λ)**
A	3.69%
B	1.19%
C	0.96%
D	0.42%
E	0.37%
F	0.20%
G	0.22%
H	0.35%

Calculating the WPS

We can fit the geometric distribution results to obtain the WPS, using Microsoft excel sum-product function.

Table 3: Values of different parameters.

Policy	Policyholders	Withdrawals	p	q	λ
A	4473	165	0.27094	0.72906	0.03689
B	8140	97	0.15928	0.84072	0.01192
C	13004	125	0.20525	0.79475	0.00961
D	20119	85	0.13957	0.86043	0.00422
E	28416	104	0.17077	0.82923	0.00366
F	13511	27	0.04433	0.95567	0.00200
G	2291	5	0.00821	0.99179	0.00218
H	285	1	0.00164	0.99836	0.00351
	90239	609			
Total					

Where p is withdrawal risk in a particular policy divided by total withdrawals.

By applying the NBD we are able to get the result for each policy.

For example for Policy A:
Total withdrawals is 165
Bayesian estimate $= \frac{165}{609} = 27.09\%$

Now, the predicted withdrawal probabilities for the various cases are computed via the NBD as given below using Microsoft excel.
Geometric PMF will compute the withdrawal probabilities in every policy and these results will then be injected into the convolution model. In our example of policy G, the results are given in Table 5.
The results after running the convolution model provides a matrix for every policy. The resultant matrix for our example of policy A is provided in Table 6.
Similarly, matrices of each policy have been computed that provided us with the final results. The desired withdrawal probabilities are obtained from matrix against numbers of withdrawal in a specific policy. For example for policy A, withdrawal probability is 0.52%.

Table 4: Estimated withdrawal probabilities of the NBD.

x	f(x) PolicyA	f(x) PolicyB	f(x) PolicyC	f(x) PolicyD	f(x) PolicyE	f(x) PolicyF	f(x) PolicyG	f(x) PolicyH
1	0.27094	0.15928	0.20525	0.13957	0.17077	0.04433	0.00821	0.00164
2	0.10704	0.04266	0.06696	0.03352	0.04837	0.00376	0.00013	0.00001
3	0.06343	0.01714	0.03277	0.01208	0.02055	0.00048	0.00000	0.00000
4	0.04176	0.00765	0.01782	0.00483	0.00970	0.00007	0.00000	0.00000
5	0.02887	0.00358	0.01017	0.00203	0.00481	0.00001	0.00000	0.00000
6	0.02053	0.00173	0.00597	0.00088	0.00245	0.00000	0.00000	0.00000
7	0.01487	0.00085	0.00357	0.00039	0.00127	0.00000	0.00000	0.00000
8	0.01091	0.00042	0.00217	0.00017	0.00067	0.00000	0.00000	0.00000
9	0.00808	0.00021	0.00132	0.00008	0.00036	0.00000	0.00000	0.00000
10	0.00603	0.00011	0.00082	0.00004	0.00019	0.00000	0.00000	0.00000
11	0.00453	0.00005	0.00051	0.00002	0.00010	0.00000	0.00000	0.00000
12	0.00341	0.00003	0.00032	0.00001	0.00006	0.00000	0.00000	0.00000
13	0.00259	0.00001	0.00020	0.00000	0.00003	0.00000	0.00000	0.00000
14	0.00196	0.00001	0.00012	0.00000	0.00002	0.00000	0.00000	0.00000
15	0.00150	0.00000	0.00008	0.00000	0.00001	0.00000	0.00000	0.00000
16	0.00114	0.00000	0.00005	0.00000	0.00000	0.00000	0.00000	0.00000
17	0.00087	0.00000	0.00003	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5: Estimated withdrawal probabilities of the geometric distribution.

x	f(x) PolicyA	f(x) PolicyB	f(x) PolicyC	f(x) PolicyD	f(x) PolicyE	f(x) PolicyF	f(x) PolicyG	f(x) PolicyH
1	0.03689	0.01192	0.00961	0.00422	0.00366	0.00200	0.00218	0.00351
2	0.03553	0.01177	0.00952	0.00421	0.00365	0.00199	0.00218	0.00350
3	0.03422	0.01163	0.00943	0.00419	0.00363	0.00199	0.00217	0.00348
4	0.03295	0.01150	0.00934	0.00417	0.00362	0.00199	0.00217	0.00347
5	0.03174	0.01136	0.00925	0.00415	0.00361	0.00198	0.00216	0.00346
6	0.03057	0.01122	0.00916	0.00414	0.00359	0.00198	0.00216	0.00345
7	0.02944	0.01109	0.00907	0.00412	0.00358	0.00197	0.00215	0.00344
8	0.02835	0.01096	0.00898	0.00410	0.00357	0.00197	0.00215	0.00342
9	0.02731	0.01083	0.00890	0.00408	0.00355	0.00197	0.00214	0.00341
10	0.02630	0.01070	0.00881	0.00407	0.00354	0.00196	0.00214	0.00340
11	0.02533	0.01057	0.00873	0.00405	0.00353	0.00196	0.00214	0.00339
12	0.02440	0.01044	0.00864	0.00403	0.00352	0.00195	0.00213	0.00338
13	0.02350	0.01032	0.00856	0.00402	0.00350	0.00195	0.00213	0.00336
14	0.02263	0.01020	0.00848	0.00400	0.00349	0.00195	0.00212	0.00335
15	0.02180	0.01008	0.00840	0.00398	0.00348	0.00194	0.00212	0.00334
16	0.02099	0.00996	0.00832	0.00396	0.00346	0.00194	0.00211	0.00333
17	0.02022	0.00984	0.00824	0.00395	0.00345	0.00194	0.00211	0.00332

Actual portfolio

We computed all the inputs and results of the actual portfolio as tabulated in Table 7.

Case One

First, we study the behaviour of the proposed model by doubling the **policyholders** number in the portfolios A,B,C and D. The models input data,

Table 6: Convolution Matrix.

x	f(x)	fX*0(x)	fX*1(x)	fX*2(x)	fX*3(x)	fX*4(x)	fX*5(x)	fX*6(x)	fX*7(x)	fX*17(x)	fS(x)
1	0.00821	1	0.00821	6.74E-05	5.53E-07	4.54E-09	3.73E-11	3.06E-13	2.51E-15	3.5E-36	1.81E-05
2	0.000134	0	0.000134	2.2E-06	2.7E-08	2.96E-10	3.04E-12	2.99E-14	2.87E-16	9.69E-37	2.97E-07
3	3.27E-06	0	3.27E-06	7.15E-08	1.1E-09	1.45E-11	1.73E-13	1.95E-15	2.1E-17	1.5E-37	7.29E-09
4	8.87E-08	0	8.87E-08	2.33E-09	4.18E-11	6.28E-13	8.46E-15	1.06E-16	1.25E-18	1.71E-38	1.99E-10
5	2.53E-09	0	2.53E-09	7.59E-11	1.53E-12	2.56E-14	3.79E-16	5.17E-18	6.64E-20	1.6E-39	5.68E-12
6	7.41E-11	0	7.41E-11	2.47E-12	5.49E-14	9.99E-16	1.6E-17	2.36E-19	3.24E-21	1.3E-40	1.67E-13
7	2.21E-12	0	2.21E-12	8.05E-14	1.94E-15	3.8E-17	6.53E-19	1.02E-20	1.5E-22	9.55E-42	5.01E-15
8	6.69E-14	0	6.69E-14	2.62E-15	6.76E-17	1.41E-18	2.58E-20	4.29E-22	6.62E-24	6.44E-43	1.52E-16
9	2.04E-15	0	2.04E-15	8.54E-17	2.34E-18	5.18E-20	9.99E-22	1.75E-23	2.83E-25	4.07E-44	4.65E-18
10	6.28E-17	0	6.28E-17	2.78E-18	8.05E-20	1.87E-21	3.8E-23	6.95E-25	1.18E-26	2.43E-45	1.43E-19
11	1.94E-18	0	1.94E-18	9.06E-20	2.75E-21	6.72E-23	1.42E-24	2.72E-26	4.79E-28	1.38E-46	4.45E-21
12	6.04E-20	0	6.04E-20	2.95E-21	9.37E-23	2.39E-24	5.26E-26	1.05E-27	1.92E-29	7.58E-48	1.39E-22
13	1.89E-21	0	1.89E-21	9.61E-23	3.18E-24	8.42E-26	1.93E-27	3.97E-29	7.54E-31	4.01E-49	4.33E-24
14	5.91E-23	0	5.91E-23	3.13E-24	1.08E-25	2.95E-27	7.01E-29	1.49E-30	2.93E-32	2.06E-50	1.36E-25
15	1.86E-24	0	1.86E-24	1.02E-25	3.63E-27	1.03E-28	2.53E-30	5.56E-32	1.12E-33	1.03E-51	4.28E-27
16	5.84E-26	0	5.84E-26	3.32E-27	1.22E-28	3.58E-30	9.05E-32	2.05E-33	4.27E-35	5.04E-53	1.35E-28
17	1.84E-27	0	1.84E-27	1.08E-28	4.1E-30	1.24E-31	3.22E-33	7.52E-35	1.61E-36	2.41E-54	4.27E-30
pn			0.002182	0.002178	0.002173	0.002168	0.002163	0.002159	0.002154	0.002107	

Table 7: Actual portfolio values.

Policy	Policyholders	Withdrawals	Bayes estimates	Observed WPS	Predicted WPS
A	4473	165	27.094%	3.689%	1.3523%
B	8140	97	15.928%	1.192%	0.2253%
C	13004	125	20.525%	0.961%	0.2476%
D	20119	85	13.957%	0.422%	0.0685%
E	28416	104	17.077%	0.366%	0.0753%
F	13511	27	4.433%	0.200%	0.0093%
G	2291	5	0.821%	0.218%	0.0018%
H	285	1	0.164%	0.351%	0.0006%
Total	**90239**	**609**			

observed withdrawal probabilities from the original contracts and the predicted withdrawal probability from the described case is as shown in Table 8.

When the number of policyholders is doubled in policies A,B,C and D. The observed withdrawal probabilities change and become less for each policy

Table 8: Case one values.

Policy	Policyholders	Withdrawals	Bayes estimates	Observed WPS	Predicted WPS
A	8946	165	27.094%	1.844%	0.6808%
B	16280	97	15.928%	0.596%	0.1128%
C	26008	125	20.525%	0.481%	0.1240%
D	40238	85	13.957%	0.211%	0.0343%
E	28416	104	17.077%	0.366%	0.0753%
F	13511	27	4.433%	0.200%	0.0093%
G	2291	5	0.821%	0.218%	0.0018%
H	285	1	0.164%	0.351%	0.0006%
Total	**180478**	**609**			

(halved), Bayesian estimates remains the same this is because withdrawals have same weights. Finally, the predicted withdrawal probabilities reduce (halved) with the predicted withdrawal probabilities from the actual portfolio.

Case two

In this case, we have doubled the number of **withdrawals**. The table 9 below shows that as the number of withdrawals increase, the withdrawal probability also increases. However, it is the results of the Bayesian estimate that are noteworthy. Comparing the Bayesian probabilities of case 1, with table in case 2, we find no change. This is because withdrawals increase with same weights. Finally, the predicted withdrawal probabilities increase with the predicted withdrawal probabilities from the actual portfolio.

Case three

Here, we investigate the relation between the predicted withdrawal probabilities and all other entries when both the number of **policyholders** and **withdrawals** rise. (see Table 10)
Applying this case the observed withdrawal probabilities doesn't change values because the policyholders number and the withdrawals numbers are both doubled. However, the predicted withdrawal probabilities decrease. This

Table 9: Case two values.

Policy	Policyholders	Withdrawals	Bayes estimates	Observed WPS	Predicted WPS
A	4473	330	27.094%	7.378%	2.6685%
B	8140	194	15.928%	2.383%	0.4495%
C	13004	250	20.525%	1.922%	0.4941%
D	20119	170	13.957%	0.845%	0.1369%
E	28416	208	17.077%	0.732%	0.1505%
F	13511	54	4.433%	0.4%	0.0185%
G	2291	10	0.821%	0.436%	0.0036%
H	285	2	0.164%	0.702%	0.0012%
Total	**90239**	**1218**			

Table 10: Case three values.

Policy	Policyholders	Withdrawals	Observed WPS	Bayes estimates	Predicted WPS
A	8946	330	27.094%	3.689%	0.7229%
B	16280	194	15.928%	1.192%	0.0716%
C	26008	250	20.525%	0.961%	0.1014%
D	40238	170	13.957%	0.422%	0.0434%
E	56832	208	17.077%	0.366%	0.0673%
F	27022	54	4.433%	0.200%	0.0040%
G	4582	10	0.821%	0.218%	0.0001%
H	570	2	0.164%	0.351%	0.00001%
Total	**180478**	**1218**			

shows the uniqueness of the model (as the policyholders numbers rise the overall withdrawal probability decrease even though withdrawals have increased).

Case four

The table 11 shows the models behaviour if withdrawals are doubled only in policies E, F, G and H. The result is an increase in predicted WDs in these policies and a decrease in policies A,B,C and D.

Table 11: Case four.

Policy	Policyholders	Withdrawals	Observed WPS	Bayes estimates	Predicted WPS
A	4473	165	22.118%	3.689%	1.0367%
B	8140	97	13.003%	1.192%	0.1778%
C	13004	125	16.756%	0.961%	0.1931%
D	20119	85	11.394%	0.422%	0.0543%
E	28416	208	27.882%	0.732%	0.2822%
F	13511	54	7.239%	0.400%	0.0312%
G	2291	10	1.340%	0.436%	0.0059%
H	285	2	0.268%	0.702%	0.0019%
Total	**90239**	**746**			

4 Conclusion

In this manuscript we applied convolutions approach to derive the withdrawal probability for lapse rates obtained from a major Kenyan insurer for new policyholders with indexation date 2018. We have computed the Bayesian probabilities and predicted WPD for each case by using these two estimates. We computed the predicted distributions for each case with the convolutions technique. Further, we examined various cases to check the models behavior. The performance of the model was very well justified. We are able to explain the impact on withdrawal of different scenarios. We advice organizations to apply this methodology in risk management since its very practical.

Future research involves application of machine learning algorithms such as convolutions neural networks to better assess the withdrawal probabilities. We hope to explore this point in a future paper.

COMPETING INTERESTS DISCLAIMER:
Authors have declared that they have no known competing financial interests OR non-financial interests OR personal relationships that could have appeared to influence the work reported in this paper.

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